10:9. FINDING AN INITIAL BASIC FEASIBLE SOLUTION

There are several methods available to obtain an initial basic feasible solution. However, we shall discuss here the following three methods :

1. North-West Corner Method,
2. Least-Cost Method, and
3. Vogel's Approximation Method (or Penalty Method).

1. North-West Corner Method (*NWC Rule*)

It is a simple and efficient method to obtain an initial basic feasible solution. Various steps of the method are :

Step 1. Select the north-west (upper left hand) corner cell of the transportation table and allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, *i.e.*, x_{11} = min. (a_1, b_1).

Step 2. If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude x_{21} = min. $(a_2, b_1 - x_{11})$ in the cell (2, 1).

If $b_1 < a_1$, we move right horizontally to the second column and make the second allocation of magnitude x_{12} = min. $(a_1 - x_{11}, b_2)$ in the cell (1, 2).

If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude.

$$x_{12} = \text{min. } (a_1 - a_1, b_1) = 0 \text{ in the cell } (1, 2).$$

or

$$x_{21} = \text{min. } (a_2, b_1 - b_1) = 0 \text{ in the cell } (2, 1).$$

Step 3. Repeat *steps 1* and *2* moving down towards the lower/right corner of the transportation table until all the rim requirements are satisfied.

SAMPLE PROBLEM

1001. *Obtain an initial basic feasible solution to the following transportation problem using the north-west corner rule :*

	D	*E*	*F*	*G*	*Available*
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	200	225	275	250	

[Pune M.B.A. 1999; Madras B.Com. 2005]

Solution. Since $\sum_i a_i = \sum_j b_j = 950$, there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows :

The transportation table of the given problem has 12 cells. Following north-west corner method, the first allocation is made in the cell (1, 1), the magnitude being x_{11} = min. (250, 200) = 200. The second allocation is made in the cell (1, 2) and the magnitude of the allocation is given by

$$x_{12} = \text{min. } (250 - 200, 225) = 50.$$

The third allocation is made in the cell (2, 2), the magnitude being x_{22} = min. (300, 225 – 50) = 175. The magnitude of fourth allocation in the cell (2, 3) is given by x_{23} = min. (300 – 175, 275) = 125. The fifth allocation is made in the cell (3, 3), the magnitude being x_{33} = min. (400, 275 – 125) = 150 and the sixth (last) allocation is made in the cell (3, 4) with magnitude x_{34} = min. (400 – 150, 250) = 250. Hence an initial basic feasible solution to the given T.P. has been obtained and is displayed in Table 10.3.

[200] 11	[50] 13	17	14	250
16	[175] 18	[125] 14	10	300
21	24	[150] 13	[250] 10	400
200	225	275	250	

Table 10.3

The transportation cost according to the above route is given by

$$z = 200 \times 11 + 50 \times 13 + 175 \times 18 + 125 \times 14 + 150 \times 13 + 250 \times 10 = 12{,}200.$$

Remarks 1. The cells which get allocation will be called **basic cells.**

2. The initial basic feasible solution obtained by means of north-west corner rule may be far from optimum, because the costs were completely ignored.

2. Least-Cost Method or Matrix Minima Method

This method takes into account the minimum unit cost and can be summarized as follows :

Step 1. Determine the smallest cost in the cost matrix of the transportation table. Let it be c_{ij}. Allocate x_{ij} = min. (a_i, b_j) in the cell (i, j).

Step 2. If $x_{ij} = a_i$ cross off the ith row of the transportation table and decrease b_j by a_i. Go to step 3.
If $x_{ij} = b_j$ cross off the jth column of the transportation table and decrease a_i by b_j. Go to Step 3.
If $x_{ij} = a_i = b_j$ cross off either the ith row or jth column but not both.

Step 3. Repeat *steps 1* and *2* for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

SAMPLE PROBLEM

1002. *Obtain an initial basic feasible solution to the following T.P. using the matrix minima method :*

	D_1	D_2	D_3	D_4	*Capacity*
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

where O_i and D_j denote ith origin and jth destination respectively.

