4

Linear Programming Problem —Simplex Method

"One step for better is good enough"

4:1. INTRODUCTION

As discussed earlier, the *Simplex Method*, also called the '*Simplex Technique*' or the '*Simplex Algorithm*' is an iterative procedure for solving a linear programming problem in a finite number of steps. The method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less (or more, as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

In the present chapter, we shall introduce and explain the computational procedure of the simplex method. The theory behind the method will be first developed and then the computational techniques explained and illustrated. The following important definitions are necessary for understanding and developing the theory that follows :

Definition **(Basic solution).** *Given a system of m simultaneous linear equations in n unknowns* $(m < n)$

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x}^T \in \mathbf{R}^n,$$

where $\mathbf{A}$ *is an* $m \times n$ *matrix of rank m. Let* $\mathbf{B}$ *be any* $m \times n$ *submatrix, formed by m linearly independent columns of* $\mathbf{A}$. *Then, a solution obtained by setting* $n - m$ *variables not associated with the columns of* $\mathbf{B}$, *equal to zero, and solving the resulting system, is called a* **basic solution** *to the given system of equations.*

The m variables, which may be all different from zero, are called **basic variables**. *The* $m \times m$ *non-singular submatrix* $\mathbf{B}$ *is called a* **basis matrix** *with the columns of* $\mathbf{B}$ *as* **basis vectors.**

Remarks. The name *basic solution*, as used above, merits a word of caution. If B is the basis sub-matrix chosen, then the basic solution to the system is

$$\mathbf{x}_B = \mathbf{B}^{-1}\,\mathbf{b}.$$

But $\mathbf{x}_B{}^T \in \mathbf{R}^m$, and as such cannot be called a solution of the given system. Truly speaking, if $\mathbf{x}_B$ is a basic solution, then a solution to the given system is $[\mathbf{x}_B{}^T, \mathbf{0}]$ where $\mathbf{x}_B{}^T \in \mathbf{R}^m$, and $\mathbf{0} \in \mathbf{R}^{n-m}$. However, we shall follow the current usage and call $\mathbf{x}_B$ a basic solution of the given system, remembering all the time that the actual solution is $[\mathbf{x}_B{}^T, \mathbf{0}]$.

Definition **(Degenerate solution).** *A basic solution to the system is called* **degenerate** *if one or more of the basic variables vanish.*

SAMPLE PROBLEMS

401. *Obtain all the basic solutions to the following system of linear equation :*

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5.$$

Solution. The given system of equations can be written in the matrix form as **Ax** = **b**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Since, rank of **A** is 2, the maximum number of linearly independent columns of **A** is 2. Thus, we can take any of the following, 2×2 sub-matrices as *basis matrix* **B** :

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

The variables not associated with the columns of **B** are x_3, x_2 and x_1 respectively, in the three different cases.

Let us first take $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_3 = 0$, and solving the system

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Thus, a basic (non-basic) solution to the given system is :

(Basic) $x_1 = 2,\ x_2 = 1$; (Non-basic) $x_3 = 0$

Similarly, the other two basic and non-basic solutions are :

(Basic) $x_1 = 5,\ x_3 = -1$; (Non-basic) $x_2 = 0$

and (Basic) $x_2 = 5/3,\ x_3 = 2/3$; (Non-basic) $x_1 = 0$

We observe that all the above three basic solutions are non-degenerate.

402. *Show that the following system of linear equations has a degenerate solution :*

$$2x_1 + x_2 - x_3 = 2$$
$$3x_1 + 2x_2 + x_3 = 3.$$

Solution. The given system of equations can be written as **Ax** = **b**, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since rank of **A** is 2, the maximum number of linearly independent columns of **A** is 2. Thus, we can take any of the following 2×2 sub-matrices of **A**, as basis matrix **B** :

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}.$$

The variables not associated with the columns of these submatrices are, respectively, x_3, x_1 and x_2.

Considering $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, a basic solution to the given system is obtained by setting $x_3 = 0$ and solving the system

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus, basic solution to the given problem is

(Basic) $x_1 = 1,\ x_2 = 0$; (Non-basic) $x_3 = 0.$

Similarly, the other two solutions are :

(Basic) $x_2 = 5/3,\ x_3 = -1/3$; (Non-basic) $x_1 = 0$,

and (Basic) $x_1 = 1,\ x_3 = 0$; (Non-basic) $x_2 = 0$.

In each of the two basic solutions, at least one of the basic variables is zero. Hence, two of the basic solutions are degenerate solutions.

Definition* (Basic feasible solution).** *A feasible solution to an L.P.P., which is also a basic solution to the problem is called a* ***basic feasible solution *to the L.P.P.*

Illustrations

1. In sample problem *401* observe that [5, 0, −1] is not a feasible solution. Only basic feasible solutions are :

(*i*) $x_2 = 5/3$ and $x_3 = 2/3$; (*ii*) $x_1 = 2$ and $x_2 = 1$.

2. In sample problem *402*, the non-degenerate solution $[0,\ 5/3,\ -1/3]$ is not feasible. Only basic feasible degenerate solutions are :

(*i*) $x_1 = 1$ and $x_2 = 0$; (*ii*) $x_1 = 1$ and $x_3 = 0$.

***Definition* (Associated cost vector).** *Let* $\mathbf{x}_B$ *be a basic feasible solution to the L.P.P. :*

$$\text{Maximize } z = \mathbf{cx} \text{ subject to : } \mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

Then, the vector

$$\mathbf{c}_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}),$$

where c_{Bi} *are components of* **c** *associated with the basic variables, is called the* ***cost vector associated*** *with the basic feasible solution* $\mathbf{x}_B$.

It is obvious that the value of the objective function for the basic feasible solution $\mathbf{x}_B$, is given by

$$z_0 = \mathbf{c}_B \mathbf{x}_B.$$

Definition* (Improved basic feasible solution).** *Let* $\mathbf{x}_B$ *and* $\hat{\mathbf{x}}_B$ *be two feasible solutions to the standard L.P.P. Then* $\hat{\mathbf{x}}_B$ *is said to be an* ***improved basic feasible solution*, as compared to* $\mathbf{x}_B$, *if*

$$\hat{\mathbf{c}}_B \hat{\mathbf{x}}_B \geq \mathbf{c}_B \mathbf{x}_B$$

where $\hat{\mathbf{c}}_B$ *is constituted of cost components corresponding to* $\hat{\mathbf{x}}_B$.

***Definition* (Optimum basic feasible solution).** *A basic feasible solution* $\mathbf{x}_B$ *to the L.P.P. :*

$$\text{Maximize } z = \mathbf{cx} \text{ subject to : } \mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

is called an ***optimum basic feasible solution*** *if* $z_0 = \mathbf{c}_B \mathbf{x}_B \geq z^*$ *where* z^* *is the value of objective function for any feasible solution.*

PROBLEMS

403. Find all the basic feasible solutions of the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

404. Find all the basic feasible solutions of the following L.P.P. without using the simplex alogorithm :

$$\text{Maximize } z = 2x_1 + 3x_2 + 4x_3 + 7x_4 \text{ subject to :}$$
$$2x_1 + 3x_2 + x_3 + 4x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$
$$x_i \geq 0,\ i = 1, 2, 3, 4$$

and choose that one which maximises z.

405. If $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 7, x_5 = 12, x_6 = 10$ is an extreme point solution of the set of equations

$$3x_1 - x_2 + 2x_3 + x_4 = 7$$
$$-2x_1 + 4x_2 + x_5 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + x_6 = 10;$$

find two more extreme point solutions of the above problem.

4:2. FUNDAMENTAL PROPERTIES OF SOLUTIONS

We are now well set to discuss the theory behind the computational procedure for obtaining an optimum basic feasible solution to a given L.P.P. The fundamental properties of the solutions to the L.P.P., that now follow, will allow us to arrive at the much awaited **simplex algorithm**.

Theorem 4-1 **(Reduction of a Feasible Solution to a Basic Feasible Solution)**. *If an L.P.P. has a feasible solution, then it also has a basic feasible solution.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to maximize

$$z = \mathbf{cx} \qquad \mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$$

subject to the constraints : $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$; where $\mathbf{A}$ is an $m \times n$ real matrix and $\mathbf{b}$, $\mathbf{c}$ are $m \times 1$ and $1 \times n$ real matrices respectively. Let $\rho(\mathbf{A}) = m$.

Since there does exist a feasible solution, we must have

$$\rho(\mathbf{A}, \mathbf{b}) = \rho(\mathbf{A}) \quad \text{and} \quad m < n.$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a feasible solution so that $x_j \geq 0$ for all j.

To be precise, let us suppose that $\mathbf{x}$ has p positive components and let the remaining $n - p$ components be zero.

Let us so relabel components that the positive components are the first p components and assume that the columns of $\mathbf{A}$ have been relabelled accordingly. Then

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_p x_p = \mathbf{b}.$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are the first p columns of $\mathbf{A}$.

Two cases now do arise :

(*i*) *The vectors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ *form a linearly independent set*. Then $p \leq m$.

If $p = m$, the given solution is a non-degenerate basic feasible solution, with $x_1, x_2, \ldots, x_p$ as the basic variables.

If $p < m$, then the set $\{ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_p \}$ can be extended to $\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_m \}$ to form a basis for the columns of $\mathbf{A}$.

Then, we have

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_m x_m = \mathbf{b},$$

where $x_j = 0$ for $j = p+1, p+2, \ldots, m$.

Thus we have, in this case, a degenerate basic feasible solution with $m - p$ of the basic variables zero.

(*ii*) *The set* $\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p \}$ *is linearly dependent*. Obviously $p > m$.

Let $\{ \alpha_1, \alpha_2, \ldots, \alpha_p \}$ be a set of constants (not all zero) such that

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_p \mathbf{a}_p = \mathbf{0}.$$

Suppose that for any index r, $\alpha_r \neq 0$. Then $\mathbf{a}_r = \sum_{j \neq r}^{p} \dfrac{\alpha_j}{\alpha_r} \mathbf{a}_j$.

$$\therefore \qquad \sum_{j \neq r}^{p} \mathbf{a}_j x_j + \left(-\sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} \mathbf{a}_j \right) x_r = \mathbf{b}$$

or

$$\sum_{j \neq r}^{p} \left(x_j - x_r \frac{\alpha_j}{\alpha_r} \right) \mathbf{a}_j = \mathbf{b}.$$

Thus, we have a solution with not more than $p - 1$ non-zero components. To ensure that these are positive, we shall choose $\mathbf{a}_r$ in such a way that

$$x_j - x_r \frac{\alpha_j}{\alpha_r} \geq 0 \quad \text{for all } j \neq r.$$

This requires that either $\alpha_j = 0$ or

$$\frac{x_j}{\alpha_j} \geq \frac{x_r}{\alpha_r}, \text{ if } \alpha_j > 0 \text{ and } \frac{x_j}{\alpha_j} \leq \frac{x_r}{\alpha_r}, \text{ if } \alpha_j < 0.$$

Thus, if we select $\mathbf{a}_r$ such that

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j}, \alpha_j > 0 \right\},$$

then for each of the $p-1$ variables $x_j - x_r \dfrac{\alpha_j}{\alpha_r}$ is non-negative, and so we have a feasible solution with not more than $p-1$ non-zero components.

Consider now this new feasible solution with not more than $p-1$ non-zero components. If the corresponding set of $p-1$ columns of $\mathbf{A}$ is linearly independent, case (*i*) applies and we have arrived at a basic feasible solution. If this set is again linearly dependent, we may repeat the process to arrive at a feasible solution with not more than $p-2$ non-zero components. The argument can be repeated. Ultimately, we get a feasible solution with associated set of column vectors of $\mathbf{A}$ linearly independent. The discussion of case (*i*) then applies and we do get a basic feasible solution.

This completes the proof.

Corollary. *There exists only finite number of basic feasible solutions to an L.P.P.*

Proof. Exercise for the reader.

Theorem 4-2 **(Extreme Point Correspondence).** *A basic feasible solution to an L.P.P. must correspond to an extreme point of the set of all feasible solutions and conversely.*

Proof. Let the L.P.P. be :

$$\text{Maximize } z = \mathbf{c}^T\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n \text{ subject to the constraints:}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ are real $m \times n$, $m \times 1$ and $n \times 1$ matrices respectively. Let $\rho(\mathbf{A}) = m$.

Let S be the set of all feasible solutions to the L.P.P. Also suppose that $\mathbf{x}$ is a basic feasible solution

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{x}_B$ is an $m \times 1$ vector, such that for a non-singular sub-matrix $\mathbf{B}$ of $\mathbf{A}$, $\mathbf{B}\mathbf{x}_B = \mathbf{b}$.

If possible, let $\mathbf{x}$ be a point of S, such that there exist $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that $\mathbf{x}_1 \neq \mathbf{x}_2$ and

$$\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \quad 0 < \lambda < 1.$$

Let
$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{v}_2 \end{bmatrix}$$

where $\mathbf{u}_1, \mathbf{u}_2$ are $m \times 1$ vectors and $\mathbf{v}_1, \mathbf{v}_2$ are $(n-m) \times 1$ vectors. Then

$$\begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{v}_2 \end{bmatrix}$$

$\therefore$
$$\mathbf{x}_B = \lambda \mathbf{u}_1 + (1 - \lambda)\mathbf{u}_2$$

and
$$\mathbf{0} = \lambda \mathbf{v}_1 + (1 - \lambda)\mathbf{v}_2 \qquad 0 < \lambda < 1.$$

Since $\mathbf{x}_1, \mathbf{x}_2$ are feasible solutions, therefore $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \geq \mathbf{0}$.

Now
$$0 < \lambda < 1 \text{ and } \mathbf{0} = \lambda \mathbf{v}_1 + (1 - \lambda)\mathbf{v}_2$$

$\therefore$ We must have
$$\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}.$$

Thus
$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{0} \end{bmatrix}$$

Suppose that there does exist a basic feasible solution $\mathbf{x}_B$, so that

(1) $$\mathbf{B}\mathbf{x}_B = \mathbf{b},\ \mathbf{x}_B \geq \mathbf{0}$$

where $\mathbf{B}$ forms a basis set for the column vectors of $\mathbf{A}$. Clearly, for any column vector $\mathbf{a}_j \in \mathbf{A}$, we have

$$\mathbf{a}_j = y_{1j}\mathbf{b}_1 + y_{2j}\mathbf{b}_2 + \ldots + y_{mj}\mathbf{b}_m = \mathbf{B}\mathbf{y}_j$$

where $\mathbf{b}_i \in \mathbf{B}$ and y_{ij} are suitable scalars.

Now, we know that if the basis vector $\mathbf{b}_r$, for which the coefficient y_{rj} is non-zero, is replaced by $\mathbf{a}_j \in \mathbf{A}$, then the new set of vectors also forms a basis.

Now, for $y_{rj} \neq 0$, we can write

$$\mathbf{b}_r = \frac{\mathbf{a}_j}{y_{rj}} - \sum_{i=1}^{m} \frac{y_{ij}}{y_{rj}}\mathbf{b}_i \qquad i \neq r$$

and therefore (1) gives

$$\mathbf{b} = \sum_{i=1}^{m} x_{Bi}\mathbf{b}_i + x_{Br}\left[\frac{\mathbf{a}_j}{y_{rj}} - \sum_{i=1}^{m}\frac{y_{ij}}{y_{rj}}\mathbf{b}_i\right] \qquad i \neq r$$

$$= \sum_{i=1}^{m}\left[x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right]\mathbf{b}_i + \frac{x_{Br}}{y_{rj}}\mathbf{a}_j, \qquad i \neq r$$

Thus, the new basic solution is $\hat{\mathbf{x}}_B$ having as its components, the variables

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\ (i = 1, 2, \ldots, m;\ i \neq r) \text{ and } \hat{x}_{Br} = \frac{x_{Br}}{y_{rj}}.$$

We shall now show that $\hat{\mathbf{x}}_B$ is feasible also, that is, the new basic variables $\hat{x}_{Bi}$ are also non-negative. Two cases do arise :

Case (I). $x_{Br} = 0$. In this case the new set of basic variables is obviously non-negative, since we have assumed the existence of a basic feasible solution $\mathbf{x}_B$.

Case (II). $x_{Br} \neq 0$. In this case we must have $y_{rj} > 0$. This requires that for the remaining $y_{ij}\ (i \neq r)$ either $y_{rj} = 0$ for $i \neq r$, or

$$\frac{x_{Bi}}{y_{ij}} \geq \frac{x_{Br}}{y_{rj}},\ \text{for } y_{ij} > 0 \text{ and } i \neq r \quad \text{and} \quad \frac{x_{Bi}}{y_{ij}} \leq \frac{x_{Br}}{y_{rj}},\ \text{for } y_{ij} < 0 \text{ and } i \neq r.$$

So, if we select the index $r\ (y_{rj} \neq 0)$ in such a way that

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{\text{Min}}\left\{\frac{x_{Bi}}{y_{ij}};\ y_{ij} > 0,\ i \neq r\right\}$$

then, the new set of basic variables are non-negative; and hence the basic solution $\hat{\mathbf{x}}_B$ is feasible.

Remark. If after replacement of basis vector, the new basis matrix is

$$\hat{\mathbf{B}} = (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \ldots, \hat{\mathbf{b}}_m)$$

where $\hat{\mathbf{b}}_i = \mathbf{b}_i$ for $i \neq r$ and $\hat{\mathbf{b}}_r = \mathbf{a}_j$; then, the new basic feasible solution is

$$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}\mathbf{b}$$

where $\hat{x}_{Bi} = x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}},\ i \neq r$ and $\hat{x}_{Br} = \dfrac{x_{Br}}{y_{rj}}$ are the basic variables.

Definition **(Net evaluation).** *Let $\mathbf{x}_B$ be a basic feasible solution to the L.P.P. :*

Maximize $z = \mathbf{cx}$, *where* $\mathbf{Ax} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

Let $\mathbf{c}_B$ be the cost vector corresponding to $\mathbf{x}_B$. For each column vector $\mathbf{a}_j$ in $\mathbf{A}$, which is not a column vector of $\mathbf{B}$, let

$$\mathbf{a}_j = \sum_{i=1}^{m} y_{ij}\mathbf{b}_i.$$

Then the number

$$z_j = \sum_{i=1}^{m} c_{Bi}\, y_{ij}$$

is called the **evaluation** *corresponding to* $\mathbf{a}_j$ *and the number* $z_j - c_j$ *is called the* **net evaluation** *corresponding to* $\mathbf{a}_j$.

Theorem 4-5 **(Improved Basic Feasible Solution).** *Let* $\mathbf{x}_B$ *be a basic feasible solution to the L.P.P. : Maximize* $z = \mathbf{cx}$ *subject to* $\mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$.

Let $\hat{\mathbf{x}}_B$ *be another basic feasible solution obtained by admitting a non-basis column vector* $\mathbf{a}_j$ *in the basis, for which the net evaluation* $z_j - c_j$ *is negative. Then* $\hat{\mathbf{x}}_B$ *is an improved basic feasible solution to the problem, that is*

$$\hat{\mathbf{c}}_B\, \hat{\mathbf{x}}_B > \mathbf{c}_B\, \mathbf{x}_B.$$

Proof. The L.P.P. is to determine $\mathbf{x}$, so as to Maximize $z = \mathbf{cx}$; $\mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, $\mathbf{b}^T \in \mathbf{R}^m$, where $\mathbf{A}$ is an $m \times n$ real matrix.

We are given that $\mathbf{x}_B$ is a basic feasible solution. Let $z_0 = \mathbf{c}_B\, \mathbf{x}_B$.

Let $\hat{\mathbf{a}}_j$ be the column vector introduced in the basis, such that $z_j - c_j < 0$.

Let $\mathbf{b}_r$ be vector removed from the basis and let $\hat{\mathbf{x}}_B$ be the new basic feasible solution, then

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \quad \text{and} \quad \hat{x}_{Br} = \frac{x_{Br}}{y_{rj}}$$

Thus, the new value of the objective function is

$$\hat{z} = \sum_{i=1}^{m} \hat{c}_{Bi}\, \hat{x}_{Bi}$$

$$= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right) + \hat{c}_{Br}\frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right) + c_j\frac{x_{Br}}{y_{rj}} \qquad (\because \hat{c}_{Br} = c_j)$$

$$= z_0 - (z_j - c_j)\frac{x_{Br}}{y_{rj}}$$

$$> z_0 \qquad [\because x_{Br}/y_{rj} > 0].$$

Hence, the new basic feasible solution $\hat{\mathbf{x}}_B$ gives an improved value of the objective function.

