UNIT - II Games & strategies.

GAME:- A competitive situation will be called a "GAME", if it has the following properties:

(i) There are finite number of competitors (participants) called players.
(ii) Each player has a finite number of strategies (alternatives) available to him.
(iii) A play of the game takes place when each player employs his strategy.
(iv) Every game results in an outcome. For example, loss or gain or draw, usually, ~~c~~alled payoff, to some player.

Two-person zero-sum game:-

When there are two competitors playing a game, it is called a "two-person game". In case the number of competitors exceeds two, say "n", then the game is termed as a "n-person game".

Games having the character that the algebraic sum of gains and losses of all players is zero are called zero-sum game. The play does not add a single paisa to the total wealth of all the players; it merely results in a new distribution of initial money among them.

Zero-Sum game with two players are called two-person Zero-Sum games. In this case, the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non-zero Sum game.

Some Basic terms of Games:-

1. **PLAYAR:-** The competitors in the game are known as players. A player may be individual, a group of individuals or an organisation.

2. STRATEGY:- A strategy for a player is defined as a set of rules or alternative course of actions available to him in advance, by which player decides the course of action that he should adopt. A strategy may be of two types:

(i) PURE STRATEGY:- If the players select the same strategy each time, then it is referred ~~as~~ to as pure-strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gain or to minimize losses.

(ii) MIXED-STRATEGY:- When ~~each~~ the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occation, then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize expected gains or to minize expected loss.

3. OPTIMUM STRATEGY:- A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

4. VALUE OF THE GAME:- It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero, and unfair if it is non-zero.

5. PAYOFF MATRIX:- when the players select their particular strategies, the payoff's (gain or loss) can be represented in the form of a matrix called the pay off matrix. Since the game is zero-sum, ∴ the gain of one player equal to the loss of other. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus it is sufficient to construct payoff only for one of the players.

Let player A have "m" strategies $A_1, A_2, \dots, A_m$ and player B have "n" strategies $B_1, B_2, \dots, B_n$. Here it is assumed that each player has his choices from amongst the pure strategies and also assume that player A is always gainer and player B is always the loser. That is, all payoff's are assumed in terms of the player A. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy A_i and player B chooses the strategy B_j.
∴ The payoff matrix for the player A is

$$\text{player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \text{player B} \\ \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

The pay off matrix to player B is $(-a_{ij})_{m \times n}$.

Example:- consider a two-person coin tossing game. Each player tosses an unbiased coin simulteneously. player B pays Rs. 7 to A if {H, H} occurs and

Rs. 4 if {T, T} occurs; otherwise player A pays Rs 3 to B. This two-person game is a zero-sum game, since the winning of one player are the losses of the other. Each player has his choices from amongst two pure strategies H and T. If we agree conventionally to express the outcome of the game interms of the payoff's to one player only, say A, then the payoff matrix interms of the pay off to player A is given below:

$$\begin{array}{ccc} & & \text{player B} \\ & & \begin{array}{cc} H & T \end{array} \\ \text{player A} & \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix} \end{array}.$$

The MAXIMIN – MINIMAX principle:-

Consider two-persons game.

For player A, minimum value in each row represents thee least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the maximin principle and the corresponding gain is called the maximin value of the game.

For the player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player B is called the minimax principle and the corresponding loss is the minimax value of the game.

SADDLE point (Equilibrium):- If the maximin value equals the minimax value, then the game is said to have saddle (Equilibrium) point and the corresponding strategies are called optimum strategies. The amount of payoff at the saddle point is known as the value of the game.

Example:-

player B

$$\begin{array}{cc|cc|c} & & B_1 & B_2 & \text{Row minimum} \\ & A_1 & 9 & 2 & 2 \\ \text{player A} & A_2 & 8 & 6 & 6 \\ & A_3 & 6 & 4 & 4 \\ \hline & \text{Column maxima} & 9 & 6 & \end{array}$$

$$\text{maximin} = \max\{2, 6, 4\} = 6.$$

$$\text{minimax} = \min\{9, 6\} = 6$$

$$\therefore \text{maximin} = 6 = \text{min max}$$

The value of the game is 6 and the optimum strategies are A_2, B_2.

