Differential Equations and Laplace Transforms

UNIT – I : ordinary differential Equations:

Exact differential equations – Equations of the first order, but of higher degree.
(Chap : 1 – sec : 3 to 7)

UNIT – II : ordinary differential Equations (cont.).

Linear differential equations with constant coefficients – special Methods of finding particular integral – Linear equations with variable coefficients, equations reducible to the linear homogeneous equats – variation of Parameters.

(chap : 2 . sec : 2 to 4, 8 to 10)

UNIT – III : ordinary differential Equations (cont..)

Simultaneous equations of the first order and first degree – Methods for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ – simultaneous linear differential equations with constant coefficients - Total differential equations.

(chap: 3 - sec: 1 to 7)

UNIT: IV : Partial differential equations:

Derivation of Partial differential equations by elimination of arbitrary constants and arbitrary function - Different integrals of partial differential equations - standard types of first order equations - Lagrange's equations.

(chap: 4 - sec: 1 to 6)

UNIT - V : Laplace Transforms:

Definition - Laplace Transforms of standard functions - Some general theorems - Inverse Laplace Transforms - Applications to first order and second order equations with constant coefficients and simultaneous linear differential equations.

Text Book:

Calculus Vol - III - S. Narayanan and T.K. Manicavachagom Pillay, S. Viswanathan Printers, 2013.

UNIT - I

Ordinary Differential Equations

A differential equation is an equation in which differential coefficients occur.

Differential equations are two types:

(i) ordinary differential equations

(ii) Partial differential equations.

Ordinary differential equations:

An ordinary differential equation is one in which a single independent variable enters, either explicitly or implicitly.

Ex: $\cancel{x\frac{\partial z}{\partial x}+y}\ \frac{dy}{dx} = 2\sin x$;

$$x^2\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + y = \sin x.$$

Partial differential equations:

A partial differential equation is one in which at least two (two (or) More) independent variables occur.

Ex:- $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z.$

Order:

The order of an ordinary differential equation is the order of the highest derivative occuring in it.

Degree:

If a differential equation can be expressed as a polynomial equation in the derivatives, the degree of the highest derivative, when the differential coefficients are cleared of radicals and fractions is called the degree of the equation.

Ex: $\left[1 + \left(\frac{dy}{dx}\right)\right]^{3/2} = a \frac{d^2y}{dx^2}$

$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)\right]^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2.$

$\therefore$ order $= 2$ //

degree $= 2$. //

Solution or integral:

A solution or integral of a differential equation is a relation that exists between the variables by means of which and the derivatives obtained therefrom, the equation is satisfied. This solution is also called the primitive of the differential equation.

Sec : 3 : Exact Differential equations:

An exact differential equation is obtained by equating an exact or perfect differential to zero.

Consider the differential equation,

$$\boxed{M dx + N dy = 0} \quad \text{— ①.}$$

If this above equation is exact, $M dx + N dy$ must have been obtained by derivating some function $u(x, y)$.

$$\therefore \boxed{du = M dx + N dy} \quad \text{— ②.}$$

But,

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy} \quad \text{— ③.}$$

comparing ② and ③,

$$\boxed{M = \frac{\partial u}{\partial x} \quad , \quad N = \frac{\partial u}{\partial y}}$$

∴ The necessary conditions for the given equation ① to be exact are,

$$\boxed{M = \frac{\partial u}{\partial x} \; ; \; N = \frac{\partial u}{\partial y}}$$

Also,

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}.$$

We know that,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

It is the criterion for $Mdx + Mdy = 0$ to exact:

3.2 : This condition is also ~~suffi~~ sufficient :

Sufficient condition for the given equation ①, is,

$$u = \int M dx + \int \left(N - \int \frac{\partial M}{\partial y} dx \right) dy + c.$$

ie.,
$$\boxed{u = \int M dx + \int N dy + c}$$

(Taking y as a constant) (Terms not containing x)

3.3 : Practical rule for solving an exact differential equation :

Integrate Mdx as if y were constant and those terms in Ndy that do not give the terms already occuring. The sum of these integrals equated to a constant gives the solution.

① Solve $(a^2-2xy-y^2)dx-(x+y)^2dy=0$.

Soln:

Given equation is,

$$(a^2-2xy-y^2)\,dx-(x+y)^2\,dy=0 \quad — ①$$

here,

$M=a^2-2xy-y^2$

$$\boxed{\frac{\partial M}{\partial y}=-2x-2y}$$

$N=-(x+y)^2$

$N=-(x^2+y^2+2xy)$

$\frac{\partial N}{\partial x}=-(2x+2y)$

$$\boxed{\frac{\partial N}{\partial x}=-2x-2y}$$

$$\therefore \boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

$\therefore$ The given equation ① is exact.

The solution of ① is,

$$\int M\,dx+\int N\,dy=c.$$

(Taking y as a constant) (Terms not containing x)

$$\int(a^2-2xy-y^2)\,dx+\int(-y^2)\,dy=c$$

$$a^2x - \frac{2x^2y}{2} - xy^2 - \frac{y^3}{3} = c$$

$$\boxed{a^2x - x^2y - xy^2 - \frac{y^3}{3} = c.}$$

which is the required solution.

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② Solve : $(2x^2y + 4x^3 - 12xy^2 + 3y^3 - xe^y + e^{2x})\,dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx = 0.$

Soln:

Given equation is,

$$(2x^2y + 4x^3 - 12xy^2 + 3y^3 - xe^y + e^{2x})\,dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx = 0 \quad \text{— ①.}$$

$$M = 12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y$$

$$\boxed{\frac{\partial M}{\partial y} = 12x^2 + 4xy - 12y^2 + 2e^{2x} - e^y} \quad \text{— ②.}$$

$$N = 2x^2y + 4x^3 - 12xy^2 + 3y^3 - xe^y + e^{2x}$$

$$\boxed{\frac{\partial N}{\partial x} = 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}} \quad \text{— ③}$$

From ② and ③,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

The given equation ① is exact. The solution of ① is,

$$\therefore \boxed{\int M\,dx + \int N\,dy = c}$$

[taking y as a constant] + [terms Not containing in x] = c

$$\int [12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y]\,dx + \int (3y^3)\,dy = c$$

$$\frac{12x^3y}{3} + \frac{2x^2y^2}{2} + \frac{4x^4}{4} - 4xy^3 + \frac{2ye^{2x}}{2} - xe^y + \frac{3y^4}{4} = c$$

$$\boxed{4x^3y + x^2y^2 + x^4 - 4xy^3 + ye^{2x} - xe^y + \frac{3y^4}{4} = c}$$

Section : 4

An integrating factor is a function by which an ordinary differential equation can be multiplied in order to make it integrable. There are definite rules for finding integrating factors. occasionally, we can guess the integrating factors.