Remark. If the given basic solution happens to be a non-degenerate one, then $x_{Br}/y_{rj} > 0$ and a definite increase in the value of z_0 occurs. If the given feasible solution happens to be a degenerate one, then an increase in the value of z_0 depends on x_{Br}/y_{rj}. In any case, $z_j - c_j < 0$ implies that $\hat{z}$ can never be less than z_0.

Corollary. If $z_j - c_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, \ldots, m$; then another basic feasible solution is obtained which gives an unchanged value of the objective function.

Proof. It follows from the above theorem that

$$\hat{z} = z_0 - (z_j - c_j)\frac{x_{Br}}{y_{rj}}, \quad y_{rj} > 0$$

$$= z_0, \quad \text{since } z_j - c_j = 0.$$

Theorem 4-6 **(Unbounded Solution).** *Let there exist a basic feasible solution to a given L.P.P. If for at least one j, for which* $y_{ij} \leq 0$ $(i = 1, 2, \ldots, m)$, *and* $z_j - c_j$ *is negative, then there does not exist any optimum solution to this L.P.P.*

Proof. Let the given L.P.P. be to determine $\mathbf{x}$ so as to

Maximize $z = \mathbf{cx}$: $\mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$ subject to the constraints :

$\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, $\mathbf{b}^T \in \mathbf{R}^m$

where $\mathbf{A}$ is an $m \times n$ real matrix.

Let a basic feasible solution to this problem be $\mathbf{x}_B$, so that

$$\mathbf{Bx}_B = \mathbf{b} \text{ and } \mathbf{x}_B \geq \mathbf{0}$$

with the value of the objective function

$$z_0 = \mathbf{c}_B \mathbf{x}_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}.$$

Now, we can write

$$\mathbf{b} = \mathbf{Bx}_B + \xi \mathbf{a}_j - \xi \mathbf{a}_j \qquad \mathbf{a}_j \in \mathbf{A}, \ \xi \text{ is a scalar}$$

$$= \sum_{i=1}^{m} x_{Bi} \mathbf{b}_i + \xi \mathbf{a}_j - \xi \sum_{i=1}^{m} y_{ij} \mathbf{b}_i$$

$$= \sum_{i=1}^{m} (x_{Bi} - \xi y_{ij}) \mathbf{b}_i + \xi \mathbf{a}_j.$$

If $\xi > 0$, then $(x_{Bi} - \xi y_{ij}) \geq 0$ since $y_{ij} \leq 0$. This shows that there exists a feasible solution whose $(m + 1)$ components may be strictly positive. But, in general, it may not be a basic solution.

The value of the objective function for these $(m + 1)$ variables is given by

$$\hat{z} = \sum_{i=1}^{m} c_{Bi} (x_{Bi} - \xi y_{ij}) + \xi c_j = \sum_{i=1}^{m} c_{Bi} x_{Bi} - \xi \left(\sum_{i=1}^{m} c_{Bi} y_{ij} - c_j \right)$$

$$= z_0 - \xi (z_j - c_j).$$

But $\quad z_j - c_j < 0$ and $\xi > 0$

$\therefore \quad \hat{z} \to +\infty$ as $\xi \to +\infty$.

Hence, there is no limit to the optimum value of z and, hence there exists an unbounded solution to the given LPP.

Remarks 1. If for some $\mathbf{a}_j$ for which $z_j - c_j > 0$ and $y_{ij} \leq 0$, $i = 1, 2, \ldots, m$, then there exists a feasible solution to the LPP, such that the $(m + 1)$ components of the solution may be strictly positive and

$$\hat{z} \to -\infty \text{ as } \xi \to +\infty.$$

2. If $y_{ij} \leq 0$ for at least one j for which $z_j - c_j = 0$, then $\hat{z} = z_0$.

***Theorem 4-7* (Conditions of Optimality).** *A sufficient condition for a basic feasible solution to an LPP to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which the column vector $\mathbf{a}_j \in \mathbf{A}$ is not in the basis $\mathbf{B}$.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to

Maximize $z = \mathbf{cx}$; $\mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$ subject to the constraints :

$\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$

where $\mathbf{A}$ and $\mathbf{b}$ are $m \times n$ and $m \times 1$ real matrices respectively. Let $\rho(\mathbf{A}) = m$ so that we can choose an $m \times m$ submatrix $\mathbf{B}$ of $\mathbf{A}$ as a basis matrix.

Let us assume that there exists a basic feasible solution $\mathbf{x}_B$ to this L.P.P. Let $\mathbf{c}_B$ be the cost vector corresponding to the basic variables.

Then $\quad \mathbf{Bx}_B = \mathbf{b}$, $\mathbf{x}_B \geq \mathbf{0}$ and $z_0 = \mathbf{c}_B \mathbf{x}_B$.

Now, for all those j for which $\mathbf{a}_j \notin \mathbf{B}$, we are given that $z_j - c_j \geq 0$.

Let $\mathbf{a}_j = \mathbf{b}_j$ for all such j for which $\mathbf{a}_j \in \mathbf{B}$. Then

$$\mathbf{y}_j = \mathbf{B}^{-1} \mathbf{b}_j = \mathbf{e}_j, \text{ the unit vector} \qquad (\text{since } \mathbf{y}_j = \mathbf{B}^{-1} \mathbf{a}_j)$$

Again,
$$\mathbf{cx} = \mathbf{c}\left(\sum_{i=1}^{k} \alpha_i \mathbf{x}_i\right) = \sum_{i=1}^{k} \alpha_i \mathbf{cx}_i = \sum_{i=1}^{k} \alpha_i z_0 = z_0$$

Hence, a convex combination of k different optimum solutions to an L.P.P. is also an optimum solution to the L.P.P.

4:3. THE COMPUTATIONAL PROCEDURE

The two fundamental conditions on which the simplex method is based are :

(*i*) *Condition of feasibility*. It assumes that if the initial (starting) solution is basic feasible, then during computation only basic feasible solutions will be obtained.

(*ii*) *Condition of optimality*. It guarantees that only better solutions will be encountered.

The computational procedure of simplex method requires the construction of the simplex table. The initial simplex table is constructed by writing out the coefficients of the variables in the objective function and constraints of LPP in a systematic tabular form.

Following is a specimen of the *simplex table* which conveniently displays the basic feasible solution $\mathbf{x}_B$, the associated cost vector $\mathbf{c}_B$, the basic variables, column vectors $\mathbf{y}_j$'s corresponding to basic as well as non-basic variables, the corresponding cost coefficients and the net evaluations in a tabular form :

			c_1	c_2	...	c_n
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	...	$\mathbf{y}_n$
c_{B1}	y_{B1}	x_{B1}	y_{11}	y_{12}	...	y_{1n}
c_{B2}	y_{B2}	x_{B2}	y_{21}	y_{22}	...	y_{2n}
⋮	⋮	⋮	⋮	⋮	...	⋮
c_{Bm}	y_{Bm}	x_{Bm}	y_{m1}	y_{m2}	...	y_{mn}
		z_0	$z_1 - c_1$	$z_2 - c_2$	...	$z_n - c_n$

The optimal solution to a General L.P.P. (when it exists) is obtained in the following major steps :

Step 1. Select an initial (starting) basic feasible solution to initiate the algorithm. (Use theorem *4-1* for the existence of a basic solution.)

Step 2. Check the objective function to see whether there is some non-basic variable that would improve the objective function if brought in the basis. If such a variable exists, go to *step 3*, otherwise stop. (Use Theorems *4-5* and *4-6* for the check.)

Step 3. Determine how large the variable found in *step 2* can be made until one of the basic variables in the current solution becomes zero. Eliminate the later variable and let the next *trial* solution contain the newly found variable instead. (Use Theorem *4-4*.)

Step 4. Check for optimality the current solution. (Use Theorem *4-7*.)

Step 5. Continue the iterations until either an optimum solution is attained or there is an indication that an unbounded solution exists.

Note. To simplify the computations, we usually consider the starting basis matrix $\mathbf{B}$ as an identity matrix $\mathbf{I}$, *i.e.*, $\mathbf{B} = \mathbf{I}$. Therefore $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j = \mathbf{I}^{-1}\mathbf{a}_j = \mathbf{I}\mathbf{a}_j = \mathbf{a}_j$,

$$\mathbf{I}_2\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = \mathbf{y}_1,$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{Ib} = \mathbf{b}, \text{ etc.}$$

The Simplex Algorithm

For the solution of any L.P.P. by **simplex algorithm**, the existence of an *initial* basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows :

Step 1. Check whether the objective function of the given L.P.P. is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximizing it by using the result

$$\text{Minimum } z = -\text{ Maximum } (-z).$$

Step 2. Check whether all b_i $(i = 1, 2, \ldots, m)$ are non-negative. If any one of b_i is negative then multiply the corresponding inequation of the constraints by -1, so as to get all b_i $(i = 1, 2, \ldots, m)$ non-negative.

Step 3. Convert all the inequations of the constraints into equations by introducing slack and/or surplus variables in the constraints. Put the costs of these variables equal to zero.

Step 4. Obtain an initial basic feasible solution to the problem in the form $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and put it in the first column of the simple table.

Step 5. Compute the net evaluations $z_j - c_j$ $(j = 1, 2, \ldots, n)$ by using the relation $z_j - c_j = \mathbf{c}_B\,\mathbf{y}_j - c_j$, where $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$.

Examine the sign $z_j - c_j$.

(*i*) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution $\mathbf{x}_B$ is an optimum basic feasible solution.

(*ii*) If at least one $(z_j - c_j) < 0$, proceed on to the next step.

Step 6. If there are more than one negative $z_j - c_j$, then choose the most negative of them. Let it be $z_r - c_r$ for some $j = r$.

(*i*) If all $(z_j - c_j) \geq 0$, then the initial basic feasible solution $\mathbf{x}_B$ is an optimum basic feasible solution.

(*ii*) If at least one $(z_j - c_j) < 0$, proceed on to next step.

Step 7. Compute the ratios $\left\{ \dfrac{x_{Bi}}{y_{ir}},\ y_{ir} > 0,\ i = 1, 2, \ldots, m \right\}$ and choose the minimum of them. Let the minimum of these ratios be x_{Bk}/y_{kr}. Then the vector $\mathbf{y}_k$ will level the basis $\mathbf{y}_B$. The common element y_{kr}, which is in the kth row and the rth column is known as the **leading element (or pivotal element)** of the table.

Step 8. Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeroes by making use of the relations :

$$\hat{y}_{ij} = y_{ij} - \frac{y_{kj}}{y_{kr}}\, y_{ir} \qquad i = 1, 2, \ldots, m+1;\ i \neq k$$

and

$$\hat{y}_{kj} = \frac{y_{kj}}{y_{kr}} \qquad j = 0, 1, 2, \ldots, n$$

Step 9. Go to *Step 5* and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

SAMPLE PROBLEMS

411. *Use simplex method to solve the following L.P.P. :*

Maximize $z = 4x_1 + 10x_2$ subject to the constraints :

$$2x_1 + x_2 \leq 50,\ 2x_1 + 5x_2 \leq 100,\ 2x_1 + 3x_2 \leq 90;\ x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Solution.

Step 1. By introducing slack variables $s_1 \geq 0$; $s_2 \geq 0$ and $s_3 \geq 0$ respectively, the set of constraints of the given L.P.P. are converted into the system of equations :

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$$

The modified objective function is to maximize

$$z = 4x_1 + 10x_2 + 0.s_1 + 0.s_2 + 0.s_3.$$

Step 2. An obvious initial basic feasible solution is given by $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

i.e., $$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$$

since $$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{B}^{-1}, \text{ and } \mathbf{b} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}.$$

Step 3. Compute $\mathbf{y}_j$ and $(z_j - c_j)$ as follows :

$$\mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

$$\mathbf{y}_3 = \mathbf{B}^{-1}\mathbf{e}_1 = \mathbf{e}_1, \ \mathbf{y}_4 = \mathbf{B}^{-1}\mathbf{e}_2 = \mathbf{e}_2 \text{ and } \mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{e}_3 = \mathbf{e}_3$$

$$z_1 - c_1 = \mathbf{c}_B\mathbf{y}_1 - c_1 = (0, 0, 0)\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - 4 = -4$$

$$z_2 - c_2 = \mathbf{c}_B\mathbf{y}_2 - c_2 = (0, 0, 0)\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} - 10 = -10$$

$$z_3 - c_3 = \mathbf{c}_B\mathbf{y}_3 - c_3 = (0, 0, 0)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0 = 0$$

$$z_4 - c_4 = \mathbf{c}_B\mathbf{y}_4 - c_4 = (0, 0, 0)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = 0$$

$$z_5 - c_5 = \mathbf{c}_B\mathbf{y}_5 - c_5 = (0, 0, 0)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 0$$

Step 4. The initial basic feasible solution is now represented in the simplex table below :

		c_j	4	10	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	50	2	1	1	0	0
0	$\mathbf{y}_4$	100	2	(5)	0	1	0
0	$\mathbf{y}_5$	90	2	3	0	0	1
	z_j		0	0	0	0	0
	$z_j - c_j$	$z\,(= 0)$	–4	–10	0	0	0

From the tableau, it is apparent that there are two $z_j - c_j$ which are negative. We choose the most negative of these, *viz.*, – 10. The corresponding column vector $\mathbf{y}_2$ enters the basis.

Step 5. Since all the entries of $\mathbf{y}_2$ are positive, we compute $\min.\left\{\frac{x_{Bi}}{y_{ir}}, y_{ir} > 0\right\}$; *i.e.*,

$\min.\left\{\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right\} = \frac{100}{5}$. This occurs for the element $y_{22} = (= 5)$. Thus the vector $\mathbf{y}_4$ will leave the basis $\mathbf{y}_B$ and the common element y_{22} becomes the leading element for the first iteration.

Step 6. Convert the leading element y_{22} to unity and all other elements of $\mathbf{y}_2$ to zeroes by making use of the following transformation :

$$\hat{y}_{ij} = y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2} ; \qquad i = 1, 2, 3, 4, \; i \neq 2$$

and

$$\hat{y}_{2j} = \frac{y_{2j}}{y_{22}} ; \qquad j = 0, 1, 2, 3, 4, 5.$$

$$\therefore \quad \hat{y}_{21} = \frac{y_{21}}{y_{22}} = \frac{2}{5} ; \; \hat{y}_{20} = \frac{y_{20}}{y_{22}} = \frac{100}{5} \text{ or } 20, \text{etc.}$$

$$\hat{y}_{10} = y_{10} - \frac{y_{20}}{y_{22}} y_{12} = 50 - \frac{100}{5} \times 1 = 30$$

$$\hat{y}_{30} = y_{30} - \frac{y_{20}}{y_{22}} y_{32} = 90 - \frac{100}{5} \times 3 = 30$$

$$\hat{y}_{31} = y_{31} - \frac{y_{21}}{y_{22}} y_{32} = 2 - \frac{2}{5} \times 3 = \frac{4}{5}$$

$$\hat{y}_{11} = y_{11} - \frac{y_{21}}{y_{22}} y_{12} = 2 - \frac{2}{5} \times 1 = \frac{8}{5}$$

$$\hat{y}_{14} = y_{14} - \frac{y_{24}}{y_{22}} y_{12} = 0 - \frac{1}{5} \times 1 = -\frac{1}{5}, \text{ and so on.}$$

Step 7. Using the above computations, the iterated simplex table is :

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	30	8/5	0	1	–1/5	0
10	$\mathbf{y}_2$	20	2/5	1	0	1/5	0
0	$\mathbf{y}_5$	30	4/5	0	0	–3/5	1
	z_j		4	10	0	2	0
	$z_j - c_j$	z (= 200)	0	0	0	2	0

The above simplex table yields a new basic feasible solution with increased value of z. Moreover, no further improvement in the value of z is possible, since all $z_j - c_j \geq 0$. Hence, we have attained our desired optimum solution with basic variables x_2, s_1 and s_3. The maximal basic feasible solution to the given *L.P.P.*, therefore, is

$$x_1 = 0, \; x_2 = 20 \text{ with maximum } z = 200.$$

Note. It is evident from the net evaluations of the optimum table that the net evaluation corresponding to non-basic variable x_1 is zero.

This is an indication that an alternative basic solution exists. Thus, we can bring $\mathbf{y}_1$ into the basis in place of $\mathbf{y}_3$ or $\mathbf{y}_5$. the resulting new basic feasible solution will also be an optimum one.

Now, by introducing $\mathbf{y}_1$ into the basis in place of $\mathbf{y}_3$, the new optimum simplex table is as follows :

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
4	$\mathbf{y}_1$	150/8	1	0	5/8	–1/8	0
10	$\mathbf{y}_2$	25/2	0	1	–1/4	1/4	0
0	$\mathbf{y}_5$	15	0	0	–1/2	–1/2	1
	z_j		4	10	0	2	0
	$z_j - c_j$	z (= 200)	0	0	0	2	0

We observe from this table that the basic feasible solution has been changed but the optimum value of z remains the same.

If two basic (optimum) feasible solutions are known, an infinite number of basic (optimum) feasible solutions can be obtained by taking any weighted average of the two solutions. Thus, if

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 20 \\ 30 \\ 0 \\ 30 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 150/8 \\ 25/2 \\ 0 \\ 0 \\ 15 \end{bmatrix} \text{ are two BFSs, then a new BFS is :}$$

$$\mathbf{x}^* = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 = \begin{bmatrix} (1-\lambda)\,150/8 \\ 20\lambda + (1-\lambda)\,25/2 \\ 30\lambda \\ 0 \\ 30\lambda + (1-\lambda)\,15 \end{bmatrix} = \begin{bmatrix} (1-\lambda)\,150/8 \\ (25-15\lambda)/2 \\ 30\lambda \\ 0 \\ 15+15\lambda \end{bmatrix}, \quad 0 \le \lambda \le 1.$$

It can be verified that the solution $\mathbf{x}^*$ will always give the same value 200 of z, for all $0 \le \lambda \le 1$.

Remark. Interpretation of the final simplex table. The simplex table helps to predict the effect of changes in the resources and the profit margin :

(*a*) *Opportunity cost.* The terms in z_j-row indicate the opportunity cost. If we do not utilize one unit of x_1, the loss of profit is 4 units and non-utilization of one unit of x_2 costs 10 units.

(*b*) *Shadow cost.* The coefficient of non-basic variables, *i.e.*, of s_1 and s_2 are called 'shadow costs'. They represent the decrease in the optimum value of the objective function resulting from a unit increase in a non-basic variable.

412. *Use simplex method to*

$$\text{Minimize } z = x_2 - 3x_3 + 2x_5 \text{ subject to the constraints :}$$

$$3x_2 - x_3 + 2x_5 \le 7, \; -2x_2 + 4x_3 \le 12,$$

$$-4x_2 + 3x_3 + 8x_5 \le 10; \; x_2 \ge 0, \; x_3 \ge 0 \text{ and } x_5 \ge 0.$$

[Garhwal M.Sc. (Math.) 2002]

Solution. Introducing slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective inequalities; and converting the objective function into that of maximization; the linear programming problem is :

$$\text{Maximize } z^* = -(x_2 - 3x_3 + 2x_5) + 0.s_1 + 0.s_2 + 0.s_3 \text{ subject to the constraints :}$$

$$3x_2 - x_3 + 2x_5 + s_1 = 7$$

$$-2x_2 + 4x_3 + 0.x_5 + s_2 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + s_3 = 10$$

$$x_2, x_3, x_5, s_1, s_2, s_3 \ge 0.$$

The set of constraints can be written in matrix form as

$$\begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix} \quad \text{or} \quad \mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{x} = [x_2 \;\; x_3 \;\; x_5 \;\; s_1 \;\; s_2 \;\; s_3]$ and $\mathbf{b} = [7 \;\; 12 \;\; 10]$.

An obvious initial basic feasible solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, where $\mathbf{B} = \mathbf{I}_3$, and $\mathbf{x}_B$ = basic variables corresponding to coloumns of basis matrix $\mathbf{B}$ (= $\mathbf{I}$).

$$\therefore \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$$

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_5$.