Theorem:- Let (a_{ij}) be the $m \times n$ payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes maximin value and if $\bar{v}$ denotes the minimax value of the game, then $\bar{v} \geq \underline{v}$.

i.e) $$\min_{1 \leq j \leq n}\left\{\max_{1 \leq i \leq m}\{a_{ij}\}\right\} \geq \max_{1 \leq i \leq n}\left\{\min_{1 \leq j \leq m}\{a_{ij}\}\right\}.$$

proof:- For $j = 1, 2, \cdots, n$, we have $\max_{1 \le i \le m} \{a_{ij}\} \ge a_{ij} \rightarrow$ (1)

ie, $\max\{a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj}\} \ge a_{ij}$ for all $j = 1, 2, \cdots, n$.

For $i = 1, 2, \cdots, m$, we have $\min_{1 \le j \le n} \{a_{ij}\} \le a_{ij} \rightarrow$ (2)

ie) $\min\{a_{i1}, a_{i2}, \cdots, a_{in}\} \le a_{ij}$ for all $i = 1, 2, \cdots, m$

In (1) & (2), if maximum attained at $i = i'$ and minimum attained at $j = j'$

From (1) and (2), we have $a_{i'j} \ge a_{ij} \ge a_{ij'}$ for all $i = 1, 2, \cdots, m$ & $j = 1, 2, \cdots, n$

$$\min_{1 \le j \le n} \{a_{i'j}\} \ge a_{ij} \ge \max_{1 \le i \le m} \{a_{ij'}\}$$

ie) $$\min_{1 \le j \le n} \{a_{i'j}\} \ge \max_{1 \le i \le m} \{a_{ij'}\}$$

Hence $$\min_{1 \le j \le n} \left\{ \max_{1 \le i \le m} \{a_{ij}\} \right\} \ge \max_{1 \le i \le m} \left\{ \min_{1 \le j \le n} \{a_{ij}\} \right\}$$

— x —

Fair ~~and Unfair~~ game: A game is said to be fair game if maximin = minimax = 0.

A game is said to be strictly determinable if maximin = minimax = v.

Rule for determining saddle point:-

1. Select the minimum element in each row and mark it [*]
2. Select the maximum element in each column & mark it [+].
3. If there appears an element in the payoff matrix marked * and + both, the position of that element ~~element~~ is a saddle point of the payoff matrix.

Problem:-

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable:

(a) Player A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 5 & 0 \\ A_2 & 0 & 2 \end{array}$$

Player B

(b) Player A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 0 & 2 \\ A_2 & -1 & 4 \end{array}$$

Player B

Solution:-

(a)

Player B

Player A

$$\begin{array}{cc|c} & & \text{Row min} \\ 5^{+} & 0^{*} & 0 \\ 0^{*} & 2^{+} & 0 \\ \hline \text{Column max} \; 5 & 2 & \end{array}$$

MAXIMIN = max of Row min = 0 = $\underline{v}$

MINIMAX = min of Column max = 2 = $\overline{v}$

$\overline{v} \neq \underline{v}$

Game is not strictly determinable

(b)

Player B

Player A

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row min} \\ \hline A_1 & 0^{*+} & 2 & 0 \\ A_2 & -1^{*} & 4^{+} & -1 \\ \hline \text{Column max} & 0 & 4 & \end{array}$$

MAXIMIN $\underline{v} = 0$

MINIMAX $\overline{v} = 0$

$\therefore \overline{v} = \underline{v} \Rightarrow$ The game is strictly determinable and fair. The optimum strategy for A is A_1 and for B is B_1.

$S_0 = \{A_1, B_1\}$.