Ex:

$\frac{1}{xy}$, $\frac{1}{x^2}$, $\frac{1}{y^2}$ are integrating factors of the equation $ydx - xdy = 0$. The following integrable combinations will help us to find the integrating factors:

(i) $d(xy) = ydx + xdy$.

(ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(iv) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2+y^2}$

(v) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2+y^2}$

(vi) $d\left(\log\left(\frac{x+y}{x-y}\right)\right) = 2\left(\frac{xdy - ydx}{(x^2-y^2)}\right)$

Problems:

1. Solve : $a(xdy + 2ydx) = xydy$.

Soln:

Given equation is,

$$axdy + 2aydx = xydy$$

$$2ay\,dx + ax\,dy - xy\,dy = 0$$

$$2ay\,dx + (ax - xy)\,dy = 0 \quad \text{--- (1).}$$

here,

$M = 2ay$, $\quad N = ax - xy$,

$\frac{\partial M}{\partial y} = 2a \quad \frac{\partial N}{\partial x} = a - y$

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

$\therefore$ The given equation is not exact.

dividing equation (1) by xy,

$$\frac{2ay}{xy}\,dx + \frac{1}{xy} \cdot ax\,dy - \frac{1}{xy} \cdot xy\,dy = 0$$

$$\frac{2a}{x}\,dx + \left(\frac{a}{y} - 1\right) dy = 0 \quad \text{--- (2).}$$

$\therefore \quad M = \frac{2a}{x} \quad N = \frac{a}{y} - 1$

$\frac{\partial M}{\partial y} = 0 \quad \frac{\partial N}{\partial x} = 0$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\therefore$ The given equation is exact.

If we multiply the integrating factor $\frac{1}{xy}$

$\therefore$ The solution of ① is,

$$\boxed{\int M dx + \int N dy = c}$$

(y as a constant) (terms Not containing x)

$$\int \frac{2a}{x} dx + \int \left(\frac{a}{y} - 1\right) dy = c$$

$$2a \int \frac{dx}{x} + a \int \frac{dy}{y} - \int dy = c$$

$$2a \log x + a \log y - y = c$$

$$a \log x^2 + a \log y - y = c$$

$$a(\log x^2 + \log y) = y + c$$

[log a + log b = log ab]

$$\boxed{a \log(x^2 y) = y + c}$$

② Solve : $(y^2 + 2x^2 y)dx + (2x^3 - xy)dy = 0$

Soln:

Given equation is,

$$(y^2 + 2x^2 y) dx + (2x^3 - xy) dy = 0 \quad \text{—①}$$

here, $M = y^2 + 2x^2 y$ | $N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y - 2x^2$ | $\frac{\partial N}{\partial x} = 6x^2 - y$.

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

$\therefore$ The given equation ① is not exact.

$\therefore$ we shall assume an integrating factor

$x^m y^n$ for suitably chosen values of m and n.

$\times\ x^m y^n \Rightarrow$

$$x^m y^n (y^2 + 2x^2 y)\,dx + x^m y^n (2x^3 - xy)\,dy = 0.$$

$$(x^m y^{n+2} + 2x^{m+2} y^{n+1})\,dx + (2x^{m+3} y^n - x^{m+1} y^{n+1})\,dy = 0 \quad \text{②}.$$

is exact.

Applying the criterion $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and

simplifying,

$$M = x^m y^{n+2} + 2x^{m+2} y^{n+1}$$

$$\boxed{\frac{\partial M}{\partial y} = (n+2)x^m y^{n+1} + 2(n+1)x^{m+2} y^n}.$$

$$N = 2x^{m+3} y^n - x^{m+1} y^{n+1}$$

$$\boxed{\frac{\partial N}{\partial x} = 2(m+3)x^{m+2} y^n - (m+1)x^m y^{n+1}}.$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$(n+2)x^m y^{n+1} + 2(n+1)x^{m+2} y^n = 2(m+3)x^{m+2} y^n - (m+1)x^m \cdot y^{n+1}$$

$\div x^m y^n$, $\Rightarrow$

$$(n+2)y + 2(n+1)x^2 = 2(m+3)x^2 - (m+1)y$$

$$(n+2)y + 2(n+1)x^2 - 2(m+3)x^2 + (m+1)y = 0$$

$$[m+n+3]y + [2n+2-2m-6]x^2 = 0.$$

$$[m+n+3]y + [2n-2m-4]x^2 = 0$$

$$\boxed{[m+n+3]y + 2[n-m-2]x^2 = 0.}$$

As this is to vanish identically,

$n+m+3 = 0$ — ③

$2(n-m-2) = 0$

$n-m-2 = 0$ — ④.

③ + ④ $\Rightarrow$ $2n+1 = 0$

$$\boxed{n = -1/2}$$

Put $n = -1/2$ in ② $\Rightarrow$ $-\frac{1}{2} + m + 3 = 0$

$m + 5/2 = 0$

$$\boxed{m = \frac{-5}{2}}$$

$\therefore$ Thus the integrating factor is,

$$x^{-5/2} y^{-1/2}$$

$\therefore$ equation ② implies,

$$[x^{-5/2} y^{-1/2+2} + 2x^{-5/2+2} y^{-1/2+1}]dx +$$

$$[2x^{-5/2+3} y^{-1/2} - x^{-5/2+1} y^{-1/2+1}]dy = 0.$$

$$[x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2}]dx + [2x^{1/2} y^{-1/2} - x^{-3/2} y^{1/2}]dy = 0 \longrightarrow ⑤.$$

The solution of equation ⑤ is,

$$\int M dx + \int N dy = c$$

(Taking y as a constant) (Terms not containing x)

$$\int (x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2}) dx + \int (0) dy = c$$

$$\frac{x^{-5/2+1}}{-5/2+1} \cdot y^{3/2} + \frac{2x^{-1/2+1}}{-1/2+1} \cdot y^{1/2} = c$$

$$-\frac{2}{3}(x^{-3/2} y^{3/2}) + 4x^{1/2} y^{1/2} = c$$

$$-2x^{-3/2} y^{3/2} + 12x^{1/2} y^{1/2} = c$$

$$\boxed{6\sqrt{xy} = x^{-3/2} y^{3/2} + c.}$$

(3). Solve : $(y^2e^x + 2xy)dx - x^2dy = 0$.