			−1	3	−2	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	7	3	−1	2	1	0	0
0	$\mathbf{y}_5$	12	−2	(4)	0	0	1	0
0	$\mathbf{y}_6$	10	−4	3	8	0	0	1
	z	0	1	−3	2	0	0	0

At least one $z_j - c_j$, *viz.*, $z_2 - c_2$ is negative and therefore the current basic feasible solution is not optimum. We choose the column corresponding to $z_2 - c_2$, *i.e.*, column vector $\mathbf{y}_2$ enters the basis $\mathbf{y}_B$ (since at least one $y_{i2} > 0$). Further, since minimum $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\}$ is $\frac{12}{4}$ (= 3), current basis vector $\mathbf{y}_5$ leaves the basis. This gives y_{22} (= 4) as the leading element. Now using E-Row operations, we convert the leading element into unity and all other elements of the entering column vector $\mathbf{y}_2$ to zero. We get the improved basic feasible solution as shown in the next simplex table.

First Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	10	(5/2)	0	2	1	1/4	0
3	$\mathbf{y}_2$	3	−1/2	1	0	0	1/4	0
0	$\mathbf{y}_6$	1	−5/2	0	8	0	−3/4	1
	z	9	−1/2	0	2	0	3/4	0

Observe that $z_1 - c_1$ is negative and thus the current basic feasible solution is not optimum. The column vector corresponding to $z_1 - c_1$ enters the next basis $\mathbf{y}_B$ (since $y_{i1} > 0$). Further, since only $y_{11} > 0$ both $y_{12} < 0$ and $y_{13} < 0$); current basis vector $\mathbf{y}_4$ leaves the basis. This gives $y_{11}\left(=\frac{5}{2}\right)$ as the leading element. Using E-Row operations, we convert the leading element into unity and all other entries in its column $\mathbf{y}_1$ to zero. The improved basic feasible solution is obtained as shown in the next simplex table.

Final Iteration. Optimum Solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
−1	$\mathbf{y}_1$	4	1	0	4/5	2/5	1/10	0
3	$\mathbf{y}_2$	5	0	1	2/5	1/5	3/10	0
0	$\mathbf{y}_6$	11	0	0	10	2/5	−1/2	1
	z	11	0	0	12/5	1/5	8/10	0

Since all $z_j - c_j \geq 0$, an optimal basic feasible solution has been attained. Thus, the optimum solution to the given L.P.P. is

$$\text{Minimum } z = -\text{Maximum } z^* = -11 \text{ with } x_2 = 4,\ x_3 = 5 \text{ and } x_5 = 0.$$

413. *Find the maximum value of* $z = 107x_1 + x_2 + 2x_3$ *subject to the constraints :*

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7,\quad 16x_1 + x_2 - 6x_3 \leq 5,\ 3x_1 - x_2 - x_3 \leq 0\,;$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution. By introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$ respectively, the set of constraints of the given linear programming problem is converted into the system of equations :

$$\begin{pmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

An obvious initial basic feasible solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, where $\mathbf{B} = \mathbf{I}_3$, $\mathbf{x}_B = [x_4 \ s_1 \ s_2]$, and $\mathbf{b} = [7/3 \quad 5 \quad 0]$.

$$\therefore \qquad \begin{bmatrix} x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}.$$

Using now simplex method, the iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_6$.

			107	1	2	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	7/3	14/3	1/3	−2	1	0	0
0	$\mathbf{y}_5$	5	16	1	−6	0	1	0
0	$\mathbf{y}_6$	0	(3)	−1	−1	0	0	1
	z_j		0	0	0	0	0	0
	$z_j - c_j$	$z\,(=0)$	−107	−1	−2	0	0	0

It is apparent from the table that there are three $z_j - c_j$ which are negative. We choose the most negative of these, *viz.*, – 107. The corresponding column vector $\mathbf{y}_1$, therefore, enters the basis $\mathbf{y}_B$ (since all $y_{i1} > 0$). Further, since minimum $\left\{\frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0\right\}$ is $\frac{0}{3}$ which occurs for the element y_{31} (= 3), vector $\mathbf{y}_6$ leaves the basis $\mathbf{y}_B$ and the element y_{31} becomes the leading element. Reducing the leading element into unity all other entries of $\mathbf{y}_1$ to zero, we get

Final Iteration. Unbounded solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	7/3	0	17/9	−4/9	1	0	−14/9
0	$\mathbf{y}_5$	5	0	19/3	−2/3	0	1	−16/3
107	$\mathbf{y}_1$	0	1	−1/3	−1/3	0	0	1/3
	z_j		107	−107/3	−107/3	0	0	107/3
	$z_j - c_j$	$z\,(=0)$	0	−110/3	−113/3	0	0	107/3

It is apparent from the table that the net evaluations $z_2 - c_2$ and $z_3 - c_3$ are negative, $z_3 - c_3$ being the most negative one. The column vector corresponding to $z_3 - c_3$ is $\mathbf{y}_3$. Now $\mathbf{y}_3$ will not enter the basis $\mathbf{y}_B$, since y_{i3} (i = 1, 2, 3) are non-positive. Instead, this is an indication that there is an unbounded solution to the given L.P.P.

PROBLEMS

414. Use simplex method to solve the L.P.P. :

(*a*) Maximize $z = 3x_1 + 2x_2$ subject to the constraints :

(*i*) $x_1 + x_2 \le 4,\ x_1 - x_2 \le 2,\ x_1 \ge 0,\ x_2 \ge 0.$ *[IAS 1992]*

(*ii*) $x_1 + x_2 \le 6,\ 2x_1 + x_2 \le 6,\ x_1, x_2 \ge 0.$

(*b*) Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$x_1 + x_2 \le 4,\ -x_1 + x_2 \le 1$ and $x_1 + 2x_2 \le 5,\ x_1 \ge 0,\ x_2 \ge 0.$

415. Use simplex method to maximise $z = 5x_1 + 3x_2$ subject to the constraints :

(*a*) $x_1 + x_2 \le 2,\ 5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12,\ x_1, x_2 \ge 0.$

[Purvanchal M.C.A. 1996]

(*b*) $x_1 \le 4,\ x_2 \le 3\ x_1 + 2x_2 \le 18$
$x_1 + x_2 \le 9,\ x_1, x_2 \ge 0.$

[Nagarjuna M.Com. 1995]

416. Use simplex method to maximize $z = x_1 + 2x_2 + 3x_3$ subject to the constraints :

$$x_1 + 2x_2 + 3x_3 \le 10,\ x_1 + x_2 \le 5,\ x_1, x_2, x_3 \ge 0.$$

417. Use simplex method to

Maximize $z = 2x_1 - x_2 + x_3$ subject to the constraints :

$$3x_1 + x_2 + x_3 \le 60,\ x_1 - x_2 + 2x_3 \le 10,$$
$$x_1 + x_2 - x_3 \le 20 \text{ and } x_1, x_2, x_3 \ge 0.$$

418. Use simplex method to

Maximize $z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints :

$$x_1 + 2x_2 + x_3 \le 430,\ 3x_1 + 2x_3 \le 460,\ x_1 + 4x_3 \le 420;\ x_1, x_2, x_3 \ge 0.$$

419. Use simplex method to solve the L.P.P. :

Maximize $z = 2x_1 + 4x_2 + x_3 + x_4$ subject to the constraints :

$$x_1 + 3x_2 + x_4 \le 4,\ 2x_1 + x_2 \le 3,\ x_2 + 4x_3 + x_4 \le 3,\ x_1, x_2, x_3, x_4 \ge 0.$$ *[Madras M.B.A. 2010]*

420. Use simplex method to

Maximize $z = 4x_1 + 3x_2 + 4x_3 + 6x_4$ subject to the constraints :

$$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80,\ 2x_1 + 2x_3 + x_4 \le 60$$
$$3x_1 + 3x_2 + x_3 + x_4 \le 80,\ x_1, x_2, x_3, x_4 \ge 0.$$

421. Using simplex method solve the L.P.P. :

Maximize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$ subject to the constraints :

$$x_1 + x_2 + x_3 + x_4 \le 15,\ 7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100,\ x_1, x_2, x_3, x_4 \ge 0.$$

[Delhi B.Sc. (Stat.) 1999]

422. Solve the following problem by the Simplex Method :

Maximize $15x_1 + 6x_2 + 9x_3 + 2x_4$ subject to the constraints :

$$2x_1 + x_2 + 5x_3 + 0.6x_4 \le 10,\ 3x_1 + x_2 + 3x_3 + 0.25x_4 \le 12$$
$$7x_1 + x_4 \le 35,\ x_j \ge 0 \text{ for } j = 1, 2, 3, 4.$$

423. Show that the L.P. P.

Maximize $z = 4x_1 + x_2 + 3x_3 + 5x_4$ subject to the constraints :

$$4x_1 - 6x_2 - 5x_3 - 4x_4 \ge -20,\ 3x_1 - 2x_2 + 4x_3 + x_4 \le 10$$
$$8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20,\ x_1, x_2, x_3, x_4 \ge 0$$

has an unbounded solution.

[Purvanchal M.C.A. 1996]

4:4. USE OF ARTIFICIAL VARIABLES

We have seen that in the computational procedure of the simplex method, it is most convenient to have the slack variables as the starting (initial) basic variables. Thus, if the original constraint is an equation or is of the type (≥) we may no longer have a *ready* starting basic feasible solution.

In order to obtain an initial basic feasible solution, we first put the given L.P.P. into its standard form and then a non-negative variable is added to the left side of each of equation that lacks the much needed starting basic variables. The so-added variable is called an *artificial variable* and plays the same role as a slack variable in providing the initial basic feasible solution. However, since such artificial variables have no physical meaning from the standpoint of the original problem, the method will be valid only if we are able to force these variables to be out or at zero level when the optimum solution is attained. In other words, to get back to the original problem, artificial variables must be driven to zero in the final solution; otherwise the resulting solution may be infeasible.

Two methods are generally employed for the solution of linear programming problems having artificial variables :

1. Two-Phase Method; and
2. Big-M Method or Method of Penalties.

TWO-PHASE METHOD

In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints (known as auxiliary L.P.P.) to get a basic feasible solution to the original L.P.P. Second phase then optimizes the original objective function starting with the basic feasible solution obtained at the end of *Phase 1*.

The iterative procedure of the algorithm may be summarise as below :

Step 1. Write the given L.P.P. into its standard form and check whether there exists a starting basic feasible solution.

(*a*) If there is a ready starting basic feasible solution, go to *Phase 2*.

(*b*) If there does not exist a ready starting basic feasible solution, go on to the next step.

PHASE 1

Step 2. Add the artificial variable to the left side of the each equation that lacks the needed starting basic variables. Construct an auxiliary objective function aimed at minimising the sum of all artificial variables.

Thus, the new objective is to

$$\text{Minimize} \quad z = A_1 + A_2 + \dots + A_n$$

i.e.,

$$\text{Maximize} \quad z^* = -A_1 - A_2 - \dots - A_n$$

where A_i ($i = 1, 2, \dots, m$) are the non-negative artificial variables.

Step 3. Apply simplex algorithm to the specially constructed L.P.P. The following three cases may arise at the least interaction :

(*a*) max $z^* < 0$ and at least one artificial variable is present in the basis with positive value. In such a case, the original L.P.P. does not possess any feasible solution.

(*b*) max $z^* = 0$ and at least one artificial variable is present in the basis at zero value. In such a case, the original L.P.P. possess the feasible solution. In order to get basic feasible solution we may proceed directly to *Phase 2* or else eliminate the artificial basic variable and then proceed to *Phase 2*.

(*c*) maximum $z^* = 0$ and no artificial variable is present in the basis. In such a case, a basic feasible solution to the original L.P.P. has been found. Go to *Phase 2*.

PHASE 2

Step 4. Consider the optimum basic feasible solution of *Phase 1* as a starting basic feasible solution for the original L.P.P. Assign actual coefficients to the variables in the objective function and a value zero to the artificial variables that appear at zero value in the final simplex table of *Phase 1*.

Apply usual simplex algorithm to the modified simplex table to get the optimum solution of the original problem.

Note : Artificial variables that do not appear in the basic solution may be deleted from the simplex table totally.

BIG-M METHOD (METHOD OF PENALTIES)

The Big-M method is an alternative method of solving a linear programming problem involving artificial variables. In this method we assign a very high penalty (say M) to the artificial variables in the objective function.

The iterative procedure of the algorithm is given below :

Step 1. Write the given L.P.P. into its standard form and check whether there exists a starting basic feasible solution.

(a) If there is a ready starting basic feasible solution, move on to *Step 3.*

(b) If there does not exist a ready starting basic feasible solution, move on to *Step 2.*

Step 2. Add artificial variables to the left side of each equation that has no obvious starting basic variables. Assign a very high penalty (say M) to these variables in the objective function.

Step 3. Apply simplex method to the modified L.P.P. Following cases may arise at the last iteration :

(a) At least one artificial variable is present in the basis with zero value. In such a case the current optimum basic feasible solution is degenerate.

(b) At least one artificial variable is present in the basis with a positive value. In such a case, the given L.P.P. does not possess an optimum basic feasible solution. The given problem is said to have a pseudo-optimum basic feasible solution.

SAMPLE PROBLEMS

424. *Use two-phase simplex method to maximize* $z = 5x_1 + 3x_2$ *subject to the constraints :*

$$2x_1 + x_2 \le 1,\quad x_1 + 4x_2 \ge 6 \quad \text{and} \quad x_1, x_2 \ge 0.$$

Solution. Introducing a slack variable $s_1 \ge 0$, a surplus variable $s_2 \ge 0$ and an artificial variable $A_1 \ge 0$ in the constraints of the L.P.P., an initial basic feasible solution is : $s_1 = 1$ and $A_1 = 6$ with I_2 as the basis matrix.

Phase I. The objective function of the auxiliary L.P.P. is to maximize $z^* = -A_1$.

Using now simplex algorithm to the auxiliary linear programming problem, the iterative simplex tables are :

Initial iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_3$.

			0	0	0	0	−1
c_B	Y_B	x_B	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	1	2	(1)	1	0	0
−1	$\mathbf{y}_5$	6	1	4	0	−1	1
	z_j	—	−1	−4	0	1	−1
	$z_j - c_j$	$z^* (= -6)$	−1	−4	0	1	0

Since $z_1 - c_1$ and $z_2 - c_2$ are negative, we choose the most negative of these, *viz.*, −4. The corresponding column vector $\mathbf{y}_2$ enters the basis. Further, since min. $\left\{\dfrac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\}$ is 1, which occurs for element y_{12}, $\mathbf{y}_3$ leaves the basis.

Final iteration. Optimum solution.

c_B	Y_B	x_B	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_2$	1	2	1	1	0	0
−1	$\mathbf{y}_5$	2	−7	0	−4	−1	1
		$z^* (= -2)$	7	0	4	1	0

Final Iteration. Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_2	30/7	0	1	−30/7	0	−1/7	4/7
0	y_4	52/7	0	0	256/7	1	2/7	−29/7
0	y_1	55/7	1	0	−6/7	0	1/14	3/14
		z^* (= 0)	0	0	0	0	0	1

Since, all $z_j - c_j \geq 0$ an optimum solution to the auxiliary L.P.P. has been reached. Furthermore, it is apparent from the table that no artificial variable appears in the basis.

Phase 2. Now, we consider the actual costs associated with the original variables. So, the objective function is

$$z = 5x_1 - 4x_2 + 3x_3 + 0 \cdot s_1 + 0 \cdot s_2$$

The iterative simplex table for this phase is :

			5	−4	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
−4	y_2	30/7	0	1	−30/7	0	−1/7
0	y_4	52/7	0	0	256/7	1	2/7
5	y_1	55/7	1	0	−6/7	0	1/14
		z^* (= 155/7)	0	0	69/7	0	13/4

Since, all $z_j - c_j \geq 0$ an optimum basic feasible solution has been reached. Hence, an optimum basic feasible solution to the given L.P.P. is

$$x_1 = 55/7,\ x_2 = 30/7,\ x_3 = 0\ ;\ \text{maximum } z = 155/7.$$

426. *Use penalty (or Big M) method to*

$$\textit{Maximize } z = 6x_1 + 4x_2 \textit{ subject to the constraints :}$$
$$2x_1 + 3x_2 \leq 30,\ 3x_1 + 2x_2 \leq 24,\ x_1 + x_2 \geq 3,$$
$$x_1 \geq 0 \textit{ and } x_2 \geq 0$$

Is the solution unique? If not, give two different solutions.

Solution. Introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and surplus variable $s_3 \geq 0$ in the constraints, the standard form of L.P.P. is :

$$\text{Maximize } z = 6x_1 + 4x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 \text{ subject to the constraints :}$$
$$2x_1 + 3x_2 + s_1 = 30,\ 3x_1 + 2x_2 + s_2 = 24,\ x_1 + x_2 - s_3 = 3,$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Clearly, we do not have a initial (starting) basic feasible solution. So, we introduce an artificial variable $A_1 \geq 0$ in the third constraint. Then, an obvious initial basic feasible solution is

$$s_1 = 30,\ s_2 = 24 \text{ and } A_1 = 3.$$

Now, corresponding to the basic variables s_1, s_2 and A_1, the matrix $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$ (where $\mathbf{B} = \mathbf{I}$ the identity matrix) and the net evaluations $z_j - c_j$ (j = 1, 2, 3, 4, 5, 6) are computed, where $\mathbf{c}_B = (0, 0, -M)$.

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_6$.

			6	4	0	0	0	−M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	30	2	3	1	0	0	0
0	y_4	24	3	2	0	1	0	0
−M	y_6	3	(1)	1	0	0	−1	1
		z (= −3M)	− M − 6	− M − 4	0	0	M	0

In the above table $z_1 - c_1$ and $z_2 - c_2$ are negative. Among these two $z_1 - c_1$ gives the most negative value (since M is very large), therefore $\mathbf{y}_1$ enters the basis. Further, min. $\left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0\right\} = \frac{3}{1}$. This indicates $\mathbf{y}_6$ leaves the basis and y_{31} becomes the leading element. Since $\mathbf{y}_6$ corresponds to artificial variable A_1, we drop it from the objective function.

First Iteration. Introduce $\mathbf{y}_5$ and drop $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	24	0	1	1	0	2
0	$\mathbf{y}_4$	15	0	−1	0	1	(3)
6	$\mathbf{y}_1$	3	1	1	0	0	−1
		z (= 18)	0	2	0	0	−6

Since $z_5 - c_5 < 0$, $\mathbf{y}_5$ enters the basis. Further, min. $\left\{\frac{x_{Bi}}{y_{i5}}, y_{i5} > 0\right\} = \frac{15}{3}$.

$\therefore$ $\mathbf{y}_4$ leaves the basis and y_{25} becomes the leading element.

Final Iteration. Optimum Solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	14	0	5/3	1	−2/3	0
0	$\mathbf{y}_5$	5	0	−1/3	0	1/3	1
6	$\mathbf{y}_1$	8	1	2/3	0	1/3	0
		z (= 48)	0	0	0	2	0

Since all $z_j - c_j \geq 0$, an optimum solution has been reached. Thus, an optimum basic feasible solution to the given L.P.P. is

$$x_1 = 8 \text{ and } x_2 = 0 \text{ with max. } z = 48.$$

Alternative Solution. It is evident from the net evaluations of the optimum table that the net evaluation corresponding to non-basic variable x_2 is zero. This is an indication that the current solution is not unique and an alternative solution exists. Thus we can bring $\mathbf{y}_2$ into the basis in place of $\mathbf{y}_3$ or $\mathbf{y}_5$. The resulting new basic feasible solution will also be an optimum one.

Therefore, by introducing $\mathbf{y}_2$ into the basis in place of $\mathbf{y}_3$, the new optimum simplex table is

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
4	$\mathbf{y}_2$	42/5	0	1	3/5	−2/5	0
0	$\mathbf{y}_5$	39/5	0	0	1/5	1/5	1
6	$\mathbf{y}_1$	12/5	1	0	−2/5	3/5	0
		z (= 48)	0	0	0	2	0

We observe from this table that a different basic feasible solution results, but the optimum value of z remains the same. Hence, the two basic (optimum) feasible solutions to the given L.P.P. are :

$$\mathbf{x}_1 = [8, 0] \text{ and } \mathbf{x}_2 = [12/5, 42/5].$$

427. *Maximize* $z = 3x_1 + 2x_2$ *subject to the constraints :*

$$2x_1 + x_2 \leq 2, \; 3x_1 + 4x_2 \geq 12, \; x_1, x_2 \geq 0.$$

Solution. Introducing slack variable $s_1 \geq 0$, surplus variable $s_2 \geq 0$ and an artificial variable $A_1 \geq 0$, the reformulated L.P.P. can be written as :

Maximize $z = 3x_1 + 2x_2 + 0 \cdot s_1 + 0 \cdot s_2 - MA_1$ subject to the constraints :

$$2x_1 + x_2 + s_1 = 2, \; 3x_1 + 4x_2 - s_2 + A_1 = 12, \; x_1, x_2, s_1, s_2 \geq 0 \text{ and } A_1 \geq 0.$$

An obvious starting basic feasible solution is : $s_1 = 2$ and $A_1 = 12$.