Teacher's Signature

2. Solve the game whose payoff matrix is given by

$$\begin{array}{c|c} & \text{player } B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{pmatrix} \end{array}$$

Solution

$$\begin{array}{cccc|c} & & \text{player } B & & \\ & B_1 & B_2 & B_3 & \text{Row min} \\ \text{player } A \quad A_1 & 1^{*+} & 3 & 1^{*+} & 1 \\ A_2 & 0 & -4^{*} & -3 & -4 \\ A_3 & 1^{+} & 5^{+} & -1^{*} & -1 \\ \text{Column max} & 1 & 5 & 1 & \end{array}$$

MAXIMIN $= \underline{V} =$ max of Row min $= 1$

MINIMAX $= \overline{V} =$ min of Column max $= 1$

$\overline{V} = \underline{V} = 1 \Rightarrow$ Game is strictly determinable & value of the game is 1.

optimum strategy for A is $\{A_1\}$

and optimum strategy for B is $\{B_1, B_3\}$.

3. Determine the range of p and q that will make the payoff element a_{22}, a saddle point for the game whose payoff matrix (a_{ij}) is given below:

$$\begin{array}{cc} & \text{player } B \\ \text{player } A & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \end{array}.$$

Solution:- Ignore p & q, determine maximin & minmax value of the payoff matrix.

	B_1	B_2	B_3	Row min
A_1	2^*	4	5	2
A_2	10^+	7^{*+}	q	7
A_3	4^*	p	8^+	4
Column max	10	7	8	

MAXMIN = $\underline{V} = 7$, MINIMAX = $\bar{V} = 7$

Thus there exists saddle point at the position (2,2).

$\therefore$ The Condition on p and q are $q \geq 7$, ~~$p \geq 4$~~ $p \leq 7$, ~~$q \leq 8$~~.

(e) The range for p & q are $p \leq 7$, $q \geq 7$.

~~$4 \leq p \leq 7$, and $7 \leq q \leq 8$.~~

Home worke :

1704, 1705, 1706, 1707, 1708
1709, 1710.

1705) consider the game G with the payoff matrix

$$\begin{array}{ccc} & B_1 & B_2 \\ A_1 & 2 & 6 \\ A_2 & -2 & \mu \end{array}$$

player B, player A

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 2 & 6 \\ -2 & \mu \end{pmatrix} \end{array}$$

a) Show that G is strictly determinable whatever μ may be

(b) Determine the value of G.

Soln:-

$$\begin{array}{cc} \begin{pmatrix} 2^{*+} & 6^{+} \\ -2^{*} & \mu \end{pmatrix} & \begin{array}{c} \text{Row min} \\ 2 \\ -2 \end{array} \\ \begin{array}{cc} 2 & 6 \end{array} & \end{array}$$

column max 2 6

Forget μ, find the rowminimum and column maximum

Maximin $\underline{v}$ = max of Row minimum = 2

Minimax $\overline{v}$ = min of column maximum = 2

$\underline{v} = \overline{v} = 2$. The Saddle point is at (1, 1) position &

this imposes no condition on μ.

$\therefore$ G is strictly determinable whatever μ may be.

value of G is $\overline{v} = \underline{v} = 2$.

1707) For what value of λ, the game with the following payoff matrix is strictly determinable?

Player B

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{pmatrix} \end{array}$$

Soln:-

Player B

$$\begin{array}{cc|ccc|c} & & B_1 & B_2 & B_3 & \text{Row min} \\ \text{Player A} & A_1 & \lambda & 6^{+} & 2^{*+} & 2 \\ & A_2 & -1^{+} & \lambda & -7^{*} & -7 \\ & A_3 & -2^{*} & 4 & \lambda & -2 \end{array}$$

Column max $\quad -1 \quad 6 \quad 2$

Ignore λ and find row minimum & column maximum

~~$\underline{v} \neq$ maximin $= 2$~~, ~~$\bar{v} \neq$ minimax~~

In first row, 2 is minimum when $\lambda \geq 2$

In third column, 2 is maximum when $\lambda \leq 2$

This imposes the condition that $\lambda = 2$.

In this case 1st column max is 2.

$\therefore \quad \underline{v} = \bar{v} = 2$. when $\lambda = 2$.

