Unit - 4.

Go through:-

*. The Co-efficient Matrix A is then an ~~AxA~~ $n \times n$ Square matrix.

*. The equations will be $Ax = B$.

*. It can be classified into three methods:

1. Direct Methods.
2. Indirect Methods.
3. Iterative Methods.

Gauss Elimination Method:-

*. It is a Simple and Systematic Methods used to Solve linear Simultaneous equation.

*. It is a direct method. Written in Matrix form.

*. The Augumented Matrix is converted into equivalent upper triangular matrix using elementary formation.

*. Substituting the known Values of y and z in equation (1) .. we get corresponding x Values of x. This procedure is called back Substitution.

* Pivoting:

When any of the pivoting element is Zero (0), the order of the equations is changed, i.e., the rows are interchanged such that the pivot is non-zero. This interchange of rows in the matrix is called pivoting.

Example :-

Solve :

$$3x - y = 2$$
$$x + 3y = 4.$$

Using gaussian elimination Method.

Solution :-

Given data:-

The given equation can be rewritten as;

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The augmented matrix will be;

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 10/3 & 10/3 \end{bmatrix} R_2 \to \left(\frac{-1}{3} R_1\right) + R_2$$

$$3x - y = 2 \rightarrow (1)$$

$$\frac{10}{3} y = \frac{10}{3} \rightarrow (2).$$

By solving equation (2). we get;

$$\frac{10}{3} y = \frac{10}{3}$$

$$y = \frac{10}{3} \times \frac{3}{10}$$

$$\boxed{y = 1}$$

Substitute $y = 1$ in (1).

$$3x - y = 2$$

$$3x - 1 = 2$$

$$3x = 3$$

$$\boxed{x = 1}$$

$\therefore$ The solutions are $x = 1$; $Y = 1$.

Example:

Solve the following system of equations by Gauss elimation method.

$$x + 2y + z = 3.$$

$$2x + 3y + 3z = 10.$$

$$3x - y + 2z = 13.$$

Solution :-

The System of equations can be written in matrix from as:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 3 \end{bmatrix}$$

The augument matrix is;

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -2 & 4 \end{bmatrix} \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-3)R_1 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{bmatrix} R_3 \to -7R_2 + R_3.$$

It can be written as;

$$x + 2y + z = 3 \to (1).$$
$$-y + z = 4 \to (2)$$
$$-8z = -24. \to (3)$$

By Solving back Substitutions:

$$\cancel{-}8z = \cancel{-}24$$
$$z = 24/8$$
$$z = 3.$$

Sub $z = 3$ in (2).

$$-y + 3 = 4$$
$$-y = 4 - 3$$
$$-y = 1$$
$$\boxed{y = -1}$$

Sub $y = -1$ in and $z = 3$ in (1).

$$x + 2(-1) + 3 = 3.$$

$$x - 2 + 3 = 3$$

$$x + 1 = 3$$
$$x = 3 - 1$$
$$\boxed{x = 2.}$$

$\therefore$ The Solutions are $x = 2$; $y = -1$, $z = 3$.

Example:

Solve by Gauss - elimination method, the equations

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33.$$

Solution:

The augmented matrix will be;

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} R_2 \to \left(\frac{-1}{7}\right) R_2 \\ R_3 \to \frac{R_3}{9}. \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & -3 \end{array}\right] R_3 \to R_3 - R_2.$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] R_3 \to \frac{-R_3}{3}$$

By using Back Substitution we get:

$2x + y + 4z = 12 \to (1).$

$y + 2z = 4 \to (2)$

$\boxed{z = 1}. \to (3)$

Sub $z = 1$ in (2).

$y + 2(1) = 4.$

$y = 4 - 2$

$\boxed{y = 2}.$

Sub $y = 2$ and $z = 1$ in ①.

$2x + 2 + 4(1) = 12.$

$2x + 6 = 12$

$2x = 12 - 6$

$2x = 6$

$\boxed{x = 3}.$

$\therefore$ The Solutions are $x = 3$; $y = 2$; $z = 1$.

Iterative Methods:

Sufficient Conditions:-

Each equation of the System must Posses one large co-efficient and the large co-efficient must be attached to a different unknows in that equations. This Condition will be Satisfied if the large co-efficient are along the leading diagonal of the co-efficient Matrix.

Gauss - Seidal Method of iteration :

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

* Iteration method is a self - correcting method i.e, any error made in computation is corrected in the subsequent iterations.