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_3$.

			3	2	0	0	$-M$
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	2	2	(1)	1	0	0
$-M$	$\mathbf{y}_5$	12	3	4	0	-1	1
	z	$-12M-3$	$-3M-2$	$-4M$	0	M	0

Clearly, $z_2 - c_2$ is the most negative and hence $\mathbf{y}_2$ enters the current basis. Further

$$\min.\left\{\frac{x_B}{y_{i2}},\ y_{i2} > 0\right\} = \frac{2}{1}.$$ So. $\mathbf{y}_3$ leaves the current basis.

Final Iteration. No solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
2	$\mathbf{y}_2$	2	2	1	0	1	0
$-M$	$\mathbf{y}_5$	4	-5	0	-4	-1	1
	z	$-4M+4$	$5M+1$	0	$4M+2$	M	0

Here, the coefficient of M in each $z_j - c_j$ is non-negative and an artificial vector appears in the basis, not at the zero level. Thus the given L.P.P. does not possess any feasible solution.

428. *Use penalty (or Big M) method to*

$$\textit{Maximize}\quad z = x_1 + 2x_2 + 3x_3 - x_4 \quad \textit{subject to the constraints :}$$

$$x_1 + 2x_2 + 3x_3 = 15,\ 2x_1 + x_2 + 5x_3 = 20,\ x_1 + 2x_2 + x_3 + x_4 = 10,\ x_1, x_2, x_3, x_4 \geq 0.$$

[I.A.S. 1995; Garhwal M.Sc. (Math.) 2002]

Solution. From the constraints of the problem, we observe that we do not have requisite identity column to get the starting **B** as an identity matrix. So, we introduce artificial variables $A_1 \geq 0$ and $A_2 \geq 0$ in the first and second constraints respectively. An initial basic feasible solution, then, is

$$A_1 = 15,\ A_2 = 20 \ \text{ and } \ x_4 = 10$$

Now, corresponding to basic variables A_1, A_2 and x_4, the basis matrix $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$ and the net evaluations $z_j - c_j$ (j = 1, 2, 3, 4, 5, 6) are computed, where $\mathbf{c}_B = (-M\ \ -M\ \ -1)$. The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_5$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$-M$	$\mathbf{y}_6$	15	1	2	3	0	0	1
$-M$	$\mathbf{y}_5$	20	2	1	(5)	0	1	0
-1	$\mathbf{y}_4$	10	1	2	1	1	0	0
	z	$-35M-10$	$-3M-2$	$-3M-4$	$-8M-4$	0	0	0

Since, the most negative $z_j - c_j$ $(= z_3 - c_3)$ corresponds to $\mathbf{y}_3$, it enters the basis. Further,

$$\min.\left\{\frac{x_{Bi}}{y_{i3}},\ y_{i3} > 0\right\} = \frac{20}{5},$$ the current basis vector $\mathbf{y}_5$ leaves the basis and y_{23} becomes the leading element.

As $\mathbf{y}_5$ corresponds to an artificial variable A_2, we drop the $\mathbf{y}_5$ column from subsequent simplex tables.

First Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_6$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_6$
$-M$	$\mathbf{y}_6$	3	$-1/5$	(7/5)	0	0	1
3	$\mathbf{y}_3$	4	2/5	1/5	1	0	0
-1	$\mathbf{y}_4$	~~6~~	3/5	9/5	0	1	0
	z	$-3M+6$	$\frac{M}{5}-\frac{2}{5}$	$\frac{-7M}{5}-\frac{16}{5}$	0	0	0

Clearly, $z_2 - c_2$ is the only negative $z_j - c_j$ (since M is very large) and hence $\mathbf{y}_2$ enters the basis. Further, min. $\left\{\frac{x_{Bi}}{y_{i2}}, y_{i2} > 0\right\}$ corresponds to $\mathbf{y}_6$. So, $\mathbf{y}_6$ leaves the basis and y_{12} becomes the leading element. Again, $\mathbf{y}_6$ corresponds to artificial variable A_1 and therefore we drop the artificial column $\mathbf{y}_6$ in the subsequent tables.

Second Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
2	$\mathbf{y}_2$	15/7	–1/7	1	0	0
3	$\mathbf{y}_3$	25/7	3/7	0	1	0
–1	$\mathbf{y}_4$	15/7	(6/7)	0	0	1
	z	90/7	–6/7	0	0	0

Clearly, $z_1 - c_1 < 0$ and, therefore, $\mathbf{y}_1$ enters the basis. Further, min. $\left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0\right\}$ corresponds to $\mathbf{y}_4'$. So, $\mathbf{y}_4$ leaves the basis and y_{31} becomes the leading element.

Final Iteration. Optimum Solution.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
2	$\mathbf{y}_2$	15/6	0	1	0	1/6
3	$\mathbf{y}_3$	15/6	0	0	1	–3/6
1	$\mathbf{y}_1$	15/6	1	0	0	7/6
	z	15	0	0	0	1

Since, all $z_j - c_j$ are positive, therefore, an optimum basic feasible solution has been attained. Thus, an optimum solution to the given L.P.P. is

$$\text{Maximum } z = 15\,;\ x_1 = x_2 = x_3 = 5/2 \text{ and } x_4 = 0.$$

PROBLEMS

429. Use two-phase simplex method to

(*a*) Maximize $z = 10x_1 + 20x_2$ subject to the constraints :

$2x_1 + x_2 = 1,\ x_1 + 2x_2 = 5,\ x_1 \geq 0,\ x_2 \geq 0.$ **[Madurai M.Com. 2002]**

(*b*) Minimize $z = 2x_1 + 4x_2$ subject to the constraints :

$2x_1 + x_2 \geq 14,\ x_1 + 3x_2 \geq 18,\ x_1 + x_2 \geq 12\,;\ x_1\ x_2 \geq 0.$ **[Panjab Tech. Univ. M.B.A. (Dec.) 2010]**

430. Use two-phase simplex method to

Maximize $z = 3x_1 - x_2$ subject to the constraints :

$2x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 2,\ x_2 \leq 4\,;\ x_1, x_2 \geq 0.$ **[Meerut M.Sc. (Math.) 1998]**

431. Use two-phase simplex method to

Maximize $z = 5x_1 + 8x_2$ subject to the constraints :

$3x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4,\ x_1 + x_2 \leq 5,\ x_1, x_2 \geq 0.$

432. Use two-phase simplex method to

Minimize $z = x_1 + x_2 + x_3$ subject to the constraints :

$x_1 - 3x_2 + 4x_3 = 5,\ x_1 - 2x_2 \leq 3,\ 2x_2 + x_3 \geq 4$

$x_1 \geq 0,\ x_2 \geq 0$ and x_3 is unrestricted.

433. Use two-phase simplex method to

Maximize $z = x_1 + 2x_2 + 3x_3$ subject to the constraints :

$x_1 - x_2 + x_3 \geq 4,\ x_1 + x_2 + 2x_3 \leq 8,\ x_1 - x_3 \geq 2,\ x_1, x_2, x_3 \geq 0.$

5

Duality in Linear Programming

"Man is simply to recognise that there are different approaches for solving problems"

5:1. INTRODUCTION

Associated with every linear programming problem (maximization or minimization) there always exist another Linear Programming problem which is based upon the same data and having the same solution. The original problem is called the **primal problem** while the associated one is called its **dual problem**. It is important to note that either of the two linear programming problems can be treated as primal and the other as its dual. The two problems, thus, constitute a primal-dual pair.

The concept of duality is based on the fact that any linear programming problem must be first put in its standard form before solving the problem by simplex method. Since, all the primal-dual computations are obtained directly from the simplex table, it is logical that we define the dual that may be constituent with the standard form of the primal.

5:2. GENERAL PRIMAL-DUAL PAIR

Based on the standard form of primal, there are two important primal-dual pairs :

Definition 1. **(Standard primal problem)**

Maximize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ *subject to the constraints :*

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i\,; \qquad i = 1, 2, \dots, m$$

$$x_j \geq 0\,; \qquad j = 1, 2, \dots, n$$

Dual Problem

Minimize $z^* = b_1w_1 + b_2w_2 + \dots + b_mw_m$ *subject to the constraints :*

$$a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m \geq c_j\,; \qquad j = 1, 2, \dots, n$$

w_i $(i = 1, 2, \dots, m)$ unrestricted.

Note that x_j's are the primal variables, w_i's the dual variables and the other constants have their usual meanings.

Definition 2. **(Standard primal problem)**

Minimize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ *subject to the constraints :*

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i\,; \qquad i = 1, 2, \dots, m$$

$$x_j \geq 0\,; \qquad j = 1, 2, \dots, n$$

Dual Problem

Maximize $z^* = b_1w_1 + b_2w_2 + \dots + b_mw_m$ *subject to the constraints :*

$$a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m \leq c_j\,; \qquad j = 1, 2, \dots, n$$

w_i $(i = 1, 2, \dots, m)$ unrestricted.

Remark. From the above definitions one may easily observe the following:

(*a*) For every primal constraint, there is a dual variable.

(*b*) For every primal variable, there is a dual constraint.

(*c*) The coefficients of the dual variables in the constraints are the same as the coefficients of the primal variables except that they are transposed, *i.e.*, columns in primal coefficient matrix become the rows in the dual coefficient matrix.

(*d*) The number of dual constraints is exactly equal to the number of primal variables, whereas the number of dual variables is exactly equal to the number of primal constraints.

(*e*) The objective coefficients of the primal variables become the right-hand side constants of dual constraints, whereas the right-hand side constants of the dual constraints become the objective coefficients of the primal problem.

The information regarding the primal-dual objective, type of constraints and the signs of the dual variables may be summarized in the following table :

Standard primal objective	*Dual*		
	Objective	*Constraints*	*Variables*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

5:3. FORMULATING A DUAL PROBLEM

Various steps involved in the formulation of a primal-dual pair are :

Step 1. Put the given linear programming problem into its standard form. Consider it as the primal problem.

Step 2. Identify the variables to be used in the dual problem. The number of these variables equals the number of constraint equations in the primal.

Step 3. Write down the objective function of the dual, using the right-hand side constants of the primal constraints.

If the primal problem is of maximization type, the dual will be a minimization problem and vice-versa.

Step 4. Making use of dual variable identified in *Step 2*, write the constraints for the dual problem.

(*a*) If the primal is a maximization problem, the dual constraints must be all of '$\geq$' type. If the primal is a minimization problem, the dual constraints must be all of '$\leq$' type.

(*b*) The column coefficients of the primal constraints become the row coefficients of the dual constraints.

(*c*) The coefficients of the primal objective function becomes the right-hand side constants of the dual constraints.

(*d*) The dual variables are defined to be unrestricted in sign.

Step 5. Using *steps 3* and *4*, write down the dual of the given L.P.P.

Note : The dual constraint corresponding to an artificial variable in the standard form of the primal is always *redundant*, hence it is never necessary to consider the dual constraint associated with an artificial variable.

Remarks. 1. If the given linear programming problem is in its canonical form, the primal-dual pair is said to be *symmetric*.

2. If the given linear programming problem is in its standard form, the primal-dual pair is said to be *unsymmetric*.

5:4. PRIMAL-DUAL PAIR IN MATRIX FORM

Standard Primal Problem

Definition 1. **(Standard primal problem).** *Find* $\mathbf{x}^T \in \mathbf{R}^n$ *so as to maximize* $z = \mathbf{cx},\ \mathbf{c} \in \mathbf{R}^n$ *subject to the constraints :*

$$\mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0},\ \mathbf{b} \in \mathbf{R}^m$$

where **A** *is an* $m \times n$ *real matrix.*

Dual Problem. *Find* $\mathbf{w}^T \in \mathbf{R}^m$ *so as to minimize* $z^* = \mathbf{b}^T\mathbf{w},\ \mathbf{b} \in \mathbf{R}^m$ *subject to the constraints :*

$$\mathbf{A}^T\mathbf{w} \geq \mathbf{c}^T,\ \mathbf{c} \in \mathbf{R}^n$$

where $\mathbf{A}^T$ *is the transpose of an* $m \times n$ *real matrix* **A** *and* **w** *is unrestricted in sign.*

Definition 2. **(Standard primal problem).** Find $\mathbf{x}^T \in \mathbf{R}^n$ so as to *minimize* $z = \mathbf{cx},\ \mathbf{c} \in \mathbf{R}^n$ *subject to the constraints :*

$$\mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0},\ \mathbf{b}^T \in \mathbf{R}^m$$

where **A** *is an* $m \times n$ *real matrix.*

Dual Problem. *Find* $\mathbf{w}^T \in \mathbf{R}^n$, *so as to maximize* $z^* = \mathbf{b}^T\mathbf{w},\ \mathbf{b} \in \mathbf{R}^m$ *subject to the constraints :*

$$\mathbf{A}^T\mathbf{w} \leq \mathbf{c}^T,\ \mathbf{c} \in \mathbf{R}^n$$

where $\mathbf{A}^T$ *is the transpose of an* $m \times n$ *real matrix* **A** *and* **w** *is unrestricted in sign.*

SAMPLE PROBLEMS

501. *Formulate the dual of the following linear programming problem :*

$$\text{Maximize } z = 5x_1 + 3x_2 \text{ subject to the constraints :}$$

$$3x_1 + 5x_2 \leq 15,\ 5x_1 + 2x_2 \leq 10,\ x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Solution. ***Standard Primal.*** Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$, the standard linear programming problem is :

$$\text{Maximize } z = 5x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2 \text{ subject to the constraints :}$$

$$3x_1 + 5x_2 + s_1 + 0 \cdot s_2 = 15,\ 5x_1 + 2x_2 + 0 \cdot s_1 + s_2 = 10,\ x_1, x_2, s_1, s_2 \geq 0$$

Dual. Let w_1 and w_2 be the dual variables corresponding to the primal constraints. Then, the dual problem will be :

$$\text{Minimize } z^* = 15w_1 + 10w_2 \text{ subject to the constraints :}$$

$$3w_1 + 5w_2 \geq 5,\ 5w_1 + 2w_2 \geq 3$$

$$\left.\begin{aligned} w_1 + 0 \cdot w_2 \geq 0 \\ 0 \cdot w_1 + w_2 \geq 0 \end{aligned}\right\} \Rightarrow w_1 \geq 0 \text{ and } w_2 \geq 0$$

$$w_1 \text{ and } w_2 \text{ unrestricted (redundant).}$$

The dual variables "w_1 and w_2 unrestricted" are dominated by $w_1 \geq 0$ and $w_2 \geq 0$. Eliminating redundancy, the restricted variables are $w_1 \geq 0$ and $w_2 \geq 0$.

502. *Write the dual of the L.P.P. :*

$$\text{Minimize } z = 4x_1 + 6x_2 + 18x_3 \text{ subject to the constraints :}$$

$$x_1 + 3x_2 \geq 3,\ x_2 + 2x_3 \geq 5 \text{ and } x_1, x_2, x_3 \geq 0.$$ **[Delhi B.Sc. (Stat.) 2006]**

Solution. ***Standard Primal.*** Introducing surplus variables $s_1 \geq 0$ and $s_2 \geq 0$, the standard form of the L.P.P. is:

$$\text{Minimize } z = 4x_1 + 6x_2 + 18x_3 + 0 \cdot s_1 + 0 \cdot s_2 \text{ subject to the constraints :}$$

$$x_1 + 3x_2 - s_1 = 3,\ 0 \cdot x_1 + x_2 + 2x_3 - s_2 = 5,\ x_1, x_2, x_3, s_1, s_2 \geq 0$$

Dual. If w_1 and w_2 be the dual variables corresponding to each primal constraint, the dual problem will be

$$\text{Maximize } z^* = 3w_1 + 5w_2 \text{ subject to the constraints :}$$
$$w_1 + 0 \cdot w_2 \leq 4,\ 3w_1 + w_2 \leq 6,\ 0 \cdot w_1 + 2w_2 \leq 18,$$
$$-w_1 + 0 \cdot w_2 \leq 0, \text{ and } 0 \cdot w_1 - w_2 \leq 0,$$
$$w_1 \text{ and } w_2 \text{ unrestricted (redundant).}$$

Eliminating redundancy, the dual problem is :

$$\text{Maximize } z^* = 3w_1 + 5w_2 \text{ subject to the constraints :}$$
$$w_1 \leq 4,\ 3w_1 + w_2 \leq 6,\ 2w_2 \leq 18;\ w_1 \geq 0 \text{ and } w_2 \geq 0.$$

503. *Write the dual of the following linear programming problem :*

$$\text{Minimize } z = 3x_1 - 2x_2 + 4x_3 \text{ subject to the constraints :}$$
$$3x_1 + 5x_2 + 4x_3 \geq 7,\ 6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10,\ x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2,$$
$$x_1 \geq 0,\ x_2 \geq 0, \text{ and } x_3 \geq 0.$$

[Visvesvaraya M.B.A. (Dec.) 2011]

Solution. ***Standard Primal.*** Introducing surplus variables $s_1 \geq 0$, $s_2 \geq 0$, $s_4 \geq 0$, $s_5 \geq 0$ and a slack variable $s_3 \geq 0$, the standard form of L.P.P. is :

$$\text{Minimize } z = 3x_1 - 2x_2 + 4x_3 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_4 + 0 \cdot s_5 \text{ subject to the constraints :}$$
$$3x_1 + 5x_2 + 4x_3 - s_1 = 7$$
$$6x_1 + x_2 + 3x_3 - s_2 = 4$$
$$7x_1 - 2x_2 - x_3 + s_3 = 10$$
$$x_1 - 2x_2 + 5x_3 - s_4 = 3$$
$$4x_1 + 7x_2 - 2x_3 - s_5 = 2$$
$$x_1, x_2, x_3, s_1, s_2, s_3, s_4, s_5 \geq 0.$$

Dual. If w_j $(j = 1, 2, 3, 4, 5)$ are the dual variables corresponding to the five primal constraints in the given order, the dual of the standard primal L.P.P. is :

$$\text{Maximize } z^* = 7w_1 + 4w_2 + 10w_3 + 3w_4 + 2w_5 \text{ subject to the constraints :}$$
$$3w_1 + 6w_2 + 7w_3 + w_4 + 4w_5 \leq 3$$
$$5w_1 + w_2 - 2w_3 - 2w_4 + 7w_5 \leq -2$$
$$4w_1 + 3w_2 - w_3 + 5w_4 - 2w_5 \leq 4$$
$$-w_1 \leq 0,\ -w_2 \leq 0,\ w_3 \leq 0,\ -w_4 \leq 0,\ -w_5 \leq 0$$
$$w_j\ (j = 1, 2, 3, 4, 5) \text{ are unrestricted in sign.}$$

Eliminating redundancy, the dual variables are : $w_1 \geq 0$, $w_2 \geq 0$, $w_3 \leq 0$, $w_4 \geq 0$ and $w_5 \geq 0$.

Note. In the primal, if we multiply the third constraint throughout by -1, the dual variable w_3 will be ≥ 0 instead of ≤ 0.

504. *Obtain the dual of the following L.P.P. :*

$$\text{Maximize } z = 2x_1 + 3x_2 + x_3 \text{ subject to the constraints :}$$
$$4x_1 + 3x_2 + x_3 = 6,\ x_1 + 2x_2 + 5x_3 = 4;\ x_1, x_2, x_3 \geq 0$$

[Madras B.E. 1999]

Solution. Clearly, the given linear programming problem is in standard form. Considering it as standard primal, its dual is :

$$\text{Minimize } z^* = 6w_1 + 4w_2 \text{ subject to the constraints :}$$
$$4w_1 + w_2 \geq 2,\ 3w_1 + 2w_2 \geq 3,\ w_1 + 5w_2 \geq 1:\ w_1 \text{ and } w_2 \text{ are unrestricted,}$$

where w_1 and w_2 are dual variables.