Games without saddle points - Mixed Strategies

When there is saddle point, both players must determine optimal mixture of strategies to find a saddle point. The optimal strategy mixure for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

Maximin - Minimax criterion:- consider an $m \times n$ game (a_{ij}) without saddle point, i.e) strategies are mixed. Let $p_1, p_2, \cdots, p_m$ be the probabilities with which player A will play his moves $A_1, A_2, \cdots, A_m$ respectively; let $q_1, q_2, \cdots, q_n$ be the probabilities with which player B will play ~~move~~ his moves $B_1, B_2, \cdots, B_n$ respectively.

obviously $p_1 + p_2 + \cdots + p_m = 1$ & $q_1 + q_2 + \cdots + q_n = 1$.

The expected pay-off for the player A will be given by $E(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = p^T A q$ where $p^T = (p_1 \; p_2 \cdots p_m)$, $A = (a_{ij})$ & $q^T = (q_1 \; q_2 \cdots q_n)$

For player A, $\underline{V} = \max_p \min_q E(p, q) = \max_p \left\{ \min_{1 \le j \le n} \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right\}$

i.e) $\underline{V} = \max_p \left\{ \min \left\{ \sum_{i=1}^{m} p_i a_{i1}, \sum_{i=1}^{m} p_i a_{i2}, \cdots, \sum_{i=1}^{m} p_i a_{in} \right\} \right\}$

Here $\sum_{i=1}^{m} p_i a_{ij}$ denotes the net expected gain of A when the player B will play the move B_j.

For player B, $\overline{V} = \min_q \max_p E(p, q) = \min_q \left\{ \max_{1 \le i \le m} \left\{ \sum_{j=1}^{n} q_j a_{ij} \right\} \right\}$

$\sum_{j=1}^{n} q_j a_{ij}$ denotes the net expected loss of the player B when player A will play the move A_i.

Here $\bar{V} \geq \underline{V}$ holds good in general.

Defn: A pair of strategies (p, q) for which $\bar{V} = \underline{V} = V$ is called a saddle point of $E(p, q)$ and such pair (p, q) are called optimal strategies.

Theorem:- For any 2×2 two-person zero-sum without saddle point having the payoff matrix for the player A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

, the optimum mixed strategies

$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ are determined

by $p_1 = \dfrac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$, $p_2 = 1 - p_1$, $q_1 = \dfrac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$,

$q_2 = 1 - q_1$ and the value v of the game to A is

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}.$$

Note: The above formulae are valid only for 2×2 games without saddle point.

Problem:

1) For the game with payoff matrix $\begin{array}{cc} & P_2 \\ P_1 & \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} \end{array}$, determine the optimum strategies and the value of the game.

Soln

			Row min
	5^{+}	1^{*}	1
	3^{*}	4^{+}	3
column max	5	4	

maximin $= 3 \neq 4 =$ minimax

$\therefore$ There is no saddle point.

$\therefore$ The optimal mixed strategies $S_{P_1} = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix}$

and $S_{P_2} = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix}$ are given by

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{1}{5}, \quad p_2 = 1 - p_1 = \frac{4}{5},$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{3}{5}, \quad q_2 = 1 - q_1 = \frac{2}{5}.$$

The value of the game for the player P_1 is

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{17}{5}.$$

$$\therefore S_{P_1} = \begin{bmatrix} 1 & 2 \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}, \quad S_{P_2} = \begin{bmatrix} 1 & 2 \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}.$$

2) Consider a "modified" form of "matching biased coins" game problem. The matching player is paid Rs. 8 if the two coins turn both heads and Re. 1 if the coins turn both tails. The non-matching player is paid Rs. 3 when two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?

Soln: The payoff matrix for the matching player is given by

$$\begin{array}{ccc|c} & \multicolumn{2}{c}{\text{Non-matching player}} & \\ & H & T & \text{Row min} \\ \text{Matching player} \quad H & 8^{+} & -3^{*} & -3 \\ T & -3^{*} & 1^{+} & -3 \\ \text{column max} & 8 & 1 & \end{array}$$

There is no saddle point.

The optimum mixed strategies are given by

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{4}{15}, \quad p_2 = 1 - p_1 = \frac{11}{15}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{4}{15}, \quad q_2 = 1 - q_1 = \frac{11}{15}$$

The expected value of the game is

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = -\frac{1}{15}.$$

$$S_{match} = \begin{bmatrix} H & T \\ \frac{4}{15} & \frac{11}{15} \end{bmatrix}, \quad S_{non\text{-}match} = \begin{bmatrix} H & T \\ \frac{4}{15} & \frac{11}{15} \end{bmatrix}$$

clearly we would like to be a non-matching player.

