Unit - 4

Integration:

Meaning:

Basic Integration: is the inverse process of differentiation.

Ex: + Differentiation of total revenue function gives ~~total~~ marginal revenue function.

+ Integration of marginal Revenue gives total revenue function.

By Integrating x^n with respect to x we generally get $\frac{x^{n+1}}{n+1} + c$, where c is constant. Sometimes a condition may be given based on which a definite value can be determined for c.

$\frac{x^{n+1}}{n+1} + c$ is known as the indefinite integral of x^n with respect to x.

This is written in a symbol as

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

$\int x^n dx$ is read integral of x^n with respect to x.

* In general, let $\int f(x)\,dx = F(x) + c$ is called the integrand. $F(x)$ is called the integral or antiderivative of $f(x)$.

The process of finding the integral of a function is called integration.

* Some standard Forms of Integrals:

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ $\quad\because \dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1} + c\right) = x^n$.

2. $\int \dfrac{1}{x}\,dx = \log x + c$ $\quad\because \dfrac{d}{dx}\log(\log x + c) = \dfrac{1}{x} = x^{-1}$.

3. $\int e^x\,dx = e^x + c$ $\quad\because \dfrac{d}{dx}(e^x + c) = e^x$.

4. $\int a^x\,dx = \dfrac{a^x}{\log_e a} + c$, if $a > 0$.

5. $\int k f(x)\,dx = k \int f(x)\,dx$.

6. $\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$.

Note: i) when $n > 0$ in the 1st form.

$\int 1\,dx = x + c$ $\quad\because x^0 = 1$.

ii) $\int k\,dx = kx + c$.

Ex ①: Evaluate $\int x^{1/2}\,dx$

Soln: $\int x^{1/2}\,dx = \dfrac{x^{1/2+1}}{1/2+1} + c = \dfrac{x^{3/2}}{3/2} + c$

reciprocal

$= \dfrac{2}{3}\, x^{3/2} + c$ $\quad \Big| \ \dfrac{1}{2} + 1 = 3/2.$ //

② Evaluate $\int x^2\,dx$.

Soln:

$\int x^2\,dx = \dfrac{x^{2+1}}{2+1} + c = \dfrac{x^3}{3} + c.$ //

③ Integrate with respect to x for the foll: $\int \left(2^x + \dfrac{4}{x} - \dfrac{a}{x^{1/3}}\right) dx.$

Soln: $= \int 2^x\,dx + 4\int \dfrac{1}{x}\,dx - a\int \dfrac{1}{x^{1/3}}\,dx.$

$= \dfrac{2^x}{\log 2} + 4\log x - a\,\dfrac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c$

$\dfrac{1}{x^{1/3}}$

$x^{-1/3}$

$= \dfrac{2^x}{\log 2} + 4\log x - a\,\dfrac{x^{2/3}}{-2/3} + c.$

$= \dfrac{2^x}{\log 2} + 4\log x - \dfrac{3a}{2}\, x^{2/3} + c$ //

④ Evaluate: $\int (2x^2+5x^3+4x^4)\,dx$.

Soln:

$\int (2x^2+5x^3+4x^4)\,dx = 2x^2dx+5x^3dx+4x^4dx$.

$= 2\int x^2dx + 5\int x^3dx + 4\int x^4dx$.

$= \frac{2x^3}{3}+\frac{5x^4}{4}+\frac{4x^5}{5}+c$ //.

⑤ Evaluate: $\int (x+\frac{1}{x})^2\,dx$.

Soln: $\int (x+\frac{1}{x})^2\,dx$. $(a+b)^2 = a^2+2ab+b^2$

$= \int (x^2+2\cancel{x}\frac{1}{\cancel{x}}+\frac{1}{x^2})\,dx$.

$= \int x^2dx + \int 2dx + \int \frac{1}{x^2}dx$.

$= \int x^2dx + \int 2dx + \int x^{-2}dx$.

$= \frac{x^3}{3}+2x+\frac{x^{-2+1}}{-2+1}+c$.

$= \frac{x^3}{3}+2x+\frac{x^{-1}}{-1}+c$.

$= \frac{x^3}{3}+2x-\frac{1}{x}+c$ //.

(6) Integrate $e^x - 1$ with respect to x.

Soln:

$$\int (e^x - 1)\,dx = \int e^x\,dx - \int 1\,dx$$

$$= e^x - x + C$$

(7) Integrate: $\dfrac{x^3 - x + 4}{x^2}$ with respect to x.

Soln:

$$\int \left(\frac{x^3 - x + 4}{x^2}\right) dx = \int \frac{x^3}{x^2}\,dx - \int \frac{x}{x^2}\,dx + \int \frac{4}{x^2}\,dx. \quad \cancel{\int \frac{4}{x^2}\,dx.}$$

$$= \int x\,dx - \int \frac{1}{x}\,dx + 4\int x^{-2}\,dx.$$

$$= \frac{x^{1+1}}{1+1} - \log x + 4\,\frac{x^{-2+1}}{-2+1} + c.$$

$$= \frac{x^2}{2} - \log x + 4\,\frac{x^{-1}}{-1} + c.$$

$$= \frac{x^2}{2} - \log x - 4x^{-1} + c.$$

$$= \frac{1}{2}x^2 - \log x - \frac{4}{x} + c.$$

Methods of Integration by parts:

let us consider, so far, integral consists of sum or differences or quotient of functions, when the integral is a product of functions. It is to be integrated by parts.

$\therefore$ The rule is $\int u\,dv = uv - \int v\,du$.