Solution. The transportation table of the given T.P. has 12 cells. Following the least-cost method, the first allocation is made in the cells (3, 1), the magnitude being $x_{31} = 4$. This satisfies the requirement at destination D_1 and thus we cross off the first column from the table. The second allocation is made in the cell (2, 4) of magnitude x_{34} = min. (6, 8) = 6. Cross off the fourth column of the table. This yields Table 10.4. There is, again, a tie for the third allocation. We choose arbitrarily the cell (1, 2) and allocate x_{12} = min. (6, 6) = 6, there. Cross off either the second column or the first row. We choose to cross off the first row of the table. The next allocation of magnitude $x_{32} = 0$ is made in cell (3, 2). Cross off the second column getting Table 10.5.

1	2	3	4	6
4	3	2	[6] 0	8
[4] 0	2	2	[4] 1	10
4	6	8	6	

Table 10.4

	[6] 2	3		6
	3	2	[6]	2
[4]	[0] 2	2		6
	6	8		

Table 10.5

We choose arbitrarily again, to make the next allocation in cell (2, 3) of magnitude x_{23} = min. (2, 8) = 2. Cross off the second row. This gives Table 10.6. The last allocation of magnitude x_{33} = min. (6, 6) = 6 is made in the cell (3, 3).

1	[6] 2	3	4	
4	3	[2] 2	[6] 0	2
[4] 0	[$\in$] 2	[6] 2	1	6
		8		

Table 10.6

1	[6] 2	3	4
4	3	[2] 2	[6] 0
[4] 0	[$\in$] 2	[6] 2	1

Table 10.7

Now, all the rim requirements have been satisfied and hence an initial feasible solution has been determined. This solution is displayed in transportation Table 10.7.

Since the cells do not form a loop, the solution is basic one. Moreover the solution is degenerate also. The transportation cost according to the above route is given by

$$z = 6 \times 2 + 2 \times 2 + 6 \times 0 + 4 \times 0 + 2 \times \in + 6 \times 2 = 28 + 2\in = 28 \text{ as } \in \to 0.$$

3. Vogel's Approximation Method (*VAM*)

The *Vogel's Approximation Method* takes into account not only the least cost c_{ij} but also the costs that *just exceed* c_{ij}. The steps of the method are given below :

Step 1. For each row of the transportation table identify the *smallest* and the *next-to-smallest* costs. Determine the *difference* between them for each row. Display them alongside the transportation table by enclosing them in parenthesis against the respective rows. Similarly, compute the differences for each column.

Step 2. Identify the row or column with the largest difference among all the rows and columns. If a tie occurs, use any arbitrary tie-breaking choice. Let the greatest difference correspond to *i*th row and let c_{ij} be the smallest cost in the *i*th row. Allocate the maximum feasible amount x_{ij} = min. (a_i, b_j) in the (i, j)th cell and cross off either the *i*th row or the *j*th column in the usual manner.

Step 3. Recompute the column and row differences for the reduced transportation table and go to step 2. Repeat the procedure until all the rim requirements are satisfied.

Remarks 1. A row or column "difference" indicates the minimum unit penalty incurred by failing to make an allocation to the smallest cost cell in that row or column.

2. It will be seen later that *VAM* determines an initial basic feasible solution which is very close to the optimum solution, that is, the number of iterations required to reach the optimum solution is smaller in this case.

SAMPLE PROBLEM

1003. *Use Vogel's Approximation Method to obtain an initial basic feasible solution transportation problem :*

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

Solution. Following *VAM*, the differences between the smallest and next-to-smallest costs i row and each column are computed and displayed inside the parenthesis against the respective and columns. The largest of these differences is (5) and is associated with the first colu the transportation table. Since the minimum cost in the first column is $c_{11} = 11$, we al x_{11} = min. (250, 200) = 200 in the cell (1, 1). This exhausts the requirement of the first colum therefore, we cross off the first column. The row and column differences are now computed f resulting reduced transportation Table 10.8, the largest of these is (5) which is associated wi second column. Since c_{12} (= 13) is the minimum cost we allocate x_{12} = min. (50, 225) = 50.