505. *Obtain the dual problem of the following primal problem :*

Minimize $z = x_1 - 3x_2 - 2x_3$ *subject to the constraints :*

$3x_1 - x_2 + 2x_3 \le 7,\ 2x_1 - 4x_2 \ge 12,\ -4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \ge 0$ *and* x_3 *is unrestricted.* *[Delhi B.Sc. (Stat.) 1995]*

Solution. ***Standard Primal.*** Introducing a slack variable $s_1 \ge 0$ and a surplus variable $s_2 \ge 0$, the primal problem is restated as

Minimize $z = \mathbf{cx}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$,

where $\mathbf{x} = [x_1, x_2, x_3', x_3'', s_1, s_2]$, $\mathbf{c} = [1, -3, -2, 2, 0, 0]$, $\mathbf{b} = [7, 12, 10]$ and

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 & -2 & 1 & 0 \\ 2 & -4 & 0 & 0 & 0 & -1 \\ -4 & 3 & 8 & -8 & 0 & 0 \end{pmatrix}, \text{ when } x_3 = x_3' - x_3''.$$

Dual. If $\mathbf{w} = (w_1, w_2, w_3)$ are the dual variables, the dual problem of the primal is

Maximize $z^* = 7w_1 + 12w_2 + 10w_3$ subject to the constraints :

$$3w_1 + 2w_2 - 4w_3 \le 1$$

$$-w_1 - 4w_2 + 3w_3 \le -3 \Rightarrow w_1 + 4w_2 - 3w_3 \ge 3$$

$$\left.\begin{aligned} 2w_1 + 8w_3 &\le -2 \\ -2w_1 - 8w_3 &\le 2 \end{aligned}\right\} \Rightarrow -2w_1 - 8w_3 = 2$$

$$\left.\begin{aligned} w_1 &\le 0 \\ -w_2 &\le 0 \end{aligned}\right\} \Rightarrow w_1 \le 0 \text{ and } w_2 \ge 0$$

w_1, w_2 and w_3 unrestricted.

Eliminating redundancy, dual variables are $w_1 \le 0$, $w_2 \ge 0$ and w_3 unrestricted. This is re-written as follows :

Maximize $z^* = 7w_1 + 12w_2 + 10w_3$ subject to the constraints :

$3w_1 + 2w_2 - 4w_3 \le 1,\ w_1 + 4w_2 - 3w_3 \ge 3,\ -2w_1 - 8w_3 = 2;$

$w_1 \le 0$ and $w_2 \ge 0$, w_3 unrestricted.

Note. In the primal, if we multiply the first inequation by -1; the dual variable w_1 will be ≥ 0 instead of ≤ 0.

PROBLEMS

506. Formulate dual of the following L.P.P. :

(*a*) Maximize $z = 4x_1 + 2x_2$ **subject to the constraints :**

$x_1 + x_2 \ge 3,\ x_1 - x_2 \ge 2;\ x_1 \ge 0$ and $x_2 \ge 0$. ***[Annamalai M.B.A. (Nov.) 2009]***

(*b*) Maximize $z = 2000x + 3000y$ subject to the constraints :

$6x + 9y \le 100,\ 2x + y \le 20;\ x \ge 0$ and $y \ge 0$. ***[Osmania M.B.A. (Sept.) 2001]***

507. Formulate the dual of the following L.P.P. :

Maximize $z = 10x_1 + 8x_2$ subject to the constraints :

$x_1 + 2x_2 \ge 5,\ 2x_1 - x_2 \ge 12,\ x_1 + 3x_2 \ge 4,$

$x_1 \ge 0$ and x_2 is unrestricted. ***[Osmania M.B.A. 1999]***

508. Obtain the dual problem of the following L.P.P. :

Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$ subject to the constraints :

$5x_1 + 6x_2 - x_3 \le 3,\ -2x_1 + x_2 + 4x_3 \le 4,\ x_1 - 5x_2 + 3x_3 \le 1,$

$-3x_1 - 3x_2 + 7x_3 \le 6,\ x_1, x_2, x_3 \ge 0.$

Similarly, the second set of conditions is equivalent to

$$x_j^o \left(\sum_{i=1}^{m} a_{ji} w_i^o - c_j \right) = 0, \qquad j = 1, 2, \ldots, n$$

Corollary 2. *For optimal feasible solutions of the primal and dual systems, whenever the ith variable is strictly positive in either system, the ith relation of its dual is an equality.*

Proof. It follows from corollary *1*, that

$$w_i^o > 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j^o = b_i \qquad \text{(ith primal relation)}$$

and

$$x_j^o > 0 \Rightarrow \sum_{i=1}^{m} a_{ji} w_i^o = c_j \qquad \text{(jth dual relation)}$$

Corollary 3. *For optimal feasible solutions of the primal and dual systems, whenever ith relation of either system is satisfied as a strict inequality, then the ith variable of its dual vanishes.*

Proof. If follows from corollary *1* that

$$\sum_{j=1}^{n} a_{ij} x_j^o < b_i \Rightarrow w_i^o = 0$$

and

$$\sum_{i=1}^{m} a_{ji} w_i^o > c_j \Rightarrow x_j^o = 0.$$

Remarks. The conditions of corollary *1* can also be written as

$$w_i^o \,.\, x_{n+i} = 0 \qquad i = 1, 2, \ldots, m$$

and

$$x_j^o \,.\, w_{m+j} = 0 \qquad j = 1, 2, \ldots, n$$

where x_{n+1} is the *i*th slack variable in the primal problem and w_{m+j} is the *j*th surplus variable in the dual.

Thus, the theorem relates the variables of one problem to the slack or surplus variables of the other. The above relations are called 'complementary slackness' because they imply that whenever a constraint in one of the problems holds with strict inequality (so that there is *slack* in the constraint), the *complementary* dual variable vanishes.

5:7. DUALITY AND SIMPLEX METHOD

Since any L.P.P. can be solved by using simplex method, the method is applicable to both the primal as well as to its dual. The fundamental theorem of duality suggests that an optimum solution to the *associated* dual can be obtained from that of its primal and *vice versa.*

If primal is a maximization problem, then following are the set of rules that govern the derivation of the optimum solution :

Rule 1. Corresponding net evaluations of the starting *primal* variables
= Difference between the left and right sides of the dual constraints associated with the starting primal variables.

Rule 2. Negative of the corresponding net evaluations of the starting *dual* variables
= Difference between the left and right sides of the primal constraints associated with dual starting variables.

Rule 3. If the primal (dual) problem is unbounded, then the dual (primal) problem does not have any *feasible solution.*

Note. In *rule 2,* dual problem is to be solved by changing the objective from minimization to the maximization.

SAMPLE PROBLEMS

515. *Use duality to solve the following L.P.P. :*

Maximize $z = 2x_1 + x_2$ *subject to the constraints :*

$x_1 + 2x_2 \le 10,\ x_1 + x_2 \le 6,\ x_1 - x_2 \le 2,\ x_1 - 2x_2 \le 1;\ x_1, x_2 \ge 0.$

[Delhi B.Sc. (Stat.) 2006]

Solution. The dual problem of the given primal is

Minimize $z' = 10w_1 + 6w_2 + 2w_3 + w_4$ subject to the constraints :

$w_1 + w_2 + w_3 + w_4 \ge 2,\ 2w_1 + w_2 - w_3 - 2w_4 \ge 1,\ w_1, w_2, w_3, w_4 \ge 0$

Introducing surplus variables $x_1 \ge 0$, $x_2 \ge 0$ and artificial variables $A_1 \ge 0$, $A_2 \ge 0$, an initial basic feasible solution is $A_1 = 2$ and $A_2 = 1$. (The primal constraints associated with x_1, x_2, A_1 and A_2 are: $-x_1 \le 0$, $-x_2 \le 0$, $x_1 \le M$ and $x_2 \le M$).

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_8$.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$-M$	y_7	2	1	1	1	1	-1	0	1	0
$-M$	y_8	1	(2)	1	-1	-2	0	-1	0	1
	z^*	$-3M$	$-3M+10$	$-2M+6$	2	$M+1$	M	M	0	0

First Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_7$.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$-M$	y_7	3/2	0	1/2	(3/2)	2	-1	1/2	1	$-1/2$
-10	y_1	1/2	1	1/2	$-1/2$	-1	0	$-1/2$	0	1/2
	z^*	$-5-3M/2$	0	$1-M/2$	$7-3M/2$	$11-2M$	M	$5-M/2$	0	$-5+3M/2$

Second Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_1$.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-2	y_3	1	0	1/3	1	4/3	$-2/3$	1/3	2/3	$-1/3$
-10	y_1	1	1	(2/3)	0	$-1/3$	$-1/3$	$-1/3$	1/3	13
	z^*	-12	0	$-4/3$	0	5/3	14/3	8/3	$M-14/3$	$M-8/3$

Final Iteration. Optimum Solution.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-2	y_3	1/2	$-1/2$	0	1	3/2	$-1/2$	1/2	1/2	$-1/2$
-6	y_2	3/2	3/2	1	0	$-1/2$	$-1/2$	$-1/2$	1/2	1/2
	z^*	-10	2	0	0	1	4	2	$M-4$	$M-2$

Thus, an optimum feasible solution to the dual problem is

$w_1 = 0,\ w_2 = 3/2$ and $w_3 = 1/2$; min. $z' = -(-10) = 10$.

Also, the primal constraints associated with the dual variables A_1 and A_2 are $x_1 \le M$ and $x_2 \le M$.

Thus, using duality rules, the optimum solution to the primal is obtained below :

	A_1	A_2
Starting dual variables		
Corresponding $\{-(z_j - c_j)\}$	$-(M-4)$	$-(M-2)$
Left minus right sides of the primal constraint associated with starting dual variables	$x_1 - M$	$x_2 - M$

Making use of *Rule 2*, we get

$$x_1 - M = -M + 4 \text{ and } x_2 - M = -M + 2$$

$\therefore$ $$x_1 = 4 \text{ and } x_2 = 2$$

Hence $$\text{Maximum } z = \text{Minimum } z^* = 10.$$

516. *Consider the L.P.P. : Maximize* $z = 2x_1 + 4x_2 + 4x_3 - 3x_4$ *subject to the constraints :*

$$x_1 + x_2 + x_3 = 4, \; x_1 + 4x_2 + x_4 = 8, \; x_1, x_2, x_3, x_4 \geq 0.$$

By using x_3 *and* x_4 *as the starting variables, the optimum table is given by*

Basis	*Solution*	x_1	x_2	x_3	x_4
x_3	2	3/4	0	1	−1/4
x_2	2	1/4	1	0	1/4
z	16	2	0	0	3

Write the dual problem and find its solution from the optimum primal table.

[Delhi B.Sc. (Stat.) 2002]

Solution. Since, the primal problem is in standard form with non-negative variables, the dual problem is :

$$\text{Minimize } z^* = 4w_1 + 8w_3 \text{ subject to the constraints :}$$

$$w_1 + w_2 \geq 2, \; w_1 + 4w_2 \geq 4, \; w_1 \geq 4 \text{ and } w_2 \geq -3$$

$$w_1 \text{ and } w_2 \text{ unrestricted (redundant).}$$

Now, x_3 and x_4 are the starting variables in the primal, and their respective net evaluations are 0 and 3 in the optimum primal table. The dual constraints associated with x_3 and x_4 are $w_1 \geq 4$ and $w_2 \geq -3$. Therefore, we write this information as below :

Starting primal variables	x_3	x_4
Corresponding net evaluations	0	3
Left minus right sides of the dual constraint associated with primal starting variables	$w_1 - 4$	$w_2 - (-3)$

Making use of *Rule 1*, we get

$$w_1 - 4 = 0 \text{ and } w_2 - (-3) = 3, \; \textit{i.e.}, \; w_1 = 4 \text{ and } w_2 = 0$$

The optimum value of the dual objective is $\text{Min. } z^* = 16 = \text{max. } z.$

517. *Consider the following LP problem :*

$$\textit{Maximize } z = 3x_1 + 2x_2 + 5x_3 \textit{ subject to :}$$

$$x_1 + 2x_2 + x_3 \leq a_1, \; 3x_1 + 2x_3 \leq a_2, \; x_1 + 4x_2 \leq a_3$$

'*are* a_1, a_2 *and* a_3 *are constants. For specific values of* a_1, a_2 *and* a_3 *the optimal solution is*

x_1	x_2	x_3	x_4	x_5	x_6	*Solution*
4	0	0	c_1	c_2	0	1,350
b_1	1	0	1/2	−1/4	0	100
b_2	0	1	0	1/2	0	c_3
b_3	0	0	−2	1	1	20

ıd c_i*'s are constants. Determine :*

values of a_1, a_2 *and* a_3 *that yield the given optimal solution.*

values of b_1, b_2, b_3 *and* c_1, c_2, c_3 *in the optimal tableau.*

optimal dual solution.

[Delhi B.Sc. (Stat.) 1996]

The optimal table indicates that slack variables x_4, x_5 and x_6 are introduced in the three aints. They happen to be the starting primal basic variables also. Thus the optimal basis 'en by $\mathbf{B}^{-1} = [\mathbf{y}_4 \;\; \mathbf{y}_5 \;\; \mathbf{y}_6]$ from the optimal table.

2

Linear Programming Problem —Mathematical Formulation

"Resources are scarce and property of the society and their abuse is a social evil"

2.1. INTRODUCTION

Many business and economic situations are concerned with a problem of planning activity. In each case, there are limited resources at your disposal and your problem is to make such a use of these resources so as to yield the maximum production or to minimise the cost of production, or to give the maximum profit, etc. Such problems are referred to as the problems of constrained optimisation. *Linear programming* is a technique for determining an optimum schedule of interdependent activities in view of the available resources. *Programming* is just another word for 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. The word *linear* stands for indicating that all relationships involved in a particular problem are linear.

In the present chapter, some applications of linear programming problems and their mathematical formulations are discussed. The concepts are then extended to the general linear programming problem.

2.2. LINEAR PROGRAMMING PROBLEM

A Linear Programming Problem (LPP) consists of three *components*, namely the (*i*) decision variables (activities), (*ii*) the objective (goal), and (*iii*) the constraints (restrictions).

(*i*) The *decision variables* refer to the activities that are competing one another for sharing the resources available. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. All the decision variables are considered as continuous, controllable and non-negative.

(*ii*) A linear programming problem must have an objective which should be clearly identifiable and measurable in quantitative terms. It could be of profit (sales) maximisation, cost (time) minimisation, and so on. The relationship among the variables representing objective must be linear.

(*iii*) There are always certain limitations (or constraints) on the use of resources, such as labour, space, raw material, money, etc. that limit the degree to which an objective can be achieved.

Such constraints must be expressed as linear inequalities or equalities in terms of decision variables.

Basic assumptions. The following four basic assumptions are necessary for all linear programming problems:

(*i*) *Certainty.* In all LPP's, it is assumed that all the parameters, such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit decision variable must be known and fixed. In other words, this assumption means that all the coefficients in the objective function as well as in the constraints are completely known with certainty and do not change during the period of study.

(*b*) *Divisibility (or continuity)*. This implies that solution values of the decision variables and resources can take on any non-negative values, including fractional values of the decision variables. For instance, it is possible to produce 4.35 quintals of wheat or 17.35 thousand kilometers of cloth or 6.52 thousand kilolitres of milk, so these variables are divisible. But it is not possible to produce 2.6 refrigerators. Such variables are not divisible and hence are to be assigned integer values. When it is necessary to have integer variables, the integer programming problem is considered to attain the desired values.

(*c*) *Proportionality*. This requires the contribution of each decision variable in both the objective function and the constraints to be directly proportional to the value of the variable. For example, if production of one unit of a particular product uses 3 hours of a particular resource, then the production of 6 units of that product uses 3×6, *i.e.*, 18 hours of that resource.

(*d*) *Additivity*. The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum of the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision variable respectively. For example, the total profit earned by the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed by A and B must be equal to the sum of resources used for A and B individually.

2:3. MATHEMATICAL FORMULATION OF THE PROBLEM

The procedure for mathematical formulation of a linear programming problem consists of the following major steps :

Step 1. Study the given situation to find the *key decisions* to be made.

Step 2. Identify the *variables* involved and designate them by symbols x_j (j = 1, 2, ...).

Step 3. State the *feasible alternatives* which generally are : $x_j \geq 0$, for all j.

Step 4. Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables.

Step 5. Identify the objective function and express it as a linear function of the decision variables.

2:4. ILLUSTRATIONS ON MATHEMATICAL FORMULATION OF LPPs

Here are some problems from real life, which have been put in the mathematical format.

SAMPLE PROBLEMS

201. (Product Allocation Problem). *A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of clothes namely suitings, shirtings and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.*

Formulate the linear programming problem to find the product mix to maximize the profit.

Mathematical Formulation

The data of the problem is summarized below :

	Departments			Profit
	Weaving (in minutes)	*Processing (in minutes)*	*Packing (in minutes)*	*(Rs. per metre)*
Suitings	3	2	1	2
Shirtings	4	1	3	4
Woollens	3	3	3	3
Availability (minutes)	60 × 60	40 × 60	80 × 60	

Step 1. The key decision is to determine the weekly rate of production for the three types of clothes.

Step 2. Let us designate the weekly production of suitings, shirtings and woollens by x_1 metres, x_2 meters and x_3 metres respectively.

Step 3. Since it is not possible to produce negative quantities, feasible alternatives are sets of values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step 4. The constraints are the limited availability of three operational departments. One metre of suiting requires 3 minutes of weaving. The quantity being x_1 metres, the requirement for suiting alone will be $3x_1$ units. Similarly, x_2 metres of shirting and x_3 metres of woollen will require $4x_2$ and $3x_3$ minutes respectively. Thus, the total requirement of weaving will be $3x_1 + 4x_2 + 3x_3$, which should not exceed the available 3600 minutes. So, the labour constraint becomes $3x_1 + 4x_2 + 3x_3 \leq 3600$.

Similarly, the constraints for the processing department and packing departments are $2x_1 + x_2 + 3x_3 \leq 2400$ and $x_1 + 3x_2 + 3x_3 \leq 4800$ respectively.

Step 5. The objective is to maximize the total profit from sales. Assuming that whatever is produced is sold in the market, the total profit is given by the linear relation $z = 2x_1 + 4x_2 + 3x_3$.

The linear programming problem can thus be put in the following mathematical format :

Find x_1, x_2 and x_3 so as to maximize

$$z = 2x_1 + 4x_2 + 3x_3$$

subject to the constraints :

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$
$$2x_1 + x_2 + 3x_3 \leq 2400$$
$$x_1 + 3x_2 + 3x_3 \leq 4800$$
$$x_1 \geq 0, \; x_2 \geq 0 \text{ and } x_3 \geq 0.$$

202. (Product Mix Problem). *Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available :*

Machine	*8-ounce bottles*	*16-ounce bottles*
A	100/minute	40/minute
B	60/minute	75/minute

Each machine can be run 8-hours per day, 5 days per week. Profit on a 8-ounce bottle is 25 paise and on a 16-ounce bottle is 35 paise. Weekly production of the drink cannot exceed 3,00,000 ounces and the market can absorb 25,000 8-ounce bottles and 7,000 16-ounce bottles per week. The planner wishes to maximize his profit subject, of course, to all the production and marketing restrictions. Formulate this as a linear programming problem. **[Meerut M.Sc. (Math.) 1998]**

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an Linear Programming model so as to maximize total profit to the company. **[Delhi M.B.A.** (*Nov.*), **2003]**

Mathematical Formulation

Step 1. The key decision is to produce the three type of parts for the automatic washing machine.

Step 2. Decision variables : Let x_1, x_2 and x_3 = number of type *A*, *B* and *C* parts to be produced per hour, respectively.

Step 3. Feasible alternatives : $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step 4. The constraints are on the time. All the three parts are to be processed on each of the three machines. On the drilling-machine, one type *A* part consumes 1/25th of the available hour, a type *B* part consumes 1/40th, and type *C* part consumes 1/25th of an hour. Thus the drilling-machine constraint is

$$\frac{x_1}{25} + \frac{x_2}{40} + \frac{x_3}{25} \leq 1, \text{ i.e., } 0.04x_1 + 0.025x_2 + 0.04x_3 \leq 1$$

Similarly, the other two constraints are :

$$\frac{1}{25}x_1 + \frac{1}{20}x_2 + \frac{1}{20}x_3 \leq 1, \text{ i.e., } 0.04x_1 + 0.05x_2 + 0.05x_3 \leq 1 \quad \text{(Shaping-machine)}$$

and

$$\frac{1}{40}x_1 + \frac{1}{30}x_2 + \frac{1}{40}x_3 \leq 1, \text{ i.e., } 0.025x_1 + 0.033x_2 + 0.025x_3 \leq 1 \quad \text{(Polishing-machine)}$$

Step 5. Profit must allow not only for the cost of the casting but for the cost of drilling, shaping, and polishing. Since, 25 type *A* parts per hour can be run on the drilling machine at a cost of Rs. 20, then Rs. $20 \times \frac{1}{25}$ = Re. 0.80 is the drilling cost per type *A* part. Similar reasoning for shaping and polishing gives

$$\text{Profit per type } A \text{ part} = (8-5) - \left(\frac{20}{25} + \frac{30}{25} + \frac{30}{40}\right) = 3 - 2.75 = 0.25$$

$$\text{Profit for type } B \text{ part} = (10-6) - \left(\frac{20}{40} + \frac{30}{20} + \frac{30}{30}\right) = 4 - 3.00 = 1.00$$

$$\text{Profit for type } C \text{ part} = (14-10) - \left(\frac{20}{25} + \frac{30}{20} + \frac{30}{40}\right) = 4 - 3.05 = 0.95$$

The objective is to maximize the total profit from sales, *viz.*, $0.25x_1 + x_2 + 0.95x_3$.