Soln:

Given equation is,

$$(y^2e^x + 2xy)\,dx - x^2dy = 0 \quad \text{— (1)}$$

$M = y^2e^x + 2xy$ | $N = -x^2$

$$\boxed{\frac{\partial M}{\partial y} = 2ye^x + 2x} \qquad \boxed{\frac{\partial N}{\partial x} = -2x}$$

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

The given equation (1) is not exact.

diving equation (1) by y^2,

$$\left(\frac{(y^2e^x + 2xy)}{y^2}\right)dx - \frac{x^2}{y^2}dy = 0$$

$$\left(e^x + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}dy = 0 \quad \text{— (2)}.$$

NOW,

$M = e^x + \frac{2x}{y}$ | $N = -\frac{x^2}{y^2}$

$$\boxed{\frac{\partial M}{\partial y} = \frac{-2x}{y^2}} \qquad \boxed{\frac{\partial N}{\partial x} = -\frac{2x}{y^2}}$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

The solution of ② is,

$$\int M\,dx + \int N\,dy = c.$$

(Taking y as a constant) (Terms not containing x)

$$\int \left(e^x + \frac{2x}{y}\right) dx + \int (0)\, dy = c$$

$$e^x + \frac{2x^2}{2y} = c$$

$$\boxed{\therefore e^x + \frac{x^2}{y} = c.}$$

Sec: 4 : Rules for finding integrating factors:

1) When $Mx + Ny \neq 0$, and the equation is homogeneous; $\frac{1}{Mx+Ny}$ is an integrating factor of $M\,dx + N\,dy = 0$.

2). When $Mx - Ny \neq 0$, and the equation is of the ~~fro~~ form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$; $\frac{1}{Mx-Ny}$ is an integrating factor.

3). If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alonge say $f(x)$, then the integrating factor is $e^{\int f(x)\,dx}$.

4). If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y alone say $f(y)$, then the integrating factor is $e^{\int f(y)\,dy}$.

Homogeneous Equations:

Consider $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$, where f_1 and f_2 are homogeneous functions of the same degree in x and y.

①. Solve : $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0$.

Soln:

Given equation is,

$$(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0 \quad \text{——} ①$$

$$\frac{dy}{dx}=\frac{x^2y-2xy^2}{x^3-3x^2y}$$

This is a homogeneous equation of degree 3.

$M = x^2y - 2xy^2$ | $N = -(x^3 - 3x^2y)$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ | $\frac{\partial N}{\partial x} = -(3x^2 - 6xy)$

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

The given equation ① is not exact and

given equation is homogeneous and,

$$Mx + Ny = (x^2y - 2xy^2)x + (x^3 - 3x^2y)y$$

$$= x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$

$$\boxed{Mx + Ny = x^2y^2}$$

$$\boxed{Mx + Ny \neq 0.}$$

$\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ is an integrating factor.

Multiplying ① by $\frac{1}{x^2y^2}$, we get,

$$\left(\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3}{x^2y^2} - \frac{3x^2y}{x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{—— ②}$$

here,

$M = \frac{1}{y} - \frac{2}{x}$ | $N = \frac{3}{y} - \frac{x}{y^2}$

$\boxed{\frac{\partial M}{\partial y} = \frac{-1}{y^2}}$ | $\boxed{\frac{\partial N}{\partial x} = -1/y^2}$

$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\therefore$ equation ② is exact.

The Solution of ② is,

$\int M dx + \int N dy = c$

$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \left(\frac{3}{y}\right) dy = c.$

$\frac{x}{y} - 2 \log x + 3 \log y = c$

$\frac{x}{y} - \log x^2 + \log y^3 = c$

[$\log a - \log b = \log\left(\frac{a}{b}\right)$]

$\frac{x}{y} + \log y^3 - \log x^2 = 0$

$$\boxed{\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c.}$$

(2). Solve: $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0.$

soln:

Given equation is,

$$y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0 \quad \text{--- (1).}$$

The equation (1) is of the form,

$$y f_1(xy)\,dx + x f_2(xy)\,dy = 0.$$

$M = xy^2 + 2x^2y^3$ | $N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$ | $\frac{\partial N}{\partial y} = 2xy - 3x^2y^2.$

$$\boxed{\therefore \frac{\partial N}{\partial y} \neq \frac{\partial N}{\partial x}}$$

equation (1) is not exact.

$$Mx - Ny = (xy^2 + 2x^2y^2)x - (x^2y - x^3y^2)y$$

$$= \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3$$

$$= 2x^3y^3 + x^3y^3$$

$$Mx - Ny = 3x^3y^3$$

$$\boxed{Mx - Ny = 3x^3y^3 \neq 0.}$$

$\frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$ is an integrating factor.

$\therefore$ Multiplying $\frac{1}{3x^3y^3}$ by ①, we get,

$$\frac{y}{3x^3y^3}(xy + 2x^2y^2)dx + \frac{x}{3x^3y^3}(xy - x^2y^2)dy = 0.$$

$$\frac{1}{3x^3y^2}(xy + 2x^2y^2)dx + \frac{1}{3x^2y^3}(xy - x^2y^2)dy = 0.$$

$$\left(\frac{xy}{3x^3y^2} + \frac{2x^2y^2}{3x^3y^2}\right)dx + \left(\frac{xy}{3x^2y^3} - \frac{x^2y^2}{3x^2y^3}\right)dy = 0.$$

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \quad \text{—②}.$$

here,

$M = \frac{1}{2x^2y} + \frac{2}{3x}$ | $N = \frac{1}{3xy^2} - \frac{1}{3y}$

$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2}$ | $\frac{\partial N}{\partial x} = \frac{-1}{3x^2y^2}$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\therefore$ equation ② is exact.

The solution of ② is,

$$\int M\,dx + \int N\,dy = c$$

(Taking y as a constant) (Terms not containing x)

$$\int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \int -\frac{1}{3y}\cdot dy = c$$

$$\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c.$$

$$-\frac{1}{xy} + 2\log x - \log y = c$$

$$\log x^2 - \log y - \frac{1}{xy} = c$$

[$\log a - \log b = \log\left(\frac{a}{b}\right)$]

$$\boxed{\log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = c.}$$

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③. Solve : $(y-3x^2)\,dx - x(1-xy^2)\,dy = 0$

Soln:

Given equation is,

$$(y-3x^2)\,dx - x(1-xy^2)\,dy = 0 \quad \text{—①.}$$

here,

$M = y - 3x^2$ | $N = -x(1-xy^2)$

$\frac{\partial M}{\partial y} = 1$ | $\frac{\partial N}{\partial x} = -1+2xy^2$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.}$$

The given equation ① is not exact.