[200] 11	13	17	14	250 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)
200 (5)	225 (5)	275 (1)	250 (0)	

Table 10.8

[50] 13	17	14	50 (1)
18	14	10	300 (4)
24	13	10	400 (3)
225 (5)	275 (1)	250 (0)	

Table 10.9

This exhausts the availability of first row and, therefore, we cross off the first row. Continuin this manner, the subsequent reduced transportation tables and the differences for the surviving and columns are shown below :

[175] 18	14	10	300 (4)
24	13	10	400 (3)
175 (6)	275 (1)	250 (0)	

14	[125] 10	125 (4)
13	10	400 (3)
275 (1)	250 (0)	

[275] 13	[125] 10	400
275	125	

Table. 10.10

Eventually, the basic feasible solution shown in Table 10.11 is obtained :

[200] 11	[50] 13	17	14
16	[175] 18	14	[125] 10
21	24	[275] 13	[125] 10

Table. 10.11

The transportation cost according to this route is given by

$$z = 200 \times 11 + 50 \times 13 + 175 \times 18 + 125 \times 10 + 275 \times 13 + 125 \times 10 = 12{,}075.$$

Remark. It may be observed that transportation cost by North-West Corner method was 12,200 (sample problem 1001) whereas in VAM (sample problem 1003) is 12,075.

1004. *Find the initial basic feasible solution to the following transportation problem using VAM, given the cost matrix :*

	D_1	D_2	D_3	D_4	*Supply*
S_1	20	25	28	31	200
S_2	32	28	32	41	180
S_3	18	35	24	32	110
Demand:	150	40	180	170	

[Osmania M.B.A. 2001]

Solution. Here, total demand is 540 and total supply is 490. Since, total demand ≠ total supply, we introduce a dummy row with its supply as (540 – 490), *i.e.*, 50 and take all the cost elements of this row as zero. Thus, the transportation table for the initial basic feasible solution of the given problem is :

	D_1	D_2	D_3	D_4	
S_1	20	25	28	31	200
S_2	32	28	32	41	180
S_3	18	35	24	32	110
Dummy	0	0	0	0	50
	150	40	180	170	

Table 10.12

Following VAM, the differences between the smallest and next to smallest costs in each row and each column are computed and displayed inside the parenthesis against the respective rows and columns. The largest of these differences is (31) and is associated with the fourth column of the transportation table. As the least cost in the 4th column is $c_{44} = 0$, we allocate x_{44} = min. (50, 170) = 50 in the cell (4, 4). This exhausts the supply of fourth row and, therefore we cross off this row.

The row and column differences of the reduced transportation table are now computed. The largest of these is (6) and is associated with the third row. Since least cost in third row is c_{31} = 18, we allocate x_{31} = min. (110, 150) = 110. This exhausts the supply of third row, and, therefore we cross off the third row.

The row and column differences of the resulting transportation table are now computed. The largest of these is (12) associated with the first column. As the least cost in the first column is c_{11} = 20, we allocate x_{11} = min. (200, 40) = 40. This exhausts the demand of first column, and, therefore we cross off the first column.

Again, the row and coloumn differences for the reduced transportation table are computed. The largest among these is (10) associated with fourth column. Thus x_{14} = min. (160, 120) = 120 is allocated to the cell (1, 4) being the least cost cell of the fourth column. As this allocation exhausts the demand of the 4th column, we cross off the fourth column for further consideration.

Continuing in this manner, the remaining allocations are : x_{22} = 40, x_{13} = 40 and x_{23} = 140.

The transportation table showing all the assignments is displayed in Table 10.13.