This has been obtained from the product rule of differentiation. $\frac{d}{dx}(uv) = \frac{u\,dv}{dx} + \frac{v\,du}{dx}$.

$\Rightarrow$ Integrating both sides, we get:

$$\int d(uv)\,dx = \int u\frac{dv}{\cancel{dx}}\,\cancel{dx} + \int v\frac{du}{\cancel{dx}}\,\cancel{dx}$$

$$uv = \int u\,dv + \int v\,du$$

$$\therefore \int u\,dv = uv - \int v\,du.$$

when the integrand is a product of two function, which one is to be taken as 'u' & which one is to be taken as dv are to decided.

Ex ① ~~which one is to be taken as~~

Integrate $x \log x$ w.r.to x.

Soln: $\int x \log x \, dx$ is to be found.

let $u = \log x$. | let $dv = x\,dx$

$du = \frac{1}{x} dx$ | $\therefore \int dv = \int x\,dx$

$v = \frac{x^2}{x}$.

Based on $\int u\,dv = uv - \int v\,du$.

$\int x \log x = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

Note: $\cdot \Rightarrow$ Multiply sign.

$= \frac{x^2}{2} \log x - \frac{1}{2} \int x\,dx$.

$= \frac{x^2}{2} \log x - \frac{1}{2} \cdot \frac{x^2}{2} + c$.

$= \frac{x^2}{2} \log x - \frac{x^2}{4} + c$.

② Evaluate $\int x e^{mx} dx$.

Soln: let $u = x$ | let $dv = e^{mx} dx$

$du = dx$. | $\therefore \int dv = \int e^{mx} dx$.

$v = \frac{e^{mx}}{m}$.

Based on $\int u\,dv = uv - \int v\,du$

$$= x \cdot \frac{e^{mx}}{m} - \int \frac{e^{mx}}{m} dx.$$

$$= x \cdot \frac{e^{mx}}{m} - \frac{1}{m} \int e^{mx} dx.$$

$$= x \cdot \frac{e^{mx}}{m} - \frac{1}{m} \int \left(\frac{e^{mx}}{m}\right) + c.$$

$$\therefore \int x e^{mx} dx = \frac{e^{mx}}{m}\left(x - \frac{1}{m}\right) + c$$ //

③ Evaluate $\int \left(\frac{1 + x\log x}{x}\right) e^x dx$.

Soln:
$$\int \left(\frac{1 + x\log x}{x}\right) e^x dx = \int \frac{1}{x} e^x dx + \int \log x \cdot e^x dx. \longrightarrow ①.$$

consider $\int \log x \cdot e^{x} dx$.

let $u = \log x$, $du = \frac{1}{x} dx$

let $dv = e^x \cdot dx$. $\therefore \int dv = \int e^x \cdot dx$, $v = e^x$.

Based on $\int u\,dv = uv - \int v\,du$.

$$\int \log x \cdot e^x dx = \log x \cdot e^x - \int e^x \cdot \frac{1}{x} dx.$$

By satisfying in eq ①.

$$\int \left(\frac{1+x\log x}{x}\right) \cdot e^x \cdot dx = \int \frac{1}{x} \cdot e^x dx + \log x \cdot e^x - \int \frac{1}{x} e^x dx$$

$$\cancel{e^x dx} = e^x \log x + c$$

(4) Evaluate: $\int x^2 \cdot e^x dx$ by integration by parts.

Soln: Let $u = x^2$, $du = 2x\,dx$ | let $dv = e^x dx$

$\therefore \int dv = \int e^x dx$

$v = e^x$

Based on $\int u\,dv = uv - \int v\,du$

$$\int x^2 \cdot e^x dx = x^2 e^x - \int e^x \cdot 2x\,dx$$

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx \rightarrow (1)$$

In $\int x e^x dx$

let $u = x$, $du = dx$ | let $dv = e^x dx$

$\int dv = \int e^x dx$

$v = e^x$

Hence $\int x \cdot e^x dx = x \cdot e^x - \int e^x dx$

$= x \cdot e^x - e^x$

By sub eq (1)

$$\int x^2 e^x \, dx = x^2 e^x - 2(xe^x - e^x) + c$$
$$= x^2 e^x - 2xe^x + 2e^x + c$$
$$= e^x(x^2 - 2x + 2) + c$$

Using in Economics (uses of Integrals)

Integration of integral cost function gives total cost function. Similarly, by integrating Marginal Revenue function, total revenue functions can be found. Consumers Surplus & producers Surplus can also be found by Integration.

Ex: The Marginal cost function for producing x units is $y = 23 + 16x - 3x^2$ & the total cost for producing 1 unit is 40. obtain the total cost function & the average cost function.

Soln: TC fun. $C = \int (23 + 16x - 3x^2)\, dx$

$$= 23x + \frac{\cancel{16}^{8} x^2}{\cancel{2}} - \frac{\cancel{3} x^3}{\cancel{3}} + c$$

$$= 23x + 8x^2 - x^3 + c$$

It is given that the total cost of 1 unit is 40, that in $(23x+8x^2-x^3+c)$ at $x=1=40$.

$$23+8-1+c=40.$$

$$30+c=40$$

$$c=40-30=10.$$

Hence the Total cost fn;

$$C=23x+8x^2-x^3+10$$

Av. cost fn: $\frac{C}{x}=\frac{23\cancel{x}}{\cancel{x}}+\frac{8\cancel{x}^2}{\cancel{x}}-\frac{\cancel{x^3}}{\cancel{x}}+\frac{10}{x}$.

$$\Rightarrow \frac{C}{x}=23+8x-x^2+\frac{10}{x}.$$