Graphical solution of $2\times n$ or $m\times 2$ games :-

consider pay off matrix $(a_{ij})_{2\times n}$ without saddle point.

$$\text{player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \overset{\text{player B}}{\begin{array}{cccc} B_1 & B_2 & \cdots & B_n \\ \hline \end{array}}$$

$$\begin{array}{c|cccc} & B_1 & B_2 & \cdots & B_n \\ \hline A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \end{array}$$

Let the optimum mixed strategy for the player A be $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$.

B's pure moves B_j	A's expected pay $E_j(p)$
B_1	$E_1(p_1) = a_{11}p_1 + a_{21}p_2 = a_{11}p_1 + a_{21}(1-p_1)$
B_2	$E_2(p_1) = a_{12}p_1 + a_{22}p_2 = a_{12}p_1 + (1-p_1)a_{22}$
$\vdots$	$\vdots$
B_n	$E_n(p_1) = a_{1n}p_1 + a_{2n}p_2 = a_{1n}p_1 + a_{2n}(1-p_1)$

According to the maximin criterion for mixed strategy games, player A should select the values of p_1 and p_2 so as to maximize his minimum expected payoffs.

This may be done by plotting the expected payoff lines: $E_j(p_1) = (a_{1j} - a_{2j})p_1 + a_{2j}$, $j = 1, 2, \cdots, n$.
The highest point on the lower envelopes of these lines will give the maximum of minimum expected payoff to the player A as also the maximum value of p_i.
The two lines passing through the maximum point identify the two critical moves of B which, combined with two of A, yield the 2x2 matrix that can be used to determine the optimum strategies of the two players for the original game.

Note: The mx2 games are also treated in the same way where the lower point on the upper envelops of the straight lines corresponding to B's expected payoff will give the maximum expected payoff to player B.

Problem:-
1) Solve the following 2x4 game graphically:

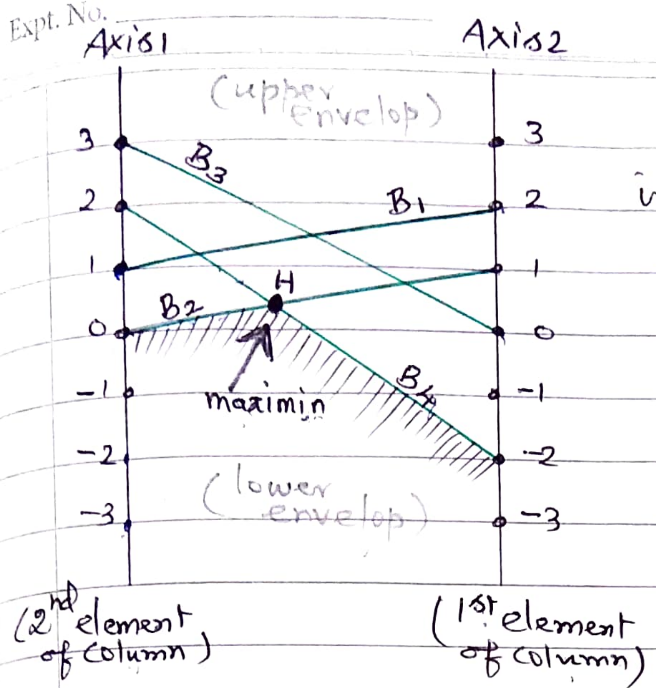
player B:

$$\text{player A}\quad \begin{array}{c c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix} \end{array}$$

Soln:- Clearly the game does not possess a saddle point.
Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$ against B.

B's pure move B_j	A's expected payoff $E_j(p_1)$
B_1	$E_1(p_1) = 2p_1 + 1p_2 = 2p_1 + (1-p_1) = p_1 + 1$
B_2	$E_2(p_1) = p_1 + 0p_2 = p_1$
B_3	$E_3(p_1) = 0p_1 + 3p_2 = -3p_1 + 3$
B_4	$E_4(p_1) = -2p_1 + 2p_2 = -2p_1 + 2(1-p_1) = -4p_1 + 2$

Expt. No.