	D_1	D_2	D_3	D_4	Row penalty
S_1	[40] 20	25	[40] 28	[120] 31	(5) (5) (5) (6) (3)
S_2	32	[40] 28	[140] 32	41	(4) (4) (4) (4) (4)
S_3	[110] 18	35	24	32	(6) (6) — — —
Dummy	0	0	0	[50] 0	(0) — — — —
Column penalty	(18) (2) (12) — —	(25) (3) (3) (3) (3)	(24) (4) (4) (4) (4)	(31) (1) (10) (10) —	

Table 10.13

The number of allocated cells in the above table are 7, which is equal to the required numb $m+n-1$ (*i.e.*, $4+4-1 = 7$), therefore, this solution is non-degenerate basic feasible. T transportation cost associated with the above solution is :

Total cost $= 40 \times 20 + 40 \times 28 + 120 \times 31 + 40 \times 28 + 140 \times 32 + 110 \times 18 + 50 \times 0 = 13{,}220.$

PROBLEMS

1005. Determine an initial basic feasible solution to the following transportation problem using North-W Corner method :

	D_1	D_2	D_3	D_4	*Availability*
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

[Madras B.Com. (Nov.) 20

1006. Consider the following transportation problem :

Source	*Destination*				*Availability*
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
Requirement	60	40	30	110	240

Determine an initial basic feasible solution using the (*a*) row minima method, and (*b*) Vogel's approximati method.

[Delhi B.Sc. (Stat.) 199

1007. Obtain an initial basic feasible solution to the following T.P. using the Vogel's approximation method

Warehouses	*Stores*				*Availability*
	I	*II*	*III*	*IV*	
A	5	1	3	3	34
B	3	3	5	4	15
C	6	4	4	3	12
D	4	–1	4	2	19
Requirement	21	25	17	17	80

[Mahatma Gandhi M.Com. 2010

1008. Find an initial basic feasible solution to the following T.P. using (*a*) North-West Corner Rule and (*b*) Vogel's Approximation Method.

Factories	*Warehouses*					*Availability*
	W_1	W_2	W_3	W_4	W_5	
F_1	20	28	32	55	70	50
F_2	48	36	40	44	25	100
F_3	35	55	22	45	48	150
Requirement	100	70	50	40	40	

[Osmania M.B.A. 1999]

1009. Consider the following transportation table showing production and transportation costs, along with the supply and demand positions of factories/distribution centres :

	M_1	M_2	M_3	M_4	*Supply*
F_1	4	6	8	13	500
F_2	13	11	10	8	700
F_3	14	4	10	13	300
F_4	9	11	13	3	500
Demand	250	350	1,050	200	

(*a*) Obtain an initial basic feasible solution by using *VAM*.

(*b*) Find out an optimal solution for the above given problem. *[Kerala M.Com. 1994]*

***Theorem 11-2** If $c_{ij} \geq 0$, such that minimum $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = 0$, then the feasible solution x_{ij} provides an optimum assignment.*

The proof is left as an exercise to the reader.

The above two theorems form the basis of *Assignment Algorithm*. By selecting suitable constants to be added to or subtracted from the elements of the cost matrix we can ensure that each $c_{ij}^* \geq 0$ and can produce at least one $c_{ij}^* = 0$ in each row and each column and try to make assignments from among these 0 positions. The assignment schedule will be optimal if there is exactly one assignment (*i.e.*, exactly one assigned 0) in each row and each column.

Remarks. It may be noted that assignment problem is a variation of transportation problem with two characteristics (*i*) the cost matrix is a square matrix, and (*ii*) the optimum solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

11:3. SOLUTION METHODS OF ASSIGNMENT PROBLEM

An assignment problem can be solved using the following four methods :

1. Complete Enumeration Method. In this method, a list of all possible assignments among the given resources and activities is prepared. Then an assignment involving the minimum cost, time or distance (or maximum profit) is selected. It represents the optimum solution. In case there are more than one assignment patterns involving the same least cost, then they all represent the optimum solutions — the problem has multiple optima.