Example :

$$27x + 6y - z = 85.$$

$$6x + 15y + 2z = 72.$$

$$x + y + 54z = 110.$$

Solution :-

The given equation can be rewritten as ;

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y).$$

we take intial values $y=0, z=0$, we get

x	y	z
	0	0.
3.14815	3.54074	1.91817
2.43218	3.57204	1.92585
2.42569	3.57294	1.92595
2.42549	3.57301	1.92595
2.42548	3.57801	1.92595
2.42548.	3.57301	1.92595.

Hence

$x = 2.4255$, $y = 3.5730$, $z = 1.9260$.

Example.

$$2x + y + 6z = 9$$

$$8x + 3y + 2z = 13$$

$$x + 5y + z = 7.$$

Soln:

The given equation can be written as:

$$x = \frac{1}{8}(13 - 3y - 2z)$$

$$y = \frac{1}{5}(7 - x - y)$$

$$z = \frac{1}{6}(9 - 2x - y).$$

x	y	z
	0	0
1.62500	1.07500	0.77917
1.02708	1.03875	0.98452
0.98934	1.00523	1.00268
0.99737	0.99999	1.00088
0.99978	0.99987	1.00010
1.00002	0.99998	1.00000

Hence

The Solution is : $x = 1$; $y = 1$; $z = 1$.

Interpolation:-

* The process of finding the Values of $f(x)$ Corresponding to non-tabulated values of x is called interpolation.

* If x lies between x_0 and x_n then the Process of finding y is called interpolation.

* If $x < x_0$ or $x > x_n$, then it is called extrapolation.

* The independent Variable x is called the argument.

Forward Differences:

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1 \quad \text{and so on} \ldots$$

$$\Delta y_{n-1} = y_n - y_{n-1}.$$ these are called

first order differences of Y.

Similarly:

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$ (By continuing this way)

$$\Delta^2 y_{n-2} = \Delta^2 y_2 - \Delta^2 y_1.$$

Forward differences table:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_1$			
		Δy_1		$\Delta^3 y_1$		
x_2	y_2		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_2		$\Delta^3 y_2$		$\Delta^5 y_1$.
x_3	y_3		$\Delta^2 y_3$		$\Delta^4 y_2$	
		Δy_3		$\Delta^3 y_3$		
x_4	y_4		$\Delta^2 y_4$			
		Δy_4				
x_5	y_5					

Backward Difference

* ∇ is called backward difference.

Similar to for Forward difference:-

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 .$$

Backward difference table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_1$			
		∇y_2		$\nabla^3 y_1$		
x_2	y_2		$\nabla^2 y_2$		$\nabla^4 y_1$	
		∇y_3		$\nabla^3 y_2$		$\nabla^5 y_1$
x_3	y_3		$\nabla^2 y_3$		$\nabla^4 y_2$	
		∇y_4		$\nabla^3 y_3$		
x_4	y_4		$\nabla^2 y_4$			
		∇y_5				
x_5	y_5					

Newton's Forward interpolation formula for equal intervals.

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 +$$

$$\dots + \frac{u(u-1)\dots(u-(n-1))}{n!}\Delta^n y_0 . \qquad \text{where } u = \frac{x - x_0}{h}$$

Newton's backward interpolation for equal intervals:

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots +$$

$$\frac{v(v+1)\dots(v+(n-1))}{n!}\nabla^n y_n . \quad \text{where } v = \frac{x - x_n}{h} .$$

Example: (1)

The population of a city in a census taken once in ten years is given below. Estimate the population in the year 1955.

Year :	1951	1961	1971	1981
Population in thousand's :	35	42	58	84.

Solution:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$.
1951	35			
		7		
			9	
1961	42.			1.
		16		
1971	58		10	
		26.		
1981	84			

By using Newton's forward difference formula;

$$u = \frac{x - x_0}{h} = \frac{1955 - 1951}{10} = \frac{4}{10} = 0.4.$$

$$y(1955) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 35 + (0.4)7 + \frac{(0.4)(0.4-1)}{2}(9) + \frac{(0.4)(0.4-1)(0.4-2)}{3\times 2}(1)$$

$$= 35 + (0.4)(7) + \frac{(0.4)(-0.6)}{2}(9) + \frac{(0.4)(-0.6)(-1.6)}{6}(1)$$

$$= 35 + (2.8) + (-1.08) + (0.064)$$

$$= 37.864 - 1.08.$$

$$= 36.784 \text{ (in thousands)}$$

$$y(1955). = 36784.$$

Example : (2).

From the following table of half-Yearly Premium for policies maturing at different ages, estimate the premium for policies maturing at age 46.