"H" is the maximin (highest) point in the lower envelop which is intersection of two lines B_2 & B_4.
∴ player B moves only B_2 & B_4; The original 2x4 game boils down the simpler 2x2 game

$$\begin{array}{cc} & \text{player } B \\ & \begin{array}{cc} B_2 & B_4 \end{array} \\ \text{player}A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \end{array}$$

The optimum strategies for A & B are given by

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \; \& \; S_B = \begin{bmatrix} B_2 & B_4 \\ q_2 & q_4 \end{bmatrix} \quad \text{where } \begin{array}{l} p_1+p_2=1 \\ q_2+q_4=1 \end{array}$$

$$p_1 = \frac{2-0}{(1+2)-(-2+0)} = \frac{2}{5}, \quad p_2 = 1-p_1 = \frac{3}{5}$$

$$q_2 = \frac{2-(-2)}{(1+2)-(-2+0)} = \frac{4}{5}, \quad q_4 = 1-q_2 = \frac{1}{5}.$$

$$v = \frac{1\times 2 - (0\times -2)}{(1+2)-(-2+0)} = \frac{2}{5}.$$

∴ The optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$,

The optimum strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & \frac{4}{5} & 0 & \frac{2}{5} \end{bmatrix}$

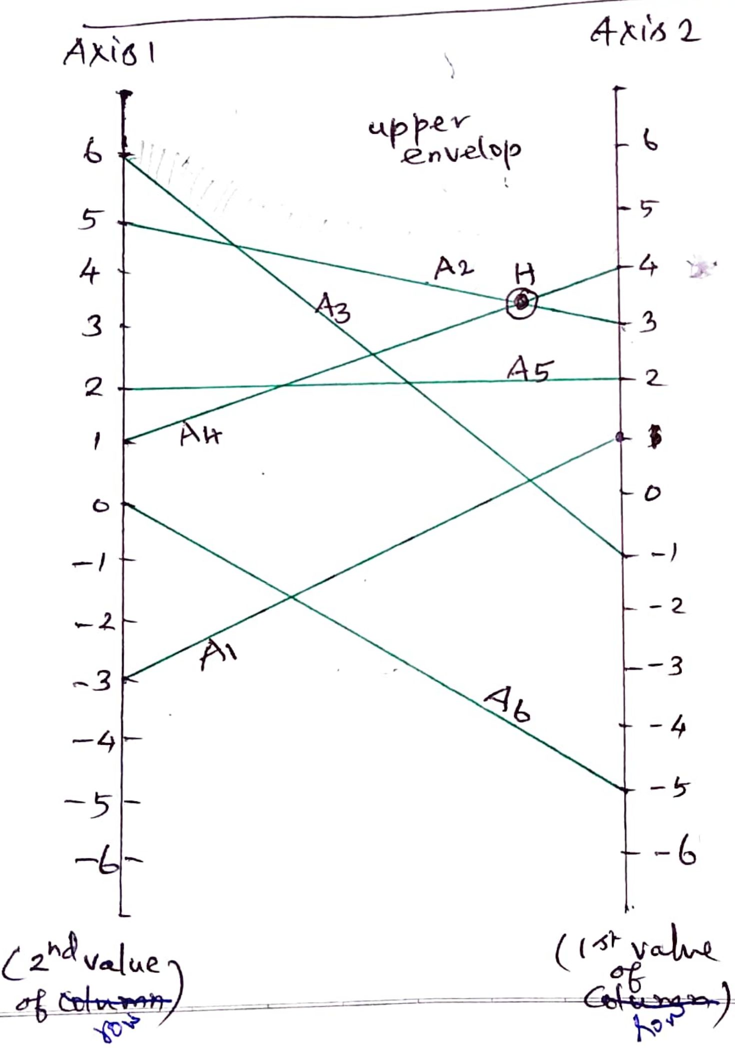
and the expected value of the game is $v = \frac{2}{5}$.