In general, if there are n jobs and n workers, there are $n!$ possible assignments. Thus, the listing and evaluation of all the possible assignments is a simple matter when n is small. When n is large, this method is not very practical. For example, if there are 8 jobs and 8 workers, we have to evaluate a total of 8 ! or 40,320 assignments. The method, therefore, is not suitable for real world situations.

2. Transportation Method. Since an assignment problem is a special case of the transportation problem, it can be solved by transportation methods discussed in the previous chapter. However, every basic feasible solution of a general assignment problem having a square payoff matrix of order n should have $m + n - 1 = n + n - 1 = 2n - 1$ assignments or basic cells. But due to the special structure of this problem, any basic solution cannot have more than n assignments. Thus, the assignment problem is inherently degenerate. In order to remove degeneracy, $(n - 1)$ dummy allocations will be required to proceed with the transportation method. However, because of the large number of dummy allocations in the solution, the transportation method becomes computationally inefficient for solving an assignment problem.

3. Simplex Method. An assignment problem can be formulated as a transportation problem which, in turn, is itself a special case of an LPP. Accordingly, an assignment problem can be formulated as an LPP with integer valued variables and may be solved using a modified simplex method or otherwise. Here, the decision variables take only one of the two values : 1 or 0.

In general let

$$x_{ij} = \begin{cases} 1 & \text{if } i\text{th person is assigned } j\text{th job} \\ 0 & \text{if } i\text{th person is not assigned } j\text{th job} \end{cases}$$

The mathematical formulation of the assignment problem as a 0–1 integer linear programming problem would be :

$$\text{Minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad \text{subject to the constraints :}$$

$$x_{i1} + x_{i2} + \ldots + x_{in} = 1 \, ; \qquad i = 1, 2, \ldots, n$$

$$x_{1j} + x_{2j} + \ldots + x_{nj} = 1 \, ; \qquad j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j.$$

As can be seen in the general mathematical formulation of the assignment problem, there are $n \times n$ decision variables and $n + n$ or $2n$ equalities/equations. In particular, for a problem involving 5 workers/jobs, there will be 25 decision variables and 10 equalities. That means a simplex table having 25 columns and 10 rows. It is difficult to solve manually and hence this approach to the solution is not considered.

4. Hungarian Assignment Method. An efficient method for solving an assignment problem is the *Hungarian Assignment Method* (also known as reduced matrix method), which is based on the concept of opportunity cost. Opportunity costs show the relative penalties associated with assignment of resource to an activity as opposed to making the best or least cost assignment. If we can reduce the cost matrix to the extent of having at least one zero in each row and each column, then it will be possible to make optimal assignments (opportunity costs are all zero).

The method of solving an assignment problem (minimization case) consists of the following steps :

Step 1. Determine the cost table from the given problem.

(*i*) If the number of sources is equal to the number of destinations, go to *step 3*.

(*ii*) If the number of sources is not equal to the number of destinations, go to *step 2*.

Step 2. Add a dummy source or dummy destination, so that the cost table becomes a square matrix. The cost entries of dummy source/destinations are always zero.

Step 3. Locate the smallest element in each row of the given cost matrix and then subtract the same from each element of that row.

Step 4. In the reduced matrix obtained in *step 3*, locate the smallest element of each column and then subtract the same from each element of that column. Each column and row now have at least one zero.

Step 5. In the modified matrix obtained in *step 4*, search for an optimal assignment as follows :

(*a*) Examine the rows successively until a row with a single zero is found. Enrectangle this zero (□) and cross off (×) all other zeros in its column. Continue in this manner until all the rows have been taken care of.

(*b*) Repeat the procedure for each column of the reduced matrix.

(*c*) If a row and/or column has two or more zeros and one cannot be chosen by inspection, then assign arbitrary any one of these zeros and cross off all other zeros of that row/column.