Age x :	45	50	55	60	65
Premium y :	114.84	96.16	83.32	74.48	68.48

Solution:-

Here $x = 46$.

By using Newton's forward difference formula;

$$u = \frac{x - x_0}{h} = \frac{46-45}{5} = \frac{1}{5} = 0.2.$$

$$y_{(46)} = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0.$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4.00		0.68.
		-8.84		-1.16	
60	74.48		2.84		
		-6.00			
65	68.48.				

$$y(46) = 114.84 + (0.2)(-18.68) + \frac{(0.2)(0.2-1)}{2}(5.84) + \frac{(0.2)(0.2-1)(0.2-2)}{6}(-1.84) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{24}(0.68)$$

$= 114.84 + (0.2)(-18.68) + (-0.4672) + (-0.8832) + (0.022848)$

$= 110.525632.$

$= 110.52.$

Example : (3)

From the given table, compute the value of $\sin 38^\circ$.

x :	0°	10°	20°	30°	40°
$\sin x$:	0	0.17365	0.34202	0.50000	0.64279

Solution :-

To find $\sin 38^\circ$, we use newton's backward difference formula.

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		0.17365			
			-0.00528		
10	0.17365				
				-0.00511	
		0.16837			
20	0.34202				0.00031
			-0.01039		
		0.15798			
30	0.50000			-0.0048	
			-0.01519		
		0.14279			
40	0.64279				

Here

$x = 38$ $\quad v = \dfrac{x - x_n}{h} = \dfrac{38 - 40}{10} = -0.2$.

$$y(38) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

Substuting the known values, we get :

$$y(38) = 0.64279 + (-0.2)(0.14279) + \frac{(-0.2)(0.8)}{2}(-0.01519) +$$

$$\frac{(-0.2)(0.8)(1.8)}{6}(-0.0048) + \frac{(0.2)(0.8)(1.8)(2.8)}{24}(0.00031)$$

$$= 0.64279 + (-0.028558) + 0.0012152 + 0.0002304 - 0.000010416$$

$$y(38) = 0.615688.$$

Example : (4)

The following data were taken from the Steam table :

Temp °C :	140	150	160	170	180.
Pressure Kgf/cm^2 :	3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature $t = 142°$ and $t = 175°$.

Solution :

t	P	ΔP	$\Delta^2 P$	$\Delta^3 P$	$\Delta^4 P$
140	3.685				
		1.169			
			0.279		
150	4.854				
		1.448		0.047	
160	6.302		0.326		0.002.
		1.774			
				0.049	
170	8.075		0.375		
		2.149			
180	10.225				

To find P when $t = 142$, By using Newton's forward interpolation formula.

$$u = \frac{t - t_0}{h} = \frac{142 - 140}{10} = 0.2.$$

$$P(142) = P_0 + u\Delta P_0 + \frac{u(u-1)}{2!}\Delta^2 P_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 P_0 + \cdots$$

$$= 3.685 + (0.2)(1.169) + \frac{(0.2)(-0.8)}{2}(0.279) + \frac{(0.2)(-0.8)(-1.8)}{6}(0.047)$$

$$+ \frac{(0.2)(-0.8)(-1.8)(-2.8)}{24}(0.002)$$

$$= 3.685 + 0.2338 + (-0.02232) + (0.048)(0.047) + (-0.0336)(0.002)$$

$$= 3.685 + 0.2338 - 0.02232 + 2.256\times10^{-3} - 6.72\times10^{-3}$$

$$= 3.921054 - 0.02904$$

$$= 3.892014.$$

To find p when $t = 175°$, we use Newton's backward interpolation formula;

$$V = \frac{t - t_n}{h} = \frac{175 - 180}{10} = \frac{-5}{10} = -0.5.$$

$$P_{(175)} = 10.225 + (0.5)(2.149) + \frac{(0.5)(-0.5+1)}{2}(0.375) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(0.049) + \frac{(0.5)(0.5+1)(-0.5+2)(-0.5+3)(0.002)}{24}$$

$$= 10.225 + (0.5)(2.149) + \frac{(-0.5)(0.5)}{2}(0.375) + \frac{(-0.5)(0.5)(1.5)}{6}(0.049) + \frac{(-0.5)(0.5)(1.5)(2.5)}{24}(0.002)$$

$$= 9.1005.$$

Example : (5)

From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks :	30-40	40-50	50-60	60-70	70-80
No. of Students:	31	42	51	35	31.