The linear programming problem, therefore, can be put in the following mathematical format :

Maximize $z = 0.25x_1 + x_2 + 0.95x_3$

subject to the constraints :

$$0.04x_1 + 0.025x_2 + 0.04x_3 \leq 1,$$
$$0.04x_1 + 0.05x_2 + 0.05x_3 \leq 1,$$
$$0.025x_1 + 0.033x_2 + 0.025x_3 \leq 1,$$

and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

205. (Blending Problem). *The manager of an oil refinery must decide on the optimum mix of two possible blending processes of which the input and output production runs are as follows :*

Process	*Input*		*Output*	
	Crude A	*Crude B*	*Gasoline X*	*Gasoline Y*
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crudes A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 4 and Rs. 5 respectively. Formulate the problem for maximising the profit. **[Madras B.Com. 2005]**

Mathematical Formulation

Step 1. The key decision is to determine the number of units of gasoline produced from process *1* and process 2.

Step 2. Decision variables: Let x_1, x_2 = number of units of gasoline produced from process *1* and 2 respectively.

Step 3. Feasible alternatives: $x_1 \geq 0$ and $x_2 \geq 0$.

Step 4. The constraints are on the availability of crude oil and demand of crude oil, *viz.*,

$6x_1 + 5x_2 \leq 250$ (*Availability of crude A*)

$4x_1 + 6x_2 \leq 200$ (*Availability of crude B*)

$6x_1 + 5x_2 \geq 150$ (*Demand of gasoline X*)

and $9x_1 + 5x_2 \geq 130$ (*Demand of gasoline Y*)

Step 5. The objective is to maximize the total profit from the production of gasoline, *viz.*, $4x_1 + 5x_2$.

The required linear programming problem, therefore, is

Maximize $z = 4x_1 + 5x_2$

subject to the constraints :

$6x_1 + 5x_2 \leq 250,\ 4x_1 + 6x_2 \leq 200,$

$6x_1 + 5x_2 \geq 150,\ 9x_1 + 5x_2 \geq 130,$

$x_1 \geq 0$ and $x_2 \geq 0$.

206. (Production Problem). *A complete unit of a certain product consists of four units of component A and three units of component B. The two components (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively, are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components per production run and the recoulting units of each component are given below :*

Department	*Input per run (units)*		*Output per run (units)*	
	Raw material I	*Raw material II*	*Component A*	*Component B*
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulate this problem as a linear programming model so as to determine the number of production runs for each department which will maximize the total number of complete units of the final product. **[Himachal B.Tech. (Mech.) June 2007]**

Mathematical Formulation

Decision variables : Let x_1 = number of production runs for department *1*,

x_2 = number of production runs for department 2, and

x_3 = number of production runs for department 3.

The objective, therefore, is to maximize $z = 1{,}600x_1 + 3{,}000x_2 + 5{,}600x_3$ subject to the constraints :

$$15x_1 + 12x_2 + 14x_3 \le 3{,}000 \quad \textit{(Machine time restriction)}$$

$$4x_1 + 3x_2 + 5x_3 \le 1{,}200 \quad \textit{(Assembly time restriction)}$$

From the balance-sheet, we observe that the cash availability

= Cash Balance + Receivables – Co-operative bank loan paid off – Top management salary *plus* fixed overheads – Interest on long-term loan – Interest on short-term loan.

= 1,40,000 + 50,000 – 40,000 – 10,000 – 2,000 – 1,200

= 1,36,800 (Interest on long-term loan is 24% p.a. paid every month, *viz.*, Rs. 2,000 p.m.)

Thus,
$$2{,}500x_1 + 4{,}500x_2 + 9{,}000x_3 \le 1{,}36{,}800 \quad \textit{(Cash Requirement Restriction)}$$

Also,
$$x_1 \ge 2,\ x_2 \ge 0 \ \text{ and } \ x_3 \ge 8.$$

All x_j (j = 1, 2, 3) are also integral valued.

PROBLEMS

221. A firm manufactures headache pills in two sizes *A* and *B*. Size *A* contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size *B* contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP.

222. Old hens can be bought for Rs. 2.00 each and young ones cost Rs. 5.00 each. The old hens lay 3 eggs per week and the young ones, 5 eggs per week, each being worth 30 paise. A hen costs Re. 1.00 per week to feed. If I have only Rs. 80.00 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6.00 per week, assuming that I cannot house more than 20 hens? Write a mathematical model of the problem.
[Pune M.B.A. 1999; Delhi B.F.I.A. 2008]

223. An animal feed company must produce 200 lbs of a mixture containing the ingredients X_1 and X_2. X_1 costs Rs. 3 per lb. and X_2 costs Rs. 8 per lb. Not more than 80 lbs. of X_1 can be used and minimum quantity to be used for X_2 is 60 lbs. Find how much of each ingredient should be used if the company wants to minimise the cost. Formulate.

224. A factory engaged in the manufacturing of pistons, rings and valves for which the profits per unit are Rs. 10, 6 and 4 respectively wants to decide the most profitable mix. It takes one hour of preparatory work, ten hours of machining and two hours of packing and allied formalities for a piston. Corresponding time requirements for rings and valves are 1, 4 and 2 and 1, 5 and 6 hours respectively. The total number of hours available for preparatory work, machining and packing and allied formalities are 100, 600 and 300 respectively. Determine the most profitable mix, assuming that what all produced can be sold. Formulate the LPP.

225. A ship is to carry 3 types of liquid cargo—*X*, *Y* and *Z*. There are 3,000 litres of *X* available, 2,000 litres of *Y* available and 1,500 litres of *Z* available. Each litre of *X*, *Y* and *Z* sold fetches a profit of Rs. 20, Rs. 35 and Rs. 40 respectively. The ship has 3 cargo holds—*A*, *B* and *C*, of capacities 2,000, 2,500 and 3,000 litres respectively. From stability considerations, it is required that each hold be filled in the same proportion. Formulate the problem of loading the ship as a linear programming problem. State clearly all decision variables and constraints.

226. A firm produces three products *A*, *B* and *C*. It uses two types of raw materials *I* and *II* of which 5,000 and 7,500 units respectively are available. The raw material requirements per unit of the products are given below :

Raw material	*Requirement per unit of product*		
	A	*B*	*C*
I	3	4	5
II	5	3	5

The labour time for each unit of product *A* is twice that of product *B* and three times that of product *C*. The entire labour force of the firm can produce the equivalent of 3,000 units. The minimum demand of the three

$$\begin{pmatrix} [illegible] & 2 & -1 \\ [illegible] & [illegible] & 3 \\ [illegible] & [illegible] & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 2 & -2 & 4 \\ 3 & 5 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 4 & 2 \\ 5 & 1 & 1 \\ 1 & 3 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 2 \\ 5 & -1 & 4 \\ 3 & 4 & 6 \end{pmatrix},$$

[illegible] $$\begin{pmatrix} [illegible] & 2 & -1 \\ [illegible] & -5 & 6 \\ 3 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 3 & -4 & -3 \\ -1 & 6 & 10 \\ 0 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 5 & -1 & 3 \\ 4 & 1 & -1 \\ 11 & -3 & 7 \end{pmatrix}, \begin{pmatrix} 3 & -4 & -3 \\ 2 & 12 & 7 \\ 5 & -10 & -5 \end{pmatrix},$$

(*iii*) $$\begin{pmatrix} -1 & 4 & 2 \\ -7 & 7 & 19 \\ 2 & 6 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -2 & 18 & 23 \\ -1 & 8 & 11 \end{pmatrix},$$ (*iv*) $$\begin{pmatrix} 11 & -2 & -15 \\ 1 & -11 & 17 \\ 0 & -12 & -21 \end{pmatrix}.$$

003. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

004. $\mathbf{B} \neq \mathbf{C}$; therefore, the matrix factors cannot be cancelled as is done in ordinary algebra.

005. $\mathbf{A}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

006. (*a*) $x = 1$, $y = 2$, and $z = 3$, (*b*) equations are inconsistent.

007. $k = 4$.

008. Consistent ; $x = -16k + 5$, $y = k$, $z = 11k - 3$, where k is an arbitrary constant.

009. $k \neq \pm 2$.

011. Any one of $\mathbf{e}_1, \mathbf{e}_2$ or $\mathbf{e}_3$.

012. (*a*) Yes, (*b*) Yes, (*c*) Yes.

013. Open half space.

015. (*i*) Convex, (*ii*) Convex, (*iii*) Convex.

018. Extreme points : $\mathbf{x}_1 = \left(\frac{1}{2}, \frac{-7}{2}\right)$, $\mathbf{x}_2 = (-11, 8)$, $\mathbf{x}_3 = (12, 8)$; $(2, 1) = \frac{14}{23}\mathbf{x}_1 + \frac{3}{23}\mathbf{x}_2 + \frac{6}{23}\mathbf{x}_3$.

019. (*i*) $\langle A \rangle = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}$.

(*ii*) $\langle A \rangle = \{\mathbf{x} \mid \mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \,;\, 0 \leq \lambda \leq 1\}$.

020. (1, 1) is the interior point of convex hull; $(1, 1) = (0, 0) \times \frac{1}{16} + (0, 1) \times \frac{1}{2} + (1, 2) \times \frac{1}{4} + (4, 0) \times \frac{3}{16}$.

023. (*i*) $\begin{pmatrix} 1 & 4 & 0 \\ 4 & 16 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, (*ii*) $\begin{pmatrix} 2 & -3 & 0 \\ -3 & 2 & 3 \\ 1 & 3 & -5 \end{pmatrix}$

024. (*i*) $2x_1^2 + 4x_2^2 - 6x_3^2 - 6x_1x_2 + 2x_1x_3 + 4x_2x_3$.

(*ii*) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$.

025. Yes.

026. (*a*) Positive definite (*b*) Negative definite.

Chapter 2

221. Minimize $z = x_1 + x_2$ subject to : $2x_1 + x_2 \geq 12$, $5x_1 + 8x_2 \geq 74$, $x_1 + 6x_2 \geq 24$; $x_1, x_2 \geq 0$.

222. Maximize $z = 0.5x_2 - 0.1x_1$ subject to : $x_1 + x_2 \leq 20$, $0.5x_2 - 0.1x_1 \geq 6$, $2x_1 + 5x_2 \leq 80$ and $x_1 \geq 0$, $x_2 \geq 0$ where x_1 = number of old hens and x_2 = number of young hens.

223. Minimize $z = 3x_1 + 8x_2$ subject to : $x_1 + x_2 = 200,\ x_1 \le 80,\ x_2 \ge 60,\ x_1, x_2 \ge 0.$

224. Maximize $z = 10x_1 + 6x_2 + 4x_3$ subject to the constraints :
$x_1 + x_2 + x_3 \le 100,\ 10x_1 + 4x_2 + 5x_3 \le 600,\ 2x_1 + 2x_2 + 6x_3 \le 300,\ x_1, x_2, x_3 \ge 0.$

225. Maximize $z = 30(x_{1A} + x_{2A} + x_{3A}) + 35(x_{1B} + x_{2B} + x_{3B}) + 40(x_{1C} + x_{2C} + x_{3C})$ subject to the constraints :
$\Sigma x_{iA} \le 2{,}000,\ \Sigma x_{iB} \le 2{,}500,\ \Sigma x_{iC} \le 3{,}000;\ (i = 1, 2, 3);\ \Sigma x_{1j} \le 3{,}000,\ \Sigma x_{2j} \le 2{,}000,$
$\Sigma x_{3j} \le 1{,}500;\ (j = A, B, C);\ x_{ij} \ge 0\ (i = 1, 2, 3 \text{ and } j = A, B, C).$

226. Maximize $z = 50x_1 + 50x_2 + 80x_3$ subject to the constraints :
$3x_1 + 4x_2 + 5x_3 \le 5{,}000,\ 5x_1 + 3x_2 + 5x_3 \le 7{,}500,\ 6x_1 + 3x_2 + 2x_3 \le 18{,}000,\ 3x_1 = 2x_2,$
$4x_2 = 3x_3,\ x_1 \ge 600,\ x_2 \ge 650$ and $x_3 \ge 500.$

227. Maximize $z = 10x_1 + 15x_2 + 12x_3 + 20x_4$ subject to the constraints :
$1.5x_1 + x_2 + 2x_3 + 1.5x_4 \le 150,\ 3x_1 + 3x_2 + 4x_3 + 3x_4 \le 400,\ 6x_1 + 5x_2 + 6x_3 + 7x_4 \le 750,$
$3x_1 + 4x_2 + 4x_3 + 3x_4 \le 500,\ x_1, x_2, x_3, x_4 \ge 0.$

228. Maximize $z = 2.75x + 4.15y$ subject to the constraints :
$50x + 30y \le 1{,}500,\ 45x + 30y \le 1{,}350,\ 30x + 45y \le 1{,}350;\ x, y \ge 0,$
where z = Total Profit = Sale Price − Purchase cost − Running cost

$$= \left[8 - \left(3 + \frac{30}{30} + \frac{22.5}{30} + \frac{22.5}{45}\right)\right]x + \left[10 - \left(4 + \frac{30}{50} + \frac{22.5}{45} + \frac{22.5}{30}\right)\right]y$$

229. Maximize $z = 0.14x_1 + 0.19x_2 + 0.23x_3 + 0.12x_4$ subject to the constraints :
$x_1 + x_2 + x_3 + x_4 \le 20{,}00{,}000,\ x_4 \ge 2{,}00{,}000,\ -30x_1 - 18x_2 + 6x_3 - 36x_4 \le 0,$
$-0.2x_1 - 0.2x_2 + 0.8x_3 - 0.2x_4 \le 0,\ x_1, x_2, x_3, x_4 \ge 0.$

230. Maximize $z = 4{,}00{,}000x_1 + 9{,}00{,}000x_2 + 5{,}00{,}000x_3 + 2{,}00{,}000x_4$ subject to the constraints :
$40{,}000x_1 + 75{,}000x_2 + 30{,}000x_3 + 15{,}000x_4 \le 8{,}00{,}000,$
$3{,}00{,}000x_1 + 4{,}00{,}000x_2 + 2{,}00{,}000x_3 + 1{,}00{,}000x_4 \ge 20{,}00{,}000,$
$40{,}000x_1 + 75{,}000x_2 \le 5{,}00{,}000,\ x_1 \ge 3,\ x_2 \ge 2,\ 5 \le x_3 \le 10,\ 5 \le x_4 \le 10;\ x_1, x_2, x_3, x_4 \ge 0.$

231. Maximize $z = 22.50x_1 + 40x_2 + 15x_3$ subject to the constraints :
$0.01x_1 + 0.03x_2 + 0.02x_3 \le 120,\ 0.2x_1 + 0.5x_2 + 0.3x_3 \le 3{,}000,$
$x_1 + x_2 + x_3 \ge 5{,}000,\ x_1 \le 3{,}000,\ x_2 \le 2{,}000,\ x_3 \le 6{,}000;\ x_1, x_2, x_3 \ge 0.$

Maximize $z = 1{,}00{,}000x_1 + 80{,}000x_2 + 50{,}000x_3$ subject to the constraints :
$7{,}500x_1 + 6{,}250x_2 + 5{,}000x_3 \le 2{,}50{,}000,\ x_1 \le 12,\ x_2 \ge 5,\ 6 \le x_3 \le 10;\ x_1, x_2, x_3 \ge 0.$

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$ subject to the constraints :
$x_1 + x_4 + x_5 + x_6 + x_7 \ge 35,\ x_2 + x_5 + x_6 + x_7 + x_1 \ge 55,\ x_3 + x_6 + x_7 + x_1 + x_2 \ge 60,$
$x_4 + x_7 + x_1 + x_2 + x_3 \ge 50,\ x_5 + x_1 + x_2 + x_3 + x_4 \ge 60,\ x_6 + x_2 + x_3 + x_4 + x_5 \le 50,$
$x_7 + x_3 + x_4 + x_5 + x_6 \ge 45;\ 0 \le x_j \le 40\ (j = 1, 2, \ldots, 7).$

Maximize $z = 4000x_1 + 3500x_2 + (3500 - 1000)x_3 + 2000x_4$ subject to the constraints :
$0.30x_1 + 0.15x_2 + 0.15x_3 + 0.75x_4 \le 600,\ \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{5}x_4 \le 100,\ \frac{1}{3}x_3 \le 40,$
$\frac{1}{12}x_1 + \frac{1}{12}(x_2 + x_3) + \frac{1}{12}x_4 \le 80,\ x_1 \le 100,\ x_2 + x_3 \le 400,\ 200 \le x_4 \le 600,\ x_1, x_2, x_3, x_4 \ge 0.$

Maximize [illegible] $+ 50(x_{3A} + x_{3B})\}$ subject to the constraints :

3

Linear Programming Problem —Graphical Solution and Extension

"Graph is the best tool to visualise any concept"

3:1. INTRODUCTION

Linear programming problems involving two decision variables can easily be solved by *graphical method*. The method also provides an insight into the concepts of *Simplex Method*—a powerful technique to solve the linear programming problems involving three or more decision variables.

3:2. GRAPHICAL SOLUTION METHOD

The major steps in the solution of a linear programming problem by graphical method are summarised as follows :

Step 1. Identify the problem — the decision variables, the objective and the restrictions.

Step 2. Set up the mathematical formulation of the problem.

Step 3. Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step 4. The feasible region obtained in *step 3* may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step 5. Find out the value of the objective function at each corner (solution) point determined in *step 4*.

Step 6. Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the *optimum feasible solution.*

Remarks. *1.* The above method is known as *Search Approach Method.*

2. Another method known as *Iso-Profit* or *Iso-cost approach*, involves the following steps :

(*a*) First *four steps* are same as in the Search Approach. In the *fifth step* we choose a convenient profit (or cost) and draw iso-profit (iso-cost) line so that it falls within the feasible region.

(*b*) Move this iso-profit (or iso-cost) line parallel to itself farther (closer) from (to) the origin.

(*c*) Identify the optimum solution as the coordinates of that point on the feasible region touched by the highest possible iso-profit line (or lower-possible iso-cost line).

(*d*) Compute the *optimum feasible solution.*

SAMPLE PROBLEMS

301. *A company makes two kinds of leather belts. Belt A is a high quality belt, and belt B is of lower quality. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires* *twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (Both A and B combined). Belt A requires a fancy buckle and only 400 buckles per day are available. There are only 700 buckles a day available for belt B. Determine the optimal product mix.* **[Madras M.B.A. 2006; Delhi M.Com. 2005; M.B.A. (Nov.) 2009]**

Solution.

Step 1. The appropriate mathematical formulation of the given linear programming problem is :

Maximize $z = 4x_1 + 3x_2$ subject to the constraints :

$2x_1 + x_2 \leq 1{,}000$ *(Time constraint)*

$x_1 + x_2 \leq 800$ *(Availability of Leather)*

$x_1 \leq 400$ and $x_2 \leq 700$ *(Availability of Buckles)*

$x_1 \geq 0$ and $x_2 \geq 0$,

where x_1 = number of belts of type A, and x_2 = number of belts of type B.

Step 2. Next we construct the graph by considering the cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2) ; any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant.

Now, the inequalities are graphed taking them as equations, *e.g.*, the first constraint $2x_1 + x_2 \leq 1000$ will be graphed as $2x_1 + x_2 = 1000$. The equation is re-written as $\frac{x_1}{500} + \frac{x_2}{1000} = 1$.

This equation indicates that when it is plotted on the graph, it cuts an x_1-intercept of 500 and x_2-intercept of 1000. These two points are then connected by a straight line which is shown in Fig. 3.1(a) as line AB. Any point (representing a combination of x_1 and x_2) that falls on this line or in the area below it, is acceptable in so far as this constraint is concerned. The region OAB formed by two axes and the line representing the equation $2x_1 + x_2 = 1000$ is the region containing acceptable values of x_1 and x_2 in respect of this constraint.