(*d*) Repeat (*a*) through (*c*) above successively until the chain of assigning (□) or cross (×) ends.

Step 6. If the number of assignments (□) is equal to n (the order of the cost matrix), an optimum solution is reached.

If the number of assignments is less than n (the order of the matrix), go to the next step.

Step 7. Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced matrix. This can be conveniently done by using a simple procedure :

(*a*) Mark (√) rows that do not have any assigned zero.

(*b*) Mark (√) columns that have zeros in the marked rows.

(*c*) Mark (√) rows that have assigned zeros in the marked columns.

(*d*) Repeat (*b*) and (*c*) above until the chain of marking is completed.

(*e*) Draw lines through all the *unmarked rows* and *marked columns*. This gives us the desired minimum number of lines.

Step 8. Develop the new revised cost matrix as follows :

(*a*) Find the smallest element of the reduced matrix not covered by any of the lines.

(*b*) Subtract this element from all the *uncovered* elements and add the same to all the elements lying at the *intersection* of any two lines.

Step 9. Go to *Step 6* and repeat the procedure until an optimum solution is attained.

SAMPLE PROBLEMS

1101. *A departmental head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take to perform each task, is given in the matrix below :*

Tasks	*Men* E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man-hours?

[Andhra B.E. (Mech. & Ind.) 1996]

Solution.

Step 1. Here, the number of tasks and the number of subordinates each equal 4, therefore the problem is balanced and we move on to *step 3.*

Step 3. Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix :

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step 4. Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

Step 5. Starting with row *1*, we enrectangle (□) (i.e., make assignment) a single zero, if any, and cross (×) all other zeros in the column so marked. Thus, we get

7	11	5	[0]
[0]	11	~~0~~	13
23	[0]	2	~~0~~
9	12	13	~~0~~

In the above matrix, we arbitrarily enrectangled a zero in column *1*, because row 2 had two zeros. It may be noted that column *3* and row *4* do not have any assignment. So, we move on to the next step.

Step 7. (*i*) Since row *4* does not have any assignment, we mark this row (√).

(*ii*) Now there is a zero in the fourth column of the marked row. So, we mark fourth column (√).

(*iii*) Further there is an assignment in the first row of the marked column. So we mark first row (√).

(*iv*) Draw straight lines through all unmarked rows and marked columns. Thus, we have

7	11	5	[0]	√
[0]	11	~~0~~	13	
23	[0]	2	~~0~~	
9	12	13	~~0~~	√
			√	

Step 8. In *step 7*, we observe that the minimum number of lines so drawn is 3, which is less th the order of the cost matrix, indicating that the current assignment is not optimum.

To increase the minimum number of lines, we generate new zeros in the modified matrix.

The smallest element not covered by the lines is 5. Subtracting this element from all uncovered elements and adding the same to all the elements lying at the intersection of the lines, we obtain the following new reduced cost matrix :

2	6	0	0
0	11	0	18
23	0	2	5
4	7	8	0

Step 9. Repeating *step 5* on the reduced matrix, we get

2	6	[0]	~~0~~
[0]	11	~~0~~	18
23	[0]	2	5
4	7	8	[0]

Now, since each row and each column has one and only one assignment, an optimal solution is reached. The optimum assignment is :

$$A \to G,\ B \to E,\ C \to F \text{ and } D \to H.$$

The minimum total time for this assignment scheduled is 17 + 13 + 19 + 10 or 59 man-hours.

1102. *A company wishes to assign 3 jobs to 3 machines in such a way that each job is assigned to some machine and no machine works on more than one job. The cost of assigning job i to machine j is given by the matrix below (ijth entry) :*

$$\text{Cost matrix : } \begin{bmatrix} 8 & 7 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 7 \end{bmatrix}$$

Draw the associated network. Formulate the network LPP and find the minimum cost of making the assignment.