2) obtain the optimal strategies for both persons & the value of the game

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

Solution:- clearly the given problem does not posses any saddle point. So let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_1 + q_2 = 1$ against player A.

A's pure move	B's expected payoff
A_1	$E_1(q_1) = q_1 - 3q_2 = 4q_1 - 3$
A_2	$E_2(q_1) = 3q_1 + 5q_2 = -2q_1 + 5$
A_3	$E_3(q_1) = -q_1 + 6q_2 = -7q_1 + 6$
A_4	$E_4(q_1) = 4q_1 + q_2 = 3q_1 + 1$
A_5	$E_5(q_1) = 2q_1 + 2q_2 = 2$
A_6	$E_6(q_1) = -5q_1 + 0q_2 = -5q_1$



player B wishes to minimize his maximum expected payoff. "H" is the lowest point on the upper envelop, which is the intersection of the lines A_2 and A_4. So player A will play only these two move A_2 & A_4.

The solution of original 6x2 game is now reduced to 2x2 game with payoff

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 3 & 5 \\ A_4 & 4 & 1 \end{array}$$

If $S_A = \begin{bmatrix} A_2 & A_4 \\ p_2 & p_4 \end{bmatrix}$ & $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ where

$p_2 + p_4 = 1$ & $q_1 + q_2 = 1$, then

$$p_2 = \frac{1-4}{(3+1)-(5+4)} = \frac{3}{5}, \quad p_4 = 1 - p_2 = \frac{2}{5}.$$

$$q_1 = \frac{1-5}{(3+1)-(5+4)} = \frac{4}{5}, \quad q_2 = 1 - q_1 = \frac{1}{5}$$

$$V = \frac{3\times1 - 5\times4}{(3+1)-(5+4)} = \frac{17}{5}$$

optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \end{bmatrix}$

& $S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{4}{5} & \frac{1}{5} \end{bmatrix}$ and the value of the game is $V = \frac{17}{5}$.

HW 3) Solve graphically:

(a)

$$\text{player A} \; \overset{\text{player B}}{\begin{pmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{pmatrix}}$$

(b)

$$\text{player A} \; \overset{\text{player B}}{\begin{pmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{pmatrix}}$$

(c)

$$\text{player A} \; \overset{\text{player B}}{\begin{pmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{pmatrix}}$$

(d)

$$\text{player A} \; \overset{\text{player B}}{\begin{pmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{pmatrix}}$$

Dominance property:

Sometimes one of the pure strategies of either player is always inferior to atleast one of the remaining ones. The superior strategies are said to be dominate the inferior ones. Clearly, a player would have no incentive to use inferior strategies which are dominated by superior ones.

In such cases of dominance, we can reduce the size of the payoff matrix by deleting those strategies which are dominated by others.

* If each elements in the k^{th} row of the payoff matrix (a_{ij}) is less than or equal to the corresponding elements in the r^{th} row, we say that k^{th} row is dominated by r^{th} row & player A will never choose the k^{th} row. So we delete k^{th} row & $P_k = P$(choosing the k^{th} row) $= 0$.

* If all the elements of the k^{th} column is greater than or equal to the corresponding elements in the r^{th} column, then k^{th} column is dominated by r^{th} column & remove k^{th} column from the payoff matrix.

* Dominated rows or column may be deleted to to reduce the size of the payoff matrix without affecting optimal strategies.

* Modified dominance property:- If some convex liner combination of rows dominates the i^{th} row, then i^{th} row will be deleted. Similar arguments follow for columns.

Problem:- 1) Two firms are competing for business under the condition so that one firm's gain is another firm's loss. Firm A's payoff matrix is given below:

$$\begin{array}{ll} & \text{Firm B} \\ & \begin{array}{ccc} \text{no ad} & \text{medium ad} & \text{Heavy ad} \end{array} \\ \text{Firm A} \begin{array}{l} \text{No advertising} \\ \text{Medium advertising} \\ \text{Heavy advertising} \end{array} & \begin{pmatrix} 10 & 5 & -2 \\ 13 & 12 & 15 \\ 16 & 14 & 10 \end{pmatrix} \end{array}$$

Suggest optimum strategies for the two firms and net outcome thereof.