Similarly, the constraint $x_1 + x_2 \leq 800$ can be plotted. The line CD in Fig. 3.1(b) represents the equation $x_1 + x_2 = 800$. The region OCD, formed by the two axes and this line represents the area in which any point would satisfy this constraint of leather availability. Further, the constraints $x_1 \leq 400$ and $x_2 \leq 700$ are also plotted on the graph which represents the area between the two axes and the lines $x_1 = 400$ and $x_2 = 700$ as shown in Fig. 3.1(b).

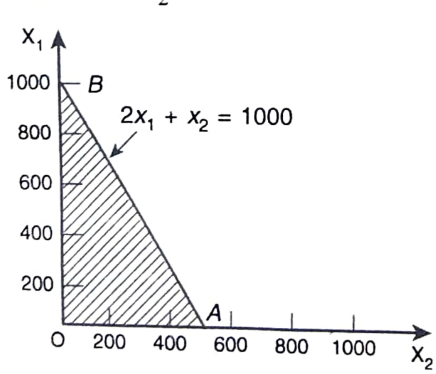

Fig. 3.1(a)

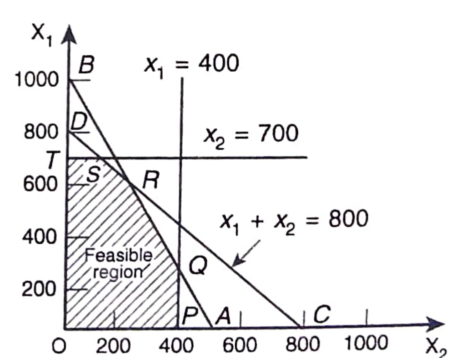

Fig. 3.1(b)

Now all the constraints have been graphed. The area bounded by all these constraints, called feasible *region* or *solution space*, is as shown in Fig. 3.1(b) by the shaded area $OPQRST$.

Step 3. The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of the extreme points are :

$O = (0, 0)$, $P = (400, 0)$, $Q = (400, 200)$, $R = (200, 600)$, $S = (100, 700)$, and $T = (0, 700)$.

Step 4. We now compute the z-values corresponding to the extreme points :

Extreme point	(x_1, x_2)	$z = 4x_1 + 3x_2$
O	(0, 0)	0
P	(400, 0)	1600
Q	(400, 200)	2200
R	(200, 600)	**2600** ← maximum
S	(100, 700)	2500
T	(0, 700)	2100

Step 5. The optimum solution is that extreme point for which the objective function has the largest value. Thus, the optimum solution occurs at the point R, *i.e.*, $x_1 = 200$ and $x_2 = 600$ with the objective function value of Rs. 2600.

Hence, to maximize profit, the company should produce 200 belts of type A and 600 belts of type B per day.

Alternative Method (Iso-profit approach)

The feasible region (solution space) obtained in *step 2* is as shown in Fig. 3.1(c) by the shaded area $OPQRST$.

Let the profit to the company (arbitrary) is Rs. 1200. The objective function then becomes :

$$4x_1 + 3x_2 = 1200.$$

We draw this equation as a straight line in the feasible region shown in Fig. 3.1(c). This line is known as *iso-profit line.*

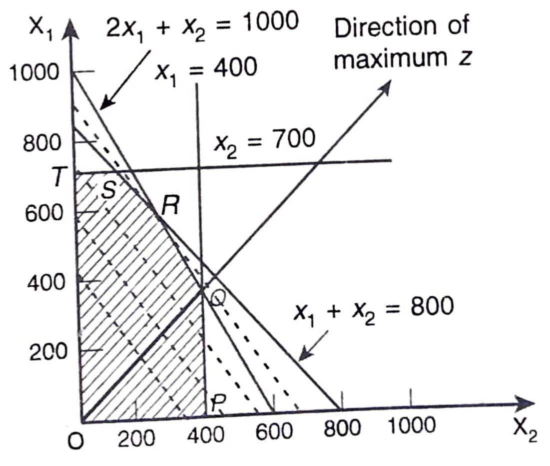

Fig. 3.1(c)

It may be noted that the iso-profit function (objective value function) is a straight line on which every point has the same total profit.

Now, we move the iso-profit line parallel to itself farther from the origin. We observe that one of the iso-profit line touches only point R before leaving the feasible region. This iso-profit line is termed as highest possible iso-profit line and point R gives the extreme point of the solution space.

Hence, the optimum feasible solution is :

$$x_1 = 200 \text{ and } x_2 = 600 \text{ with Maximum } z = \text{Rs. } 2600.$$

302. *Let us assume that you have inherited Rs. 1,00,000 from your father-in-law that can be invested in a combination of only two stock portfolios, with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the* *not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?*

Formulate this as a Linear Programming Problem and solve it by Graphic Method.

Solution. *[C.A. Final (May) 1999]*

Step 1. The appropriate mathematical formulation of the linear programming problem is :

$$\text{Maximize } z = 0.10x_1 + 0.20x_2 \text{ subject to the constraints :}$$

$$x_1 + x_2 \le 1{,}00{,}000, \quad x_1 \le 75{,}000, \quad x_2 \le 75{,}000$$

$$0.10x_1 + 0.20x_2 \ge 0.12(x_1 + x_2) \quad \text{or} \quad -0.02x_1 + 0.08x_2 \ge 0$$

$$4x_1 + 9x_2 \le 6(x_1 + x_2) \quad \text{or} \quad -2x_1 + 3x_2 \le 0$$

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0$$

where x_1 = amount invested in portfolio *1*, and x_2 = amount invested in portfolio 2.

Step 2. The first constraint $x_1 + x_2 \le 1{,}00{,}000$ can be graphed by plotting the straight line $\frac{x_1}{1{,}00{,}000} + \frac{x_2}{1{,}00{,}000} = 1$. This cuts a x_1-intercept and x_2-intercept of 1,00,000 each. The area below this line represents the feasible area in respect of this constraint. Similarly, the other constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 75{,}000$, $x_2 = 75{,}000$, $-2x_1 + 3x_2 = 0$, and $-0.02x_1 + 0.08x_2 = 0$. Here, the area below the first three lines and beyond the fourth line gives the feasible region in respect of these four constraints.

Thus, the feasible region in respect of the given problem is as shown in Fig. 3.2.

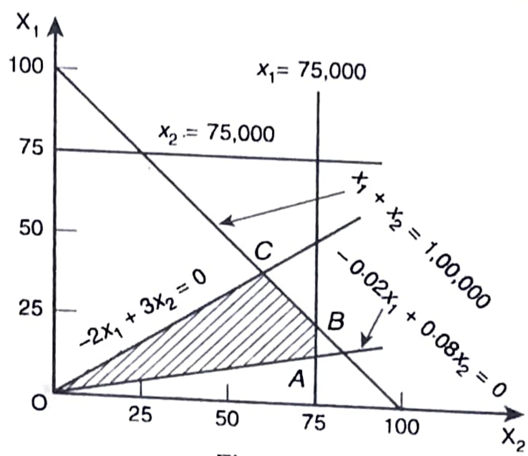

Fig. 3.2

Step 3. The coordinates of the extreme points are : O = (0, 0), A = (75,000, 18,750), B = (75,000, 25,000) and C = (60,000, 40,000).

Step 4. The z-values corresponding to the extreme points are :

Extreme point	(x_1, x_2)	$z = 0.10x_1 + 0.20x_2$
O	(0, 0)	0
A	(75,000, 18,750)	11,250
B	(75,000, 25,000)	12,500
C	(60,000, 40,000)	**14,000** ⟵ maximum

Hence, the optimum solution is :

$$x_1 = 60{,}000, \quad x_2 = 40{,}000 \quad \text{and} \quad \text{maximum return} = \text{Rs. } 14{,}000.$$

303. *A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary* *to buy two additional products, say, A and B. One unit of product A contains 36 units of X, 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs Rs. 20 per unit and product B Rs. 40 per unit.*

Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphic method. ***[C.A. Final (May) 2002]***

Solution. *Step 1.* The data of the given problem can be summarised as follows :

Nutrient constituents	*Nutrient content in product*		*Minimum amount of nutrient*
	A	*B*	
X	36	06	108
Y	03	12	36
Z	20	10	100
Cost of Product	Rs. 20	Rs. 40	

Making use of above information, the appropriate mathematical formulation of the linear programming problem is :

Minimize $z = 20x_1 + 40x_2$ subject to the constraints :

$$36x_1 + 6x_2 \geq 108,\ 3x_1 + 12x_2 \geq 36,\ 20x_1 + 10x_2 \geq 100, \text{ and } x_1, x_2 \geq 0.$$

where x_1 = number of units of product *A*, and x_2 = number of units of product *B*.

Step 2. Consider now a set of cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2), any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The constraints of the given problem are plotted as described earlier by treating them as equations :

$$36x_1 + 6x_2 = 108,\ 3x_1 + 12x_2 = 36 \text{ and } 20x_1 + 10x_2 = 100$$

or

$$\frac{x_1}{3} + \frac{x_2}{18} = 1,\ \frac{x_1}{12} + \frac{x_2}{3} \text{ and } \frac{x_1}{5} + \frac{x_2}{10} = 1.$$

The area beyond these lines represents the feasible region in respect of these constraints; any point on the straight lines or in the region above these lines would satisfy the constraints. The feasible region of the problem is as shown in Fig. 3.3.

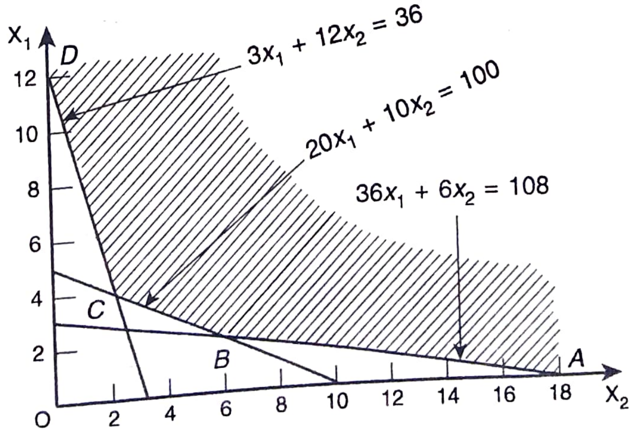

Fig. 3.3

Step 3. The coordinates of the extreme points of the feasible region are :

$$A = (0, 18),\ B = (2, 6),\ C = (4, 2) \text{ and } D = (12, 0).$$

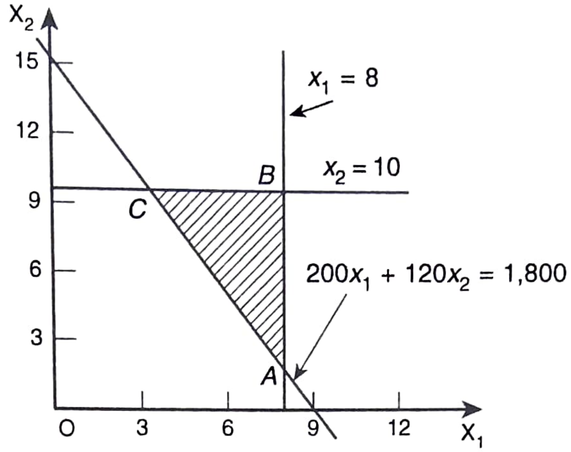

Fig. 3.4

Step 3. The coordinates of the extreme points are : $A = (8, 5/3)$, $B = (8, 10)$, and $C = (3, 10)$.

Step 4. The z-value corresponding to the extreme points are :

Extreme point	(x_1, x_2)	$z = 400x_1 + 360x_2$
A	(8, 5/3)	**3,800** ⟵ minimum
B	(8, 10)	6,800
C	(3, 10)	4,800

Hence, the optimum solution is :

$$x_1 = 8 \text{ and } x_2 = 1.7 \text{ or } 2 \text{ (approximately), minimum } z = 3{,}800$$

Thus, 8 grade *1* inspectors and 2 grade 2 inspectors should be assigned to have Rs. 3,800 as the total minimum inspection cost.

305. *Use the graphical method to solve the following LPP :*

$$\textit{Minimize } z = -x_1 + 2x_2; \textit{ subject to the constraints :}$$
$$-x_1 + 3x_2 \le 10, \quad x_1 + x_2 \le 6,$$
$$x_1 - x_2 \le 2, \quad \textit{and} \quad x_1 \ge 0, \; x_2 \ge 0.$$

Solution. The constraints are re-written in the intercept form. Thus, we write

$$\frac{x_1}{-10} + \frac{x_2}{10/3} \le 1, \quad \frac{x_1}{6} + \frac{x_2}{6} \le 1, \text{ and } \frac{x_1}{2} + \frac{x_2}{-2} \le 1$$

First, we treat these inequalities as equations and graph them as straight lines. Now, considering the inequalities, shade the feasible region for each constraint. The common region *OABCD* gives the solution space of the LPP. It may be noted that we have considered the feasible region only in the first quadrant of the cartesian axis, as $x_1 \ge 0$ and $x_2 \ge 0$.

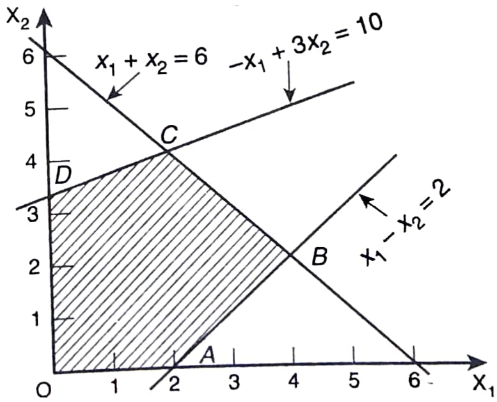

Fig. 3.5

The coordinates of the extreme points are :

$$O = (0, 0), \; A = (2, 0), \; B = (4, 2), \; C = (2, 4) \text{ and } D = \left(0, \frac{10}{3}\right).$$

The z-value corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = -x_1 + 2x_2$
O	(0, 0)	0
A	(2, 0)	**−2** ← minimum
B	(4, 2)	0
C	(2, 4)	6
D	$(0, \frac{10}{3})$	$\frac{20}{3}$

The minimum value of z occurs at the extreme point A (2, 0). Hence, the optimum solution of the LPP is :

$$x_1 = 2, \; x_2 = 0 \text{ and minimum } z = -2.$$

306. *Use the graphical method to solve the following LPP :*

Maximize $z = 2x_1 + 3x_2$; *subject to the constraints :*

$$x_1 + x_2 \leq 30, \; x_1 - x_2 \geq 0, \; x_2 \geq 3,$$
$$0 \leq x_1 \leq 20 \text{ and } 0 \leq x_2 \leq 12.$$

Solution. To graph the given inequalities, we first treat them as equations $x_1 + x_2 = 30$, $x_1 - x_2 = 0$, $x_2 = 3$, $x_1 = 20$ and $x_2 = 12$ and plot each of these equations as straight lines. We use the inequality condition of each constraint to plot the corresponding feasible region. The common region $ABCDE$ satisfying all the inequalities (feasible region) is shown in shaded Fig. 3.6.

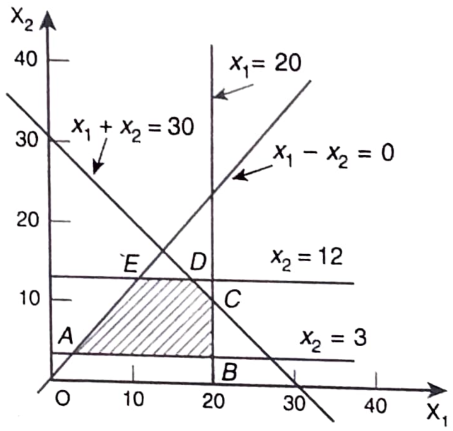

Fig. 3.6

The coordinates of the extreme points of the feasible region are :

A = (3, 3), B = (20, 3), C = (20, 10), D = (18, 12), and E = (12, 12).

The z-values corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = 2x_1 + 3x_2$
A	(3, 3)	15
B	(20, 3)	49
C	(20, 10)	70
D	(18, 12)	**72** ← maximum
E	(12, 12)	60

The maximum value of z occurs at the extreme point D (18, 12). Hence, the optimum solution is :

$$x_1 = 18, \; x_2 = 12 \text{ and maximum } z = 72.$$

307. *The advertising agency wishes to reach two types of audiences, customers with annual incomes greater than Rs. 40,000 (target audience A) and customers with annual incomes of less than Rs. 40,000 (target audience B). The total advertising budget is Rs. 2,00,000. One programme of TV advertising costs Rs. 50,000; one programme of radio advertising costs Rs. 20,000. For contract* than 4 kg. of B_1 and minimum of 2 kg. of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.

[Bangalore M.B.A. 1997]

322. Find the maximum value of $z = 50x_1 + 60x_2$ subject to the constraints :

$$2x_1 + 3x_2 \le 1500, \quad 3x_1 + 2x_2 \le 1500, \quad 0 \le x_1 \le 400, \quad 0 \le x_2 \le 400.$$

323. Maximize $z = 1.75x_1 + 1.50x_2$ subject to the constraints :

$$8x_1 + 5x_2 \le 320, \quad 4x_1 + 5x_2 \le 200, \quad 0 \le x_1 \ge 15, \quad 0 \le x_2 \ge 10.$$

324. Find the maximum value of $z = 7x_1 + 3x_2$ subject to the constraints :

$$x_1 + 2x_2 \ge 3, \quad x_1 + x_2 \le 4, \quad 0 \le x_2 \le 5/2, \quad 0 \le x_2 \le 3/2.$$

325. Show graphically that the maximum or minimum values of the objective functions for the following problem are same :

Maximize (or Minimize) $z = 5x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 6, \quad 2x_1 + 3x_2 \ge 3, \quad 0 \le x_1 \ge 3, \quad 0 \le x_2 \ge 3.$$

[Purvanchal M.C.A. 1996]

326. Find the maximum value of $z = 5x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 6, \quad 2x_1 + 3x_2 \ge 6, \quad 0 \le x_1 \le 4, \quad 0 \le x_2 \le 3.$$

[Madurai M.B.A. (DPL) 2009]

327. Solve graphically the following L.P.P. :

Maximize $z = 3x_1 + 2x_2$ subject to the constraints :

$$-2x_1 + x_2 = 1, \quad x_1 \le 2, \quad x_1 + x_2 \le 3, \quad x_1, x_2 \ge 0.$$

328. Find the minimum or/and maximum value of $z = 3x_1 + 5x_2$ subject to the constraints :

(*a*) $-3x_1 + 4x_2 \le 12, \; 2x_1 - x_2 \ge -2$
$2x_1 + 3x_2 \ge 12, \; 4 \ge x_2 \ge 0, \; 2 \ge x_2 \ge 0.$

(*b*) $x_1 \le 4, \; 2x_2 \le 6, \; 3x_1 + 2x_2 \le 18$
$x_1 + x_2 \le 9, \; x_1, x_2 \ge 0.$

3:3. SOME EXCEPTIONAL CASES

In the preceding sections, we discussed some linear programming problems which may be called 'well-behaved' problems. In each case, a solution was obtained, in some cases it took less effort while in some others it took less effort while in some others it took a little more. But a solution was finally obtained. It should not be taken as a rule. There may be an L.P.P. for which no solution exists or for which the only solution obtained is an unbounded one. Though such problems seldom occur in real situations, it will be an omission, if at this stage, the reader is not exposed to such exceptional cases.

This section considers the following three special cases that arise in the application of the graphical method :

(*i*) Alternative optima, (*ii*) Unbounded solution, (*iii*) Infeasible (or non-existing) solution.

1. **Alternative optima.** When the objective function is parallel to a binding constraint (*i.e.*, a constraint that is staisfied as an equation by the optimal solution), the objective function will assume the same optimum value at more than one solution point. For this reason they are called alternative optima. The problem given below shows that there is an infinity number of such solutions.

SAMPLE PROBLEMS

329. *Use graphical method to solve the L.P.P. :*

Maximize $z = 2x_1 + 4x_2$ *subject to the constraints :*

$$x_1 + 2x_2 \le 5, \quad x_1 + x_2 \le 4; \quad \text{and} \quad x_1, x_2 \ge 0.$$

Solution. The problem is depicted graphically in Fig. 3.8. The extreme points of the feasible region are O, A, B and C.