[GGSIP Univ. B.B.A. 2011; Madras B.Com. 200

Solution. (*a*) Network formulation of the given problem is given as under :

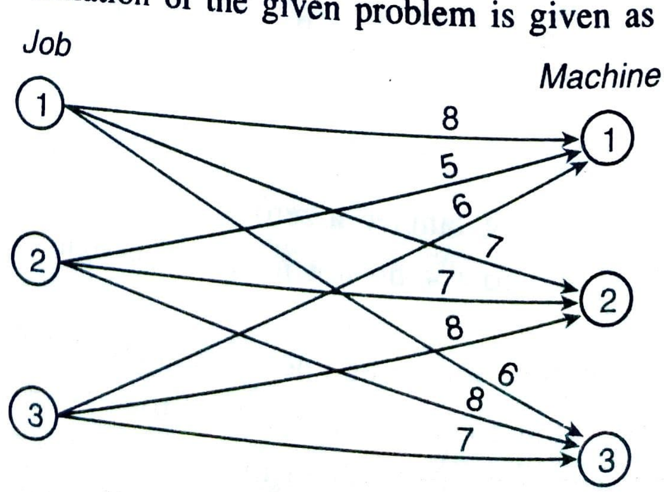

Fig. 11.1

(*b*) Linear programming formulation of the given problem is :
Minimize the total cost involved, *i.e.*,

Minimize $z = (8x_{11} + 7x_{12} + 6x_{13}) + (5x_{21} + 7x_{22} + 8x_{23}) + (6x_{31} + 8x_{32} + 7x_{33})$

subject to the constraints :

$$x_{i1} + x_{i2} + x_{i3} = 1 ; \quad i = 1, 2, 3$$
$$x_{1j} + x_{2j} + x_{3j} = 1 ; \quad j = 1, 2, 3$$
$$x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j.$$

(*c*) Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix, make assignments in rows and columns having single zeros and cross off all other zeros in rows and columns in the which the assignment has been made. See table *11.1*. Now, draw the minimum number of lines to cover all the zeros. For this, we proceed as follows:

(*i*) Mark (√) third row since it has no assignment.
(*ii*) Mark (√) first column, since third row has a zero in this column.
(*iii*) Mark (√) second row, since marked column has an assignment in the second row.
(*iv*) Since no other row or column can be marked, draw straight lines through the unmarked rows and marked column as shown in table *11.1* :

2	0	[0]	
[0]	1	3	√
~~0~~	1	1	√
√			

Table 11.1

3	[0]	~~0~~
[0]	~~0~~	2
~~0~~	~~0~~	[0]

Table 11.2

Modify the reduced cost matrix (table *11.1*) by selecting the smallest element among all the uncovered elements. Subtract this element from all the uncovered elements including itself and add it to the intersection element (1, 1) which lies at the intersection of two lines. The modified cost matrix so obtained is shown in table *11.2*.

In table *11.2*, we observe that there is no row and column which has single zero. So, we make an assignment arbitrarily at (1, 2) and cross off all zeros of first row and second column. Now, we get a single zero in the second row and therefore an assignment is made at (2, 1). Cross off all zeros in the first column. Finally, we make an assignment at (3, 3) being the single zero in the third row.

Clearly, the number of assignments in table *11.2* is equal to the order of the matrix. Hence, an optimum assignment has been attained, *viz.*,

Job *1* → Machine 2, Job 2 → Machine *1*, Job *3* → Machine *3*.

Total minimum cost will be $(7 + 5 + 7)$, *i.e.*, 19.

1103. *A pharmaceutical company is producing a single product and is selling it through five agencies located in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to despatch the product to needy cities in such a way that the travelling distance is minimised. The distance between the surplus and deficit cities (in km) is given in the following table :*

		Deficit cities				
		a	*b*	*c*	*d*	*e*
Surplus cities	*A*	85	75	65	125	75
	B	90	78	66	132	78
	C	75	66	57	114	69
	D	80	72	60	120	72
	E	76	64	56	112	68

[Delhi M.Com. 2009]

Determine the optimum assignment schedule.