Solution:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	31				
		42			
Below 50	73		9		
		51		-25	
Below 60	124		-16		37.
		35		12	
Below 70	159		-4		
		31			
Below 80	190				

$$u = \frac{x - x_0}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5.$$

$$y(45) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2}(9) + \frac{(0.5)(-0.5)(-1.5)}{6}(-25) +$$

$$\frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(37)$$

$$= 47.8672$$

$$\simeq 48.$$

$\therefore$ No. of students who obtained marks below 45 = 48

$\therefore$ No. of students who obtained marks below 40 = 31

$\therefore$ No. of students who obtained marks between 40 and 45 = 48 - 31 = 17

Lagrange's interpolation formula:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \cdot y_0 +$$

$$\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \cdot y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_n)} \cdot y_2 + \cdots$$

$$\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\,y_n.$$

Example: (1)

Determine by Lagrange's interpolation method, the number of patients over 40 years using the following data:

Age over (x) years :	30	35	45	55.
Number (y) patients :	148	96	68	34.

Solution:-

From given data:

Here

$x_0 = 30, \quad x_1 = 35, \; x_2 = 45, \; x_3 = 55, \; y_0 = 148,$

$y_1 = 96, \; y_2 = 68, \; y_3 = 34.$

By Lagrange formula,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\,y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,y_3.$$

$$\therefore y(40) = \frac{(40-35)(40-45)(40-55)}{(30-35)(30-45)(30-55)} \times 148 +$$

$$\frac{(40-30)(40-45)(40-55)}{(35-30)(-45+35)(35-55)} \times 96 +$$

$$\frac{(40-30)(40-35)(40-55)}{(45-30)(45-35)(45-55)} \times 68 +$$

$$\frac{(40-30)(40-35)(40-55)}{(55-30)(55-35)(55-45)} \times 34$$

$$= \frac{(5)(-5)(-15)}{(-5)(-15)(-25)} \times 148 + \frac{(10)(-5)(-15)}{(5)(-10)(-20)} \times 96 +$$

$$\frac{(10)(5)(-15)}{(15)(10)(-10)} \times 68 + \frac{(10)(5)(-5)}{(25)(20)(10)} \times 34 .$$

$$\therefore y(40) = 74.71.$$

Example: (2)

Given:-

$u_0 = -4,\ u_1 = -2,\ u_4 = 220,\ u_5 = 546,\ u_6 = 1148,$

find u_2 and u_3.

Solution:-

We rewrite the given data as follows:

x :	0	1	4	5	6
$u(x)$:	-4	-2	220	546	1148.

Since the x-values are unequally spaced, we use Lagrange's interpolation for data.

$$u(x) = \frac{(x-1)(x-4)(x-5)(x-6)}{(0-1)(0-4)(0-5)(0-6)}(-4)$$

$$+ \frac{(x-0)(x-4)(x-5)(x-6)}{(1-0)(1-4)(1-5)(1-6)}(-2)$$

$$+ \frac{(x-0)(x-1)(x-5)(x-6)}{(4-0)(4-1)(4-5)(4-6)}(220)$$

$$+ \frac{(x-0)(x-1)(x-4)(x-6)}{(5-0)(5-1)(5-4)(5-6)}(546)$$

$$+ \frac{(x-0)(x-1)(x-4)(x-5)}{(6-0)(6-1)(6-4)(6-5)}(1148)$$

$$u_2 = u(2) = \frac{(1)(-2)(-3)(-4)}{(-1)(-4)(-5)(-6)}(-4) + \frac{(2)(-2)(-3)(-4)}{(1)(-3)(-4)(-5)}(-2)$$

$$+ \frac{(2)(1)(-3)(-4)}{(4)(3)(-1)(-2)}(220) + \frac{(2)(1)(-2)(-4)}{(5)(4)(1)(-1)}(546)$$

$$+ \frac{(2)(1)(-2)(-3)}{(6)(5)(2)(1)}(1148)$$

$u(2) = 12.$

$$u_3 = u(3) = \frac{(2)(-1)(-2)(-3)}{(-1)(-4)(-5)(-6)}(-4) + \frac{(3)(-1)(-2)(-3)}{(1)(-3)(-4)(-5)}(-2)$$

$$+ \frac{(3)(2)(-2)(-3)}{(4)(3)(-1)(-2)}(220) + \frac{(3)(2)(-1)(-3)}{(5)(4)(1)(-1)}(546)$$

$$+ \frac{(3)(2)(-1)(-2)}{(6)(5)(2)(1)}(1148)$$

$$u(3) = 68.$$