Solution:- First column is dominated by 2^{nd} column. So eliminate first column.

$$\begin{array}{c|c} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 5 & -2 \\ 12 & 15 \\ 14 & 10 \end{pmatrix} \end{array}$$

Again first row is dominated by 2^{nd} row. So eliminate 1^{st} row.

$$\begin{array}{c|c} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_2 \\ A_3 \end{array} & \begin{pmatrix} 12 & 15 \\ 14 & 10 \end{pmatrix} \end{array}$$

This reduced payoff do not have saddle point. The optimum strategies are mixed.

let $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}$, $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$

where $p_1+p_2+p_3 = 1 = q_1+q_2+q_3$.

Here $p_1=0$, $q_1=0$.

$$p_2 = \frac{10-14}{(12+10)-(15+14)} = \frac{4}{7}, \quad p_3 = 1-p_2 = \frac{3}{7}$$

$$q_2 = \frac{10-15}{-7} = \frac{5}{7}, \quad q_3 = 1-q_2 = \frac{2}{7}$$

$$v = \frac{12\times10 - 15\times14}{(12+10)-(15+14)} = \frac{90}{7}$$

$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & \frac{4}{7} & \frac{3}{7} \end{bmatrix}$, $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 5/7 & 2/7 \end{bmatrix}$

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& The expected value of the game is $V = \frac{90}{7}$.

2) Solve the following game:

player B

$$\text{player } A \quad \begin{array}{c|cccc|c} & I & II & III & IV & \\ \hline I & 3 & 2 & 4 & 0 & 0 \\ II & 3 & 4 & 2 & 4 & 2 \\ III & 4 & 2 & 4 & 0 & 0 \\ IV & 0 & 4 & 0 & 8 & 0 \\ \hline & 4 & 4 & 4 & 8 & \end{array}$$

Solution:- clearly, the game has no saddle point.

~~1st~~ 1 row is dominated by third row & ∴ remove 1st row

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline II & 3 & 4 & 2 & 4 \\ III & 4 & 2 & 4 & 0 \\ & 0 & 4 & 0 & 8 \end{array}$$

1st column is dominated by 3rd column & ∴ remove 1st column.

$$\begin{array}{c|ccc|} & II & III & IV \\ \hline II & 4 & 2 & 4 \\ III & 2 & 4 & 0 \\ IV & 4 & 0 & 8 \end{array}$$

None of the rows (columns) is dominated by other rows (columns) but combination (addition) of III & IV rows dominates II row. So remove II row.

$$\begin{array}{c|ccc|} & II & III & IV \\ \hline III & 2 & 4 & 0 \\ IV & 4 & 0 & 8 \end{array}$$

Now 2^{nd} column is dominated by the average of 3^{rd} & 4^{th} column. So remove 2^{nd} column.

$$\begin{array}{c|cc} & \text{III} & \text{IV} \\ \hline \text{III} & 4 & 0 \\ \text{IV} & 0 & 8 \end{array}$$

Let $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & p_3 & p_4 \end{bmatrix}$, $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_3 & q_4 \end{bmatrix}$

where $p_3 + p_4 = 1$, $q_3 + q_4 = 1$

$\therefore \quad p_3 = \dfrac{8-0}{(4+8)-(0+0)} = \dfrac{2}{3}, \quad p_4 = 1 - p_3 = \dfrac{1}{3}$

$q_3 = \dfrac{8-0}{(4+8)-(0+0)} = \dfrac{2}{3}, \quad q_4 = 1 - q_3 = \dfrac{1}{3}$

$V = \dfrac{4\times 8 - 0\times 0}{(4+8)-(0+0)} = \dfrac{32}{12} = \dfrac{8}{3}$

The optimum strategies are $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

& the expected value of the game is $V = \dfrac{8}{3}$.

HW

3) Solve the game whose payoff matrix is given below:

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 4 & -2 & 3 & -1 \\ A_2 & -1 & 2 & 0 & 1 \\ A_3 & -2 & 1 & -2 & 0 \end{array}$$

HW

4) page No: 461 & 462.

II UNIT END