Dividing equation ① by x^2,

$$\frac{1}{x^2}(y-3x^2)\,dx - \frac{x}{x^2}(1-xy^2)\,dy = 0.$$

$$\left(\frac{y}{x^2}-3\right)dx - \left(\frac{1}{x}-y^2\right)dy = 0.$$

$$\frac{y}{x^2}\,dx - 3dx - \frac{1}{x}\,dy + y^2\,dy = 0.$$

$$\frac{y\,dx - x\,dy}{x^2} - 3dx + y^2dy = 0$$

$$d\left(\frac{y}{x}\right) - 3dx + y^2dy = 0.$$

$$\boxed{\frac{y}{x} - 3x + \frac{y^3}{3} = c.}$$

4. Solve : $\frac{dy}{dx} = \frac{2x}{x^2+y^2-2y}$

Soln:

Given equation is,

$$(x^2+y^2-2y)\,dy = 2x\,dx.$$

$$(x^2+y^2)\,dy - 2y\,dy = 2x\,dx$$

$$(x^2+y^2)\,dy = d(x^2+y^2)$$

$d(x^2) = 2x\,dx$

$d(y^2) = 2y\,dy.$

$$dy = \frac{d(x^2+y^2)}{(x^2+y^2)}$$

integrating on both sides, we get,

$$\int dy = \int \frac{d(x^2+y^2)}{(x^2+y^2)}$$

$$\boxed{y = \log(x^2+y^2)+c.}$$

5. Solve: $(1+xy^2)\,dx + (1+x^2y)\,dy = 0.$

Soln:

Given equation is,

$$(1+xy^2)\,dx + (1+x^2y)\,dy = 0 \quad \text{—①}$$

$$dx + xy^2\,dx + dy + x^2y\,dy = 0$$

$$dx + dy + xy^2\,dx + x^2y\,dy = 0$$

$$d(x+y) + \frac{1}{2}d(x^2y^2) = 0.$$

$d(x^2y^2) = x^2 2y\,dy + 2xy^2\,dx = 2(x^2y\,dy + xy^2\,dx).$

integrating we get,

$$(x+y) + \frac{1}{2}(x^2y^2) = c$$

The solution of ① is,

$$\boxed{x+y+\frac{1}{2}(x^2y^2) = c.}$$

⑥. Solve : $(x^2+y^2)(xdx+ydy) = a^2(xdy-ydx)$

soln:

Given equation is,

$$(x^2+y^2)(xdx+ydy) = a^2(xdy-ydx) \quad — ①.$$

$$xdx+ydy = a^2\left(\frac{xdy-ydx}{x^2+y^2}\right)$$

$$\frac{1}{2}d(x^2+y^2) = a^2\, d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

Integrating on both sides, we get,

$$\boxed{\frac{1}{2}(x^2+y^2) = a^2\tan^{-1}\left(\frac{y}{x}\right)+c.}$$

Sec : 5 : Equations of the first order, but of higher degree:

5.1 : TYPT - A : equations solvable for dy/dx.

In this section, Let us denote,

$$\boxed{\frac{dy}{dx} = p}$$

consider the equation of the first order and the n^{th} degree in p is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0 \quad \text{--- (1)}$$

where $P_1, P_2, P_3 \dots P_n$ are functions of x and y.

suppose equation (1) can be resolved into factors of the first degree of the form,

$$(P - R_1)(P - R_2) \dots (P - R_n)$$

$\therefore$ Any relation beween x and y which makes any of these factors vanish is a solution of (1),

$$P - R_1 = 0, \quad P - R_2 = 0, \quad \dots \quad P - R_n = 0.$$

$$\therefore \phi_1(x, y, c_1) = 0, \quad \phi_2 = (x, y, c_2) = 0, \dots$$

$$\phi_n(x, y, c_n) = 0.$$

are the primitives of the equation (1). where $c_1, c_2, \dots c_n$ are arbitrary constants.

Put $c_1 = c_2 = \dots = c_n = c$. then the Solution of (1) is,

$$\boxed{\phi_1(x, y, c) \cdot \phi_2(x, y, c) \dots \phi_n(x, y, c) = 0}$$

① Solve : $x^2p^2 + 3xyp + 2y^2 = 0$.

Soln:

Given equation is,

$$x^2p^2 + 3xyp + 2y^2 = 0 \quad \text{—— ①}.$$

equation ① is solvable for 'p'.

$$x^2p^2 + xyp + 2xyp + 2y^2 = 0.$$

$$xp(xp+y) + 2y(xp+y) = 0$$

$$(xp+y)(xp+2y) = 0.$$

$xp + y = 0$	$xp + 2y = 0$
$xp = -y$	$xp = -2y$
$p = \frac{-y}{x}$	$p = \frac{-2y}{x}$
$\frac{dy}{dx} = \frac{-y}{x}$	$\frac{dy}{dx} = -\frac{2y}{x}$
$\boxed{\frac{dy}{y} = \frac{-dx}{x}}$	$\boxed{\frac{dy}{y} = \frac{-2dx}{x}}$

Integrating the above equation on both sides,

$\log y = -\log x + \log c$

$\log y + \log x = \log c$

$\log (xy) = \log c$

$\boxed{xy = c}$

$\log y = -2\log x + \log c$

$\log y + 2\log x = \log c$

$\log y x^2 = \log c$

$\boxed{yx^2 = c}$

$\therefore$ The solution of equation ① is,

$$\boxed{(xy-c)(x^2y-c) = 0.}$$

2. Solve : $p^2 + (x+y-\frac{2y}{x})p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$

Soln:

Given equation is,

$$p^2 + (x+y-\frac{2y}{x})p + (xy + \frac{y^2}{x^2} - y - \frac{y^2}{x}) = 0 \quad \text{—①}$$

$$p = \frac{-(x+y-\frac{2y}{x}) \pm \sqrt{(x+y-\frac{2y}{x})^2 - 4(1)(xy+\frac{y^2}{x^2}-y-\frac{y^2}{x})}}{2(1)}$$

$$p = \frac{-(x+y-\frac{2y}{x}) \pm \sqrt{x^2+y^2+\cancel{\frac{4y^2}{x^2}}+2xy-\cancel{\frac{4y^2}{x}}-\cancel{4y}-4xy-\cancel{\frac{4y^2}{x^2}}+\cancel{4y}+\frac{4y^2}{x}}}{2}$$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right) \pm \sqrt{x^2+y^2-2xy}}{2}$$

$$= \frac{-\left(x+y-\frac{2y}{x}\right) \pm \sqrt{(x-y)^2}}{2}$$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right) \pm (x-y)}{2}$$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right)+(x-y)}{2}$$

$$= \frac{-x-y+2y/x+x-y}{2}$$

$$= \frac{2y/x-2y}{2}$$

$$p = \frac{y}{x}-y$$

$$\frac{dy}{dx} = y\left(\frac{1}{x}-1\right)$$

$$\frac{dy}{y} = \left(\frac{1}{x}-1\right)dx \quad \text{—} \ (2)$$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right)-(x-y)}{2}$$

$$= \frac{-x-y+2y/x-x+y}{2}$$

$$= \frac{2y/x-2x}{2}$$

$$p = \frac{y}{x}-x$$

$$\frac{dy}{dx}-\frac{y}{x} = -x$$

$$\frac{dy}{dx}-\frac{y}{x} = -x \quad \text{—} \ (3).$$

Integrating (2) on both sides,

$$\log y = \log x - x + \log c$$

$$\log y - \log x = -x + \log c$$

$$\log\left(\frac{y}{x}\right) = -x + \log c$$

$$\log\left(\frac{y}{x}\right) - \log c = -x$$

$$\log\left(\frac{y}{xc}\right) = -x$$

$$e^{\log(y/xc)} = e^{-x}$$

$$\frac{y}{xc} = e^{-x}$$

$$\boxed{y = cxe^{-x}} \quad \text{---} \quad (4)$$

equation (3) is linear in y.

we know that,

$$\boxed{\frac{dy}{dx} + Py = Q(x)}$$

The solution of the equation is,

$$ye^{\int P dx} = \int Q e^{\int P dx} dx + c.$$

$\therefore$ (3) $\Rightarrow$ $P = -1/x$, $Q = -x$

$$ye^{\int -1/x \, dx} = \int -x e^{\int -\frac{1}{x} dx} dx + c$$

$$ye^{-\log x} = \int -x e^{-\log x} dx + c$$

$$y e^{\log x^{-1}} = \int (-x) e^{\log x^{-1}} dx + c.$$

$$y x^{-1} = -\int x\left(\frac{1}{x}\right) dx + c$$

$$\frac{y}{x} = -\int dx + c$$

$$\frac{y}{x} = -x + c$$

$$\boxed{y = -x^2 + cx} \quad \text{---} \; (5)$$

$\therefore$ The general solution is,

$$\boxed{(y - cxe^{-x})(y + x^2 - cx) = 0.}$$

3. Solve : $p^2 - 5p + 6 = 0$.

Soln:

Given equation is,

$$p^2 - 5p + 6 = 0 \quad \text{---} \; (1)$$

$$p^2 - 3p - 2p + 6 = 0$$

$$p(p-3) - 2(p-3) = 0$$

$$(p-3)(p-2) = 0$$

$$\boxed{p = 2, 3.}$$

$p = 2$ | $p = 3$

$\frac{dy}{dx} = 2$ | $\frac{dy}{dx} = 3$

$dy = 2dx$ | $dy = 3dx.$

Integrating on both sides,

$y = 2x + c$ | $y = 3x + c$

$\boxed{y - 2x = c}$ | $\boxed{y - 3x = c.}$

The solution of ① is,

$$\boxed{(y-2x-c)(y-3x-c) = 0.}$$

---

④. Solve : $p^2 - (\cos x + \sec x) p + 1 = 0$

Soln:

Given equation is,

$$p^2 - (\cos x + \sec x) p + 1 = 0 \quad \text{——} ①$$

$$p = \frac{(\cos x + \sec x) \pm \sqrt{(\cos x + \sec x)^2 - 4 \times 1 \times 1}}{2}$$

$$= \frac{(\cos x + \sec x) \pm \sqrt{\cos^2 x + \sec^2 x + 2\cos x \sec x - 4}}{2}$$

$$p = \frac{(\cos x + \sec x) \pm \sqrt{\cos^2 x + \sec^2 x - 2}}{2}$$

$$= \frac{(\cos x + \sec x) \pm \sqrt{(\cos x - \sec x)^2}}{2}$$

$$= \frac{(\cos x + \sec x) \pm (\cos x - \sec x)}{2}$$

$$p = \frac{\cos x + \sec x + \cos x - \sec x}{2}$$ and

$$p = \frac{\cos x + \sec x - \cos x + \sec x}{2}$$

$p = \frac{2\cos x}{2}$	$p = \frac{2\sec x}{2}$
$\frac{dy}{dx} = \cos x$	$\frac{dy}{dx} = \sec x$
$dy = \cos x\, dx$	$dy = \sec x\, dx$
$\boxed{y = \sin x + c}$	$\boxed{y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c}$

The solution of ① is,

$$\boxed{(y - \sin x - c)\left(y - \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) - c\right) = 0}$$

5. solve : $p^2 + 2yp \cot x = y^2$

Soln:

Given equation is,

$$p^2 + 2yp\cot x - y^2 = 0 \quad (1).$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cos^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x + 1}}{2}$$

$$p = -y\cot x \pm y\sqrt{\csc^2 x}$$

$$p = -y\cot x \pm y\csc x.$$

$$\frac{dy}{dx} = -y(\cot x + \csc x)$$

$$\frac{dy}{y} = -(\cot x + \csc x)\,dx$$

$$\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right)dx$$

$$\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right)dx$$

$$\frac{dy}{dx} = -y(\cot x - \csc x)$$

$$\frac{dy}{y} = (\csc x - \cot x)\,dx$$

$$\frac{dy}{y} = \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)dx$$

$$\frac{dy}{y} = \left(\frac{1-\cos x}{\sin x}\right)dx$$

$$\log y = -\log(1+\cos x) + \log c$$

$$\log y + \log(1+\cos x) = \log c$$

$$y(1+\cos x) = c$$

$$\boxed{[y(1+\cos x)] - c = 0}$$

$$\log y = \log(1-\cos x) + \log c$$

$$\log y - \log(1-\cos x) = \log c$$

$$\frac{y}{(1-\cos x)} = c$$

$$\boxed{y - c(1-\cos x) = 0.}$$

The solution is,

$$\boxed{(y(1+\cos x) - c)(y - c(1-\cos x)) = 0.}$$

6. Solve : $p^2 y + p(x-y) - x = 0$.

Soln:

Given equation is,

$$p^2 y + p(x-y) - x = 0 \quad \text{——} \quad (1).$$

$$p = \frac{-(x-y) \pm \sqrt{(x-y)^2 - 4 \times y \times (-x)}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{x^2+y^2-2xy+4xy}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{x^2+y^2+2xy}}{2y}$$

$$P = \frac{-(x-y) \pm \sqrt{(x+y)^2}}{2y}$$

$$= \frac{-(x-y) \pm (x+y)}{2y}$$

$$p = \frac{-x+y+x+y}{2y}$$

$$= \frac{2y}{2y}$$

$$P = 1$$

$$\frac{dy}{dx} = 1$$

$$dy = dx$$

$$\boxed{y = x+c}$$

$$p = \frac{-x+y-x-y}{2y}$$

$$p = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c}{2}$$

$$\boxed{y^2 + x^2 = c}$$

$\therefore$ The Solution of ① Is,

$$\boxed{(y-x-c)(x^2+y^2-c) = 0}$$

Section - 4 - Problems:-

① Solve $(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + (x^3y^3 - x^2y^2 - xy + 1)\,x\,dy = 0.$

Soln:

$$M = x^3y^4 + x^2y^3 + xy^2 + y$$

$$\frac{\partial M}{\partial y} = 4x^3y^3 + 3x^2y^2 + 2xy + 1.$$

$$N = x^4y^3 - x^3y^2 - x^2y + x.$$

$$\frac{\partial N}{\partial x} = 4x^3y^3 - 3x^2y^2 - 2xy + 1$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.}$$

Hence the given equation is not an exact.

$$\frac{1}{Mx - Ny} = \frac{1}{\cancel{x^4y^4} + x^3y^3 + x^2y^2 + \cancel{xy} - \cancel{x^4y^4} + x^3y^3 + x^2y^2 - \cancel{xy}}$$

$$= \frac{1}{2x^3y^3 + 2x^2y^2}$$

$$\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2(xy+1)} \quad \text{—— ②.}$$

If of

$\times$ ②, in ①,

$$\frac{1}{x^2y^2(xy+1)}\left[\begin{matrix}(1+x^3y^3) \\ (ydx + xdy)\end{matrix}\right] + \frac{[x^2y^2 + xy]}{x^2y^2(xy+1)} \frac{[ydx - xdy]}{y^2} = 0.$$

$$\frac{(xy+1)[(xy)^2 - (xy) + 1]}{(xy)^2(xy+1)} d(xy) + \frac{d(x/y)}{x/y} = 0$$

$$\left(1-\frac{1}{xy}+\frac{1}{(xy)^2}\right)d(xy)+\frac{d(x/y)}{x/y}=0.$$

$$xy-\log xy-\frac{1}{xy}+\log(x)-\log(y)=0$$

$$xy-1/xy+\log\left(\frac{x}{xy^2}\right)=c.$$

$$\cancel{x^2y^2+xy\log(1/y^2)=1+xy}$$

$$\frac{x^2y^2-1}{xy}+\log(1/y^2)=c$$

$$\cancel{x^2y^2-2xy\log(y)=1+cxy}$$

$$xy^2-1+xy\log(1/y^2)=c(xy)$$

H.W Sum. solve: $y\,dx-x\,dy-3x^2y^2e^{x^3}dx=0.$

2). $(2xy+y-\tan y)dx+(x^2-x\tan^2y+\sec^3y)dy=0.$

Soln:

$M=2xy+y-\tan y$

$N=x^2-x\tan^2y+\sec^3y$

$$\frac{\partial M}{\partial y}=2x+1-\sec^2y$$

$$\frac{\partial N}{\partial x}=2x-\tan^2y$$

$$=2x-(\sec^2y-1)$$

$$\frac{\partial N}{\partial x}=2x-\sec^2y+1$$

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

Hence ① is exact

$\therefore$ Solution is,

$$\int M\,dx + \int N\,dy = c.$$

(Taking y as constant) (Terms not containing x)

$$\Rightarrow \int (2xy + y - \tan y)\,dx + \int (\sec^3 y)\,dy = c$$

$$\Rightarrow x^2y + x\tan y + \frac{1}{2}\sec y\tan y + \frac{1}{2}\log(\sec y + \tan y) = c$$

---

(3). $(x^4e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$ — (1)

Soln:

$M = x^4e^x - 2xmy^2$ | $N = 2mx^2y$

$\frac{\partial M}{\partial y} = -4mxy$ | $\frac{\partial N}{\partial x} = 4mxy$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence (1) is not exact.

from rule (3),

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4mxy - 4mxy$$

$$= -8mxy.$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{-8mxy}{2mx^2y}$$

$$=-\frac{4}{x}$$

$\therefore$ The Integrating factor is,

$$e^{\int f(x)dx}=e^{\int(-4/x)dx}$$

$$=e^{-4\log x}$$

$$=e^{\log x^{-4}}$$

$$=x^{-4}$$

$$\boxed{e^{\int f(x)dx}=\frac{1}{x^4}}$$

① $\times \frac{1}{x^4}$ give,

$$\left(e^x-\frac{2my^2}{x^3}\right)dx+\left(\frac{2my\,dy}{x^2}\right)dy=0.$$

Now,

$M=e^x-\frac{2my^2}{x^3}$ | $N=\frac{2my}{x^2}$

$\frac{\partial M}{\partial y}=-\frac{4my}{x^3}$ | $\frac{\partial N}{\partial x}=-\frac{4my}{x^3}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M dx + \int N dy = c$$

$$\int \left(e^x - \frac{2my^2}{x^3}\right) dx + \int \cos dy = c$$

$$e^x + \frac{my^2}{x^2} = c.$$

$$\boxed{e^x + \frac{my^2}{x^2} = c.}$$

4. $(x^2 - yx^2) \frac{dy}{dx} + (y^2 + x^2y^2) = 0$ —— ①.