Solution. Subtracting the smallest element of each row from every element of that and subtracting the smallest element of each coloumn from every element of that coloumn, we g reduced distance table :

	a	b	c	d	e
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Table 11.3

In the reduced distance table, we make assignments in rows and coloumns having single zer cross off all other zeros is those rows and columns, where assignments have been made. No the minimum number of lines to cover all the zeros. This is done in the following steps :

(*i*) Mark (√) row '*D*' since it has no assignment.

(*ii*) Mark (√) coloumn '*C*' since row '*D*' has zero in this column.

(*iii*) mark (√) row *B* since column '*C*' has an assignment in row '*B*'.

(*iv*) Since no other rows or columns can be marked, draw straight lines through the unm rows '*A*', '*C*', and '*E*', and marked coloumn '*C*' as shown in Table *11.4*..

	a	b	c	d	e	
A	2	2	~~0~~	4	[0]	
B	6	4	[0]	10	2	√
C	[0]	1	~~0~~	1	2	
D	2	4	~~0~~	4	2	√
E	2	[0]	~~0~~	~~0~~	2	
			√			

Table 11.4

	a	b	c	d	e
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

Table 11.5

Modify the reduced distance table (Table *11.4*) by subtracting the smallest element not cover lines from all the uncovered elements and add the same at the intersection elements of the lines modified distance table so obtain is shown in Table *11.5*.

Repeat the above procedure to find the new assignment in table *11.6*.

	a	b	c	d	e	
A	2	2	2	4	[0]	√
B	4	2	[0]	8	~~0~~	√
C	[0]	1	2	1	2	√
D	~~0~~	2	~~0~~	2	~~0~~	√
E	2	[0]	2	~~0~~	2	
	√		√		√	

Table 11.6

	a	b	c	d	e
A	2	1	2	3	[0]
B	4	1	[0]	7	~~0~~
C	~~0~~	[0]	2	~~0~~	2
D	[0]	1	~~0~~	1	~~0~~
E	3	~~0~~	3	[0]	3

Table 11.7

Clearly the assignment shown in table *11.6* is also not optimum, since only four assignm made. To get the next solution, we draw the minimum number of horizontal and vertical lines to all the zeros in table *11.6*. Subtracting the smallest unconvered element (*viz.*, 1) from all unc elements and adding the same to the intersection element of two lines gives us table *11.7*.

Total ... kilometres.

1104. *A department head has four tasks to be performed and three subordinates, the subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each man, so as to minimize the total man-hours?*

Task	*Men* 1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Solution. Here we have three subordinates who have to perform four tasks. So, the given problem is unbalanced and therefore we add a dummy subordinate (column) with all its entries as zero. The resulting balanced problem is :

Subordinate		1	2	3	*Dummy*
Task	*I*	9	26	15	0
	II	13	27	6	0
	III	35	20	15	0
	IV	18	30	20	0

Now, reduce the balanced time-matrix by subtracting the smallest element of each column from all the elements of that column. In the reduced matrix, make assignments in rows and columns having single zeros and cross off all other zeros of the rows and columns, where assignment have been made. We get the following assignment solution :

	1	2	3	*Dummy*
I	[0]	6	9	⊠
II	4	7	[0]	⊠
III	26	[0]	9	⊠
IV	9	10	14	[0]

Table 11.8

The optimum assignment is

I → *1*, *II* → *3* and *III* → 2 ; while task *IV* should be assigned to a dummy man, *i.e.*, it remains to be done. The minimum time is 35 hours.

PROBLEMS

1105. Four professors are each capable of teaching any one of four different courses. Class preparation time in hours for different topics varies from professor to professor and is given in the table below. Each professor is assigned only one course. Determine an assignment schedule so as to minimize the total course preparation time for all cour...