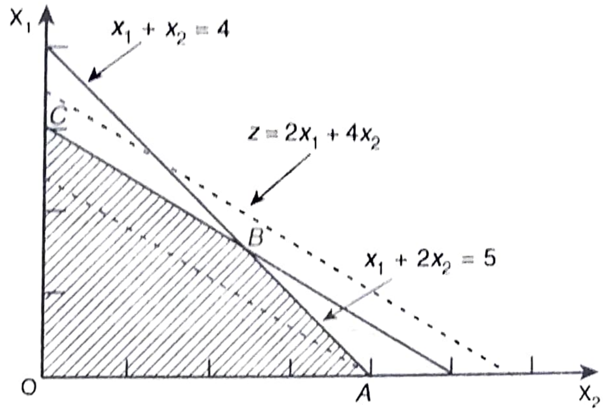

Fig. 3.8

We observe that our objective function (iso-profit line) is parallel to the line BC (or the first constraint), which forms the boundary of the feasible region. Thus as we move the iso-profit line away from the origin, it coincides with the portion BC of the constraint line which forms the boundary of the feasible region. This implies that any point including extreme points B and C on the same line between B and C is an optimal solution. Therefore, in fact an infinite number of values of x_1, x_2 give the same value of objective function.

Now, the value of objective function at each of the extreme points is evaluated as follows :

Extreme point	(x_1, x_2)	$z = 2x_1 + 4x_2$
O	(0, 0)	0
A	(4, 0)	8
B	(3, 1)	**10** ⟵ maximum
C	(0, 2.5)	**10** ⟵ maximum

Since, any point on the line segment BC gives the maximum value ($z = 10$) of the objective function, there exists an alternative optima.

2. **Unbounded solution.** When the values of the decision variables may be increased indefinitely without violating any of the constraints, the solution space (feasible region) is unbounded. The value of objective function, in such cases, may increase (for maximization) or decrease (for minimization) indefinitely. Thus, both the solution space and the objective function value are unbounded.

330. *Use graphical method to solve the following L.P.P. :*

$$\textit{Maximize } z = 6x_1 + x_2 \textit{ subject to the constraints :}$$

$$2x_1 + x_2 \geq 3, \; x_2 - x_1 \geq 0 \textit{ and } x_1, x_2 \geq 0.$$

Solution. The problem is depicted graphically in Fig. 3.9. The two extreme points of the feasible region are A and B.

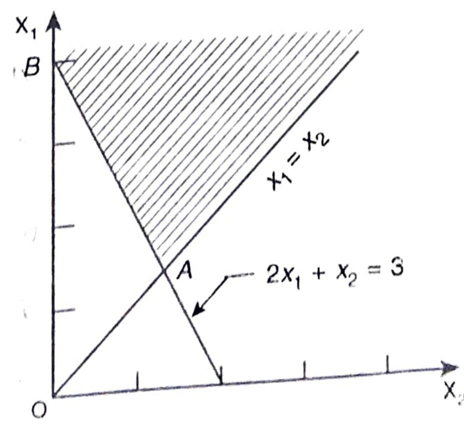

Fig. 3.9

We observe, that the feasible region (solution space) is unbounded. The value of the objective function at the extreme points $A(1, 1)$ and $B(0, 3)$ are 7 and 3 respectively.

But there exist number of points in the feasible region for which the value of the objective function is more than 7. For example, the point (3, 6) lies in the feasible region and the objective function value at this point is 24 which is more than 7. Thus both the variables x_1 and x_2 can be made arbitrarily large and the value of z also increased. Hence, the problem has an unbounded solution.

Remark. An unbounded solution means that there exist an infinite number of solutions to the given problem and the optimal value of z lies at infinity.

3. **Infeasible solution.** When the constraints are not satisfied simultaneously, the linear programming problem has no feasible solution. This situation can never occur if all the constraints are of the '$\leq$' type.

331. *Solve the following L.P.P. :*

Maximixe $z = x_1 + x_2$ *subject to the constraints :*

$$x_1 + x_2 \leq 1, \quad -3x_1 + x_2 \geq 3$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$ ***[Guru Nanak Dev Univ. B.Com. 2006]***

Solution. The problem is depicted graphically in Fig. 3.10. As shown in the figure, there is no point (x_1, x_2) which can lie in both the regions (satisfy both the constraints), there exists no solution to the given problem. Hence, there is infeasible solution.

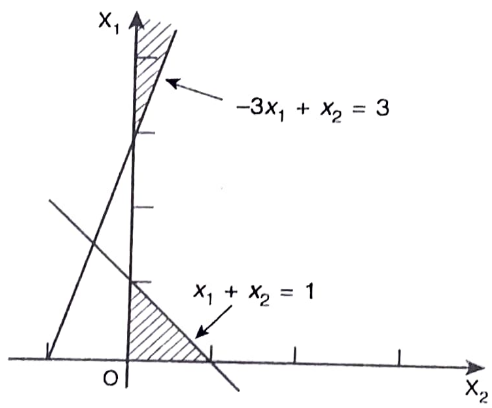

Fig. 3.10

PROBLEMS

332. Find the maximum value of $z = 10x_1 + 6x_2$ subject to the constraints :

$$5x_1 + 3x_2 \leq 30, \ x_1 + 2x_2 \leq 18 \text{ and } x_1, x_2 \geq 0.$$

333. Find the maximum value of $z = x_1 + 2x_2$ subject to the constraints :

$$x_1 - x_2 \leq 1, \ x_1 + x_2 \geq 3 \text{ and } x_1 \geq 0, \ x_2 \geq 0.$$

334. Find the maximum value of $z = 2x_1 + x_2$ subject to the constraints :

$$x_1 - x_2 \leq 10, \ x_1 \leq 20 \text{ and } x_1 \geq 0, \ x_2 \geq 0.$$

335. Find the maximum value of $z = 3x_1 + 2x_2$ subject to the constraints :

$$2x_1 + x_2 \leq 2, \ 3x_1 + 4x_2 \geq 12 ; \ x_1, x_2 \geq 0.$$

336. Find the maximum value of $z = 3x_1 + 2x_2$ subject to the constraints :

$$-2x_1 + 3x_2 \leq 9, \ x_1 - 5x_2 \geq -20 \text{ and } x_1, x_2 \geq 0.$$

[Panjab Tech. Univ. M.B.A. (Dec.) 2009]

337. Consider the following LPP :

Maximize $z = 3x_1 + 7x_2$ subject to the constraints :

$$4x_1 + 5x_2 \leq 20, \ 2x_1 + x_2 \leq 6, \ 2x_1 \geq 7, \ 2x_2 \leq 7 ; \ x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Demonstrate whether it is possible to get a solution to the above LPP. ***[Delhi M.B.A. (HCA) 2015]***

338. A company buying scrap metal has two types of scrap available to them. The first type of scrap metal has 20% of metal A, 10% of impurity and 20% of metal B by weight. The second type of scrap has 30% of metal A, 10% of impurity and 15% of metal B by weight. The company requires at least 120 kg. of metal A, at most 40 kg. of impurity and at least 90 kg. of metal B. The prices for the two scraps are Rs. 200 and Rs. 300 per kg. respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost. *[Nagpur, M.B.A. 1997]*

339. A manufacturing firm has a long history of production troubles. It produces two products A and B, which are equally profitable. Recently, the company has entered into contract to supply 40 units of A and 20 units of B per week to another company. The technology of the chemical process implies that production of A must always be at least as large as of B. There are two raw material constraints to be satisfied:

$$5A + 8B \le 400 \quad \text{and} \quad 55A + 50B \le 2{,}750.$$

Solve the problem graphically and comment on the solution obtained.

3:4. GENERAL LINEAR PROGRAMMING PROBLEM

We shall now consider the L.P.P. in the general context, that is, when the number of variables is more than two.

Definition 1 **(General Linear Programming Problem).** *Let z be a linear function on R^n defined by*

(a) $$z = c_1 x_2 + c_2 x_2 + \dots + c_n x_n$$

where c_j's are constants. Let (a_{ij}) be an $m \times n$ real matrix and $\{b_1, b_2, \dots, b_m\}$ be a set of constants such that

(b) $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ge \text{ or } \le \text{ or } = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ge \text{ or } \le \text{ or } = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ge \text{ or } \le \text{ or } = b_m \end{cases}$$

and finally let

(c) $$x_j \ge 0, \quad j = 1, 2, \dots, n.$$

The problem of determining an n-tuple $(x_1, x_2, \dots, x_n)$ which makes z a minimum (or maximum) and satisfies (b) and (c) is called the ***general linear programming problem.***

Objective function. *The linear function*

$$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

which is to be minimized (or maximized) is called the ***objective function*** *of the General L.P.P.*

Constraints. *The inequations (b) are called the* ***constraints*** *of the General L.P.P.*

Non-negative restrictions. *The set of inequations (c) is usually known as the set of* ***non-negative restrictions*** *of the General L.P.P.*

Definition 2 **(Solution).** *An n-tuple $(x_1, x_2, \dots, x_n)$ of real numbers which satisfies the constraints of a General L.P.P. is called a* ***solution*** *to the General L.P.P.*

Definition 3 **(Feasible solution).** *Any solution to a General L.P.P. which also satisfies the non-negative restrictions of the problem, is called a* ***feasible solution*** *to the General L.P.P.*

Definition 4 **(Optimum solution).** *Any feasible solution which optimizes (minimizes or maximizes) the bjective function of a General L.P.P. is called an* ***optimum solution*** *to the General L.P.P.*

Note. The term *optimal solution* is also used for optimum solution.

Example of a general L.P.P. **(Diet problem).** *Given the nutrient contents of a number of different foodstuffs and the daily minimum requirement of each nutrient for a diet, determine the balanced diet which staisfied the minimum daily requirements and at the same time has the minimum cost.*

Mathematical Formulation

Let there be n different types of foodstuffs available and m different types of nutrients required. Let a_{ij} denote the number of units of nutrient i in one unit of foodstuff j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Let x_j be the number of units of food j in the desired diet. Then, the total number of units of nutrient i in the desired diet is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$$

Let b_i be the number of units of the minimum daily requirement of nutrient i. Then, we must have

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i \qquad i = 1, 2, \ldots, m.$$

Also, each x_j must be either positive or zero. Thus, we also have

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n.$$

Finally, consider the cost. Let c_j be the cost per unit of food j. Thus, the total cost of the diet is given by

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Thus, the problem of selecting the best diet reduces to the following mathematical form :

Find an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers, such that

(*a*) $$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i \qquad i = 1, 2, \ldots, n$$

(*b*) $$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

and for which the expression (objective function)

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

may be a minimum (least).

This is a General L.P.P. It is general in the sense that the data b_i, a_{ij} and c_j are parameters, which for different sets of values will give rise to different problems.

Slack and Surplus Variables

Definition 1 (**Slack variables**). *Let the constraints of a General L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i = 1, 2, \ldots, k$$

Then, the non-negative variables x_{n+i} *which satisfy*

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \qquad i = 1, 2, \ldots, k$$

*are called **slack variables**.*

Definition 2 (**Surplus variables**). *Let the constraints of a General L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad i = k+1, k+2, \ldots, l$$

Then, the non-negative variables x_{n+i} *which satisfy*

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \qquad i = k+1, k+2, \ldots, l$$

*are called **surplus (or negative slack) variables.***

3:5. CANONICAL AND STANDARD FORMS OF L.P.P.

After the formulation of linear programming problem (L.P.P.), the next step is to obtain its solution. But for the solution of any linear programming problem, the problem must be available in a particular form. Two forms are dealt with here, the *canonical form* and the *standard form*.

The Canonical Form

The general formulation of linear programming problem discussed in the previous section can always be put in the following form :

Maximize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the constraints :

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i; \qquad i = 1, 2, \ldots, m$$

$$x_1, x_2, \ldots, x_n \ge 0$$

by making use of some elementary transformations. This form of L.P.P. *is called the* **canonical form** of L.P.P. The characteristics of this form are :

(*i*) *The objective function is of the maximization type.*

The minimization of a function, $f(x)$, is equivalent to the maximization of the negative expression of this function, $-f(x)$, *i.e.*,

$$\text{Minimize } f(x) = -\text{Maximize}\{-f(x)\}$$

For example, the linear objective function

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

is equivalent to

$$\text{Maximize } h = -z = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$$

with $z = -h$.

(*ii*) *All the constraints are of the "≤" type, except for the non-negative restrictions.*

An inequality of "≥" type can be changed to an inequality of the "≤" type by multiplying both sides of the inequality by –1.

For example, the linear constraint

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ge b_i$$

is equivalent to

$$-a_{i1}x_1 - a_{i1}x_2 - \ldots - a_{in}x_n \le -b_i$$

An equation may be replaced by two weak inequalities in opposite directions. For example,

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$$

is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i$$

and

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ge b_i.$$

(*iii*) *All the variables are non-negative.*

A variable which is *unrestricted* in sign (*i.e.*, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x_j is unrestricted in sign, it can be replaced by $(x_j' - x_j'')$, where x_j' and x_j'' are both non-negative, *i.e.*,

$$x_j = x_j' - x_j'', \qquad \text{where } x_j' \ge 0 \text{ and } x_j'' \ge 0.$$

The Standard Form

The general linear programming problem in the form

Maximize or Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the constraints :

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i, \qquad i = 1, 2, \ldots, n$$

$$x_1, x_2, \ldots, x_n \ge 0$$

is known as in **standard form.** The characteristics of this form are :

(*i*) All the constraints are expressed in the form of equations, except for the non-negative restrictions.

(*ii*) The right hand side of each constraint equation is non-negative.

The inequality constraint can be changed into equation by introducing a non-negative variable on the left hand side of such constraint. It is to be added (slack variable) if the constraint is of "$\le$" type and subtracted (surplus variable) if the constraint is of "$\ge$" type.

In matrix notation the standard form of L.P.P. can be expressed as :

Maximize or Minimize $z = \mathbf{cx}$ *(objective function)*

subject to the constraints :

$\mathbf{Ax} = \mathbf{b}$, *(constraints)*

$\mathbf{x} \ge \mathbf{0}$ *(non-negative restrictions)*

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, $\mathbf{b}^T = (b_1, b_2, \ldots, b_m)$ and $\mathbf{A} = (a_{ij})$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Remarks 1. The coefficients of slack and/or surplus variables in the objective function are always assumed to be zero, so that the conversion of the constraints to a system of simultaneous linear equations does not change the objective function under consideration.

2. The linear programming form :

$$\text{Maximize } z = \mathbf{cx} \text{ subject to the constraints : } \mathbf{Ax} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$$

is known as the **canonical form** of the L.P.P.

Theorem 2-1. *The set of feasible solutions to an L.P.P. is a convex set.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to maximize the linear function $z = \mathbf{c}^T\mathbf{x}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$.

Let $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ be two feasible solutions to this problem, so that

$$\mathbf{Ax}^{(1)} = \mathbf{b} ;\ \mathbf{Ax}^{(2)} = \mathbf{b} ;\ \mathbf{x}^{(1)} \ge \mathbf{0} \text{ and } \mathbf{x}^{(2)} \ge \mathbf{0}.$$

Now, consider convex combination of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, namely,

$$\mathbf{x} = \lambda\mathbf{x}^{(1)} + (1 - \lambda)\,\mathbf{x}^{(2)}, \quad 0 \le \lambda \le 1.$$

Clearly

$$\begin{aligned}\mathbf{Ax} &= \mathbf{A}\,[\lambda.\mathbf{x}^{(1)} + (1 - \lambda)\,\mathbf{x}^{(2)}] = \lambda.\mathbf{Ax}^{(1)} + (1 - \lambda)\,\mathbf{Ax}^{(2)} \\ &= \lambda\mathbf{b} + (1 - \lambda)\,\mathbf{b} = \mathbf{b}\end{aligned}$$

Again, since $\mathbf{x}^{(1)} \ge \mathbf{0}$, $\mathbf{x}^{(2)} \ge \mathbf{0}$, and λ, $1 - \lambda \ge 0$, $\therefore\ \mathbf{x} \ge \mathbf{0}$.

Hence, $\mathbf{x}$ is also feasible solution to the problem. Thus, the set

$$S = \{\mathbf{x} \mid \mathbf{x} \text{ is a feasible solution to an L.P.P.}\}$$

is a convex set.

Remark. In general a convex set S is either empty or unbounded or closed.

The *empty set* occurs when the constraints are not satisfied simultaneously. In this case the system yields no solution. An unbounded set implies that the region of feasible solution is not constrained in at least one direction.

A *closed set* implies that the region of feasible solution is a convex polyhedron, since it is defined by the intersections of a finite number of linear constraints.

SAMPLE PROBLEMS

340. *Rewrite in standard form the following linear programming problem :*

$$\textit{Minimize } z = 2x_1 + x_2 + 4x_3 \textit{ subject to the constraints :}$$

$$-2x_1 + 4x_2 \le 4,\ x_1 + 2x_2 + x_3 \ge 5,\ 2x_1 + 3x_3 \le 2,$$

$$x_1, x_2 \ge 0 \textit{ and } x_3 \textit{ unrestricted in sign.}$$

Solution. As the constraints are in the form of inequalities, we introduce slack variables $x_4 \ge 0$ and $x_5 \ge 0$ in the first and third inequalities and a surplus variable $x_6 \ge 0$ in the second inequality of the constraints.

Further since x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$, where $x_3' \geq 0$ and $x_3'' \geq 0$. Thus the constraints of the problem become :

$$-2x_1 + 4x_2 + x_4 = 4$$
$$x_1 + 2x_2 + (x_3' - x_3'') - x_6 = 5$$
$$2x_1 + 3(x_3' - x_3'') + x_5 = 2$$

The non-negative restrictions are :

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \text{ and } x_6 \geq 0.$$

The objective function of the problem is converted into that of maximization by multiplying it by (–1). Also we assign a cost zero to each slack/surplus variable. Thus, the revised objective function is :

$$\text{Maximize } z^* = -2x_2 - x_2 - 4(x_3' - x_3'') + 0.x_4 + 0.x_5 + 0.x_6$$

where $z^* = -z$.

Thus, the given LPP in standard form is :

Maximize $z^* = -2x_1 - x_2 - 4x_3' + 4x_3'' + 0.x_4 + 0.x_5 + 0.x_6$ subject to the constraints :

$$-2x_1 + 4x_2 + x_4 = 4,\ x_1 + 2x_2 + x_3' - x_3'' - x_6 = 5$$
$$2x_1 + 3x_3' - 3x_3'' + x_5 = 2,\ x_j \geq 0 ;\ j = 1, 2, 3', 3'', 4, 5, 6.$$

341. *Reduce the following LPP to its standard form :*

Maximize $z^* = 3x_1 + 4x_2 + 6x_3$ *subject to the constraints :*

$$2x_1 + x_2 + 2x_3 \geq 6,\ 3x_1 + 2x_2 = 8,$$
$$7x_1 - 3x_2 + 5x_3 \geq 9 ;\ x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \text{ unrestricted in sign.}$$

Solution. Since, the first and third constraints are inequalities with ≥ sign, we introduce surplus variables $x_4 \geq 0$ and $x_5 \geq 0$ in the respective inequalities.

Further since x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$, where $x_3' \geq 0$ and $x_3'' \geq 0$.

Thus, the constraints of the given LPP are :

$$2x_1 + x_2 + 2(x_3' - x_3'') + x_4 = 6,$$
$$3x_1 + 2x_2 = 8,$$
$$7x_1 - 3x_2 + 5(x_3' - x_3'') + x_5 = 9$$

The objective function will be :

$$\text{Maximize } z = 3x_1 + 4x_2 + 6(x_3' - x_3'').$$

∴ The given LPP in its standard form is :

Maximize $z = 3x_1 + 4x_2 + 6x_3' - 6x_3'' + 0 \cdot x_4 + 0 \cdot x_5$ subject to the constraints :

$$2x_1 + x_2 + 2x_3' - 2x_3'' + x_4 = 6,\ 3x_1 + 2x_2 = 8$$
$$7x_1 - 3x_2 + 5x_3' - 5x_3'' + x_5 = 9$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ x_4 \geq 0 \text{ and } x_5 \geq 0.$$

PROBLEMS

342. Reduce the following L.P.P. to its standard form :

Maximize $z = x_1 - 3x_2$ subject to the constraints :

$$-x_1 + 2x_2 \leq 15,\ x_1 + 3x_2 = 10$$

x_1 and x_2 unrestricted in sign.

343. Write down the following LPP in standard form :

Maximize $z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints :

$$2x_1 - 3x_2 \leq 3,\ x_1 + 2x_2 + 3x_3 \geq 5,\ 3x_1 + 2x_3 \leq 2$$
$$x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \geq 0.$$