<u>soln:</u>

$$(x^2 - yx^2) dy + (y^2 + x^2y^2) dx = 0$$

$M = y^2 + x^2y^2$	$N = x^2 - yx^2$
$\frac{\partial M}{\partial y} = 2y + 2yx^2$	$\frac{\partial N}{\partial x} = 2x - 2xy$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence ① is not exact.

$$x^2(1-y) dy + y^2(1+x^2) dx = 0.$$

$\div x^2y^2$

$$\int \frac{(1-y)}{y^2}\,dy + \int \frac{(1+x^2)}{x^2}\,dx = 0.$$

$$\int \left(\frac{1}{y^2} - \frac{1}{y}\right) dy + \int \left(\frac{1}{x^2} + 1\right) dx = 0.$$

$$-\frac{1}{y} - \log y - \frac{1}{x} + x = c$$

$$\boxed{\frac{1}{y} + \log y + \frac{1}{x} - x = c.}$$

Sec : 5.2 - TYPE : B

If given differential equation is in the form $f(x, y, p) = 0$, then it cannot be resolved into rational linear factors. as in 5.1. It may be either solved for y or x.

Sec : 5.3 : Equations solvable for y :

$f(x, y, p) = 0$ can be put in the form

$$\boxed{y = F(x, p)} \quad \text{——} \quad (1)$$

Differentiating w.r.to x,

$$\frac{dy}{dx} = p = \phi\left(x, p, \frac{dp}{dx}\right)$$

This is an equation in the two variables p and x. This can be integrated by any of the foregoing methods. Hence we get,

$$\boxed{\psi(x, p, c) = 0 \quad \text{--- (2)}}$$

Eliminating p between (1), (2), we get the solutions.

---

(1). Solve : $xp^2 - 2yp + x = 0$.

Soln:

Given equation is

$$xp^2 - 2yp + x = 0 \quad \text{--- (1)}$$

Solving for 'y', we get,

$$2yp = xp^2 + x$$

$$y = \frac{xp^2 + x}{2p} = x\frac{(p^2+1)}{2p}$$

$$y = \frac{x(p^2+1)}{2p} \quad \text{--- (2)}$$

differentiating equation ② by 'x' we get,

$$\frac{dy}{dx} = \frac{p^2+1}{2p} + \frac{x}{2} \cdot \frac{p(2p\, dp/dx - (p^2+1)\, dp/dx}{p^2}$$

$$p = \frac{p^2+1}{2p} + \frac{x}{2} \cdot \frac{2p^2-(p^2+1)}{p^2} \frac{dp}{dx}.$$

$$= \frac{p^2+1}{2p} + \frac{x}{2} \cdot \frac{2p^2-p^2-1}{p^2} \cdot \frac{dp}{dx}.$$

$$= \frac{p^2+1}{2p} + \frac{x}{2} \cdot \frac{p^2-1}{p^2} \cdot \frac{dp}{dx}.$$

$$\frac{2p^2-p^2-1}{2p} = \frac{x(p^2-1)}{2p^2} \cdot \frac{dp}{dx}.$$

$$\frac{p^2-1}{2p} = \frac{x(p^2-1)}{2p^2} \cdot \frac{dp}{dx}.$$

$$\frac{p}{x} = \frac{dp}{dx}$$

$$\frac{dx}{x} = \frac{dp}{p}$$

integrating on bothsides, we get,

$$\log c + \log x = \log p$$

$$\Rightarrow \log p = \log cx.$$

$$\boxed{p = cx.}$$

Put $p = cx$ in ①,

$$x(cx)^2 - 2y(cx) + x = 0$$

$$xc^2 \cdot x^2 - 2ycx + x = 0$$

$$x(c^2x^2 - 2yc + 1) = 0$$

$$c^2x^2 + 1 = 2yc$$

$\therefore$ The solution of ① is

$$\boxed{2yc = c^2x^2 + 1.}$$

Sec: 5.4 Equations solvable for x.

Let $f(x, y, p) = 0$ be in this case put in the form $\boxed{x = F(y, p)}$ — ①.

differentiating w.r. to 'y',

$$\boxed{\frac{dx}{dy} = \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right)}$$

Integrating the above equation, we get,

$$\boxed{\psi(y, p, c) = 0} \text{ — ②}$$

Eliminating p between ①, ②, we get the solution of ①.

1. Solve : $x = y^2 + \log p$

Soln:

Given equation is,

$$x = y^2 + \log p \quad \text{—— (1)}$$

differentiating equation (1) by y,

$$\frac{dx}{dy} = \frac{1}{p} = 2y + \frac{1}{p} \cdot \frac{dp}{dy}$$

$$\therefore \frac{1}{p} = 2y + \frac{1}{p} \cdot \frac{dp}{dy}$$

$$\frac{dp}{dy} + 2py = 1$$

This is linear in p, we get,

$$pe^{\int 2y\,dy} = \int 1 \cdot e^{\int 2y\,dy} dy + c \cdot \quad ye^{\int p\,dx} = \int q e^{\int p\,dx} dx$$

$$\boxed{pe^{y^2} = \int e^{y^2} dy + c} \quad \text{—— (2)}$$

Eliminating p between (1), (2), we get the solution of given equation.

Home work Sumes:

HW (1). $y^2 = (1 + p^2)$, (3). $p^3 - 4xyp + 8y^2 = 0$.

(2). $x(1 + p^2) = 1$,

sec 6.1 : Clairaut's form

The equation known as clairaut's is of the form,

$$\boxed{y = px + f(p)} \quad \text{---} \ ①.$$

differentiating w.r.to x,

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\therefore \ p = p + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}.$$

$$x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0.$$

$$\frac{dp}{dx}[x + f'(p)] = 0.$$

$$\therefore \frac{dp}{dx} = 0 \quad (\text{or}) \quad x + f'(p) = 0.$$

$$\frac{dp}{dx} = 0 \Rightarrow dp = 0 \Rightarrow \boxed{p = c} \text{ (constant)}$$

$\therefore$ put $p = c$ in ①,

$$\boxed{y = cx + f(c)}$$ //

$\therefore$ we have to replace p in clairaut's equation by c. The other factor

$y + f'(p) = 0$ taken along with ① give, on eliminating of p. a soultion of 1.

But this Solution is not included in the general solution ②. Such a Solution as this is called a singular solution.

---

①. solve: $y = (x-a)p - p^2$

<u>soln:</u>

Given equation is,

$$y = (x-a)p - p^2 \quad \text{--- ①.}$$

The equation ① is in clairaut's form,

$$y = px + f(p)$$

$\therefore$ put $p = c$ in ①,

$$y = xp - ap - p^2$$

$$y = xc - ac - c^2$$

$$\boxed{y = (x-a)c - c^2}$$

② Solve: $y = 2px + y^2p^2$.

Soln:

Given equation is,

$$y = 2px + y^2p^2 \quad \text{— ①.}$$

Let $X = 2x$ and $Y = y^2$ then

$dX = 2dx$ $\quad dY = 2ydy$

$$\therefore P = \frac{dY}{dX} = \frac{2ydy}{2dx} = y\frac{dy}{dx} = yp$$

$$\boxed{P = yp}$$

The equation ① transforms into,

$$y = X\frac{P}{y} + y\frac{P^2}{y^2}$$

$$\boxed{y^2 = XP + P^2}$$

$$\therefore y^2 = cX + c^2$$

$$y^2 = 2xc + c^2$$

$$\boxed{y^2 = 2xc + c^2}$$

The solution of ① is $y^2 = 2xc + c^2$ //

HOME WORK SUM

①. $y = px + \frac{2p}{\sqrt{1+p}}$, ③ $y = px + \frac{a}{p}$

②. $(y-px)(p-1) = p$, ④ $x^2(y-px) = yp^3$ (Put $X = x^2$, $Y = y^2$)

sec 6.2: Extended form of Clairaut's equation

Consider the extended form of Clairaut's equation,

$$\boxed{y = x f(p) + \phi(p)} \quad \text{--- (1).}$$

differentiating (1) w.r.to 'x', we get,

$$p = x f'(p) \frac{dp}{dx} + f(p) + \phi'(p) \frac{dp}{dx}$$

$$= f(p) + [x f'(p) + \phi'(p)] \frac{dp}{dx}$$

$$p - f(p) = [x f'(p) + \phi'(p)] \frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{x f'(p)}{p - f(p)} + \frac{\phi'(p)}{p - f(p)}$$

$$\frac{dx}{dp} + \frac{x f'(p)}{p - f(p)} = \frac{\phi'(p)}{p - f(p)}$$

This is linear in x and hence gives

$F(x, p, c) = 0$.

The elimination of p between this equation and (1) give the Solution of (1).

1. Solve : $y = xp + x(1+p^2)^{1/2}$

Soln:

Given equation is,

$$y = xp + x(1+p^2)^{1/2} \quad \text{--- (1)}$$

differentiating (1) w.r.to 'x',

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + (1+p^2)^{1/2} + x\frac{1}{2}(1+p^2)^{-1/2} \cdot 2p\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + (1+p^2)^{1/2} + \frac{xp}{(1+p^2)^{1/2}} \cdot \frac{dp}{dx}.$$

$$(1+p^2)^{1/2} + x\frac{dp}{dx} + \frac{xp}{(1+p^2)^{1/2}} \cdot \frac{dp}{dx} = 0.$$

$$(1+p^2) + (x)(1+p^2)^{1/2} \cdot \frac{dp}{dx} + xp\frac{dp}{dx} = 0.$$

$$(x(1+p^2)^{1/2} + xp)\frac{dp}{dx} = -(1+p^2)$$

$$\frac{x[(1+p^2)^{1/2} + p]\,dp}{1+p^2} = -dx.$$

$$\frac{dp}{(1+p^2)^{1/2}} + \frac{p}{1+p^2}dp = \frac{-dx}{x}$$

$$\frac{dp}{(1+p^2)^{1/2}} + \frac{p}{1+p^2}dp + \frac{dx}{x} = 0$$

Integrating, we get,

$$\int \frac{dp}{(1+p^2)^{1/2}} + \int \frac{p}{1+p^2}\, dp + \int \frac{dx}{x} = 0$$

$$\log(p+\sqrt{1+p^2}) + \frac{1}{2}\log(1+p^2) + \log x = \log c.$$

$$\log(p+\sqrt{1+p^2})(\sqrt{1+p^2}).\, x = \log c$$

$$(p+\sqrt{1+p^2})(\sqrt{1+p^2})(x) = c$$

$$\boxed{(p\sqrt{1+p^2} + 1 + p^2)\, x = c.} \quad \text{—— (2).}$$

Eliminating 'p' between (1), (2), we get, the solution.

Home Work Sum

H.W

(1). $y = 2px + p^2 y$

(2). $xp^2 - 2yp + 4x = 0.$

(3). $(y+px)^2 = py^2.$

Sec : 7.1 Equations that do not contain 'x' explicitly

Suppose an equation is of the form

$$f(y, p) = 0 \quad \text{—— ①.}$$

If this is solvable for p, then $p = \phi(y)$ and hence it is immediately integrable. If ① is solvable for y, so that $y = \phi(p)$, then the method of sec 5.3 is applied.

Sec 7.2 equations that do not contain 'y' explicitly

Let the equation be,

$$f(x, p) = 0 \quad \text{—— ①.}$$

If this solvable for p, so that $p = \phi(x)$, it is directly integrable. If ① is solvable for x, the method of sec 5.4 is applied.

Sec 7.3 : Equations homogeneous in x and y.

Let the equation be,

$$f\left(\frac{y}{x}, p\right) = 0 \quad \text{—— ①.}$$

If this solvable for p, then $p = F(y/x)$ and is immediately integrable.

If ① is solvable for $\frac{y}{x}$, so that,

$y = xF(p)$, then we proceed as in sec 6.2.

Differentiating w.r.to 'x', we get,

$$p = F(p) + xF'(p)\frac{dp}{dx}.$$

$$\frac{dx}{x} = \frac{F'(p)\,dp}{p - F(p)}$$

This is integrable and eliminate of p.

between this equation and ① is the

required solution.

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① Solve : $x^2 = 1 + p^2$

Soln:

The Given equation is,

$$x^2 = 1 + p^2 \quad \text{—①}$$

$$x = \pm\sqrt{1+p^2} \quad \text{—②}$$

here y is explicitly absent.

differentiating ② w.r.to 'y', we get,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}(1+p^2)^{-1/2} \cdot 2p \cdot \frac{dp}{dy}$$

$$\frac{1}{p} = \frac{p}{\sqrt{1+p^2}} \cdot \frac{dp}{dy}$$

$$\frac{p^2}{\sqrt{1+p^2}} dp = dy.$$

integrating on both sides, we get,

$$\int dy = \int \frac{p^2+1-1}{\sqrt{p^2+1}} dp.$$

$$y+c = \int \frac{p^2+1}{\sqrt{p^2+1}} dp - \int \frac{1}{\sqrt{p^2+1}} dp.$$

$$= \int \sqrt{p^2+1}\, dp - \int \frac{1}{\sqrt{p^2+1}} dp.$$

$$= \frac{1}{2}\left(p\sqrt{1+p^2} + \sinh^{-1} p\right) - \sinh^{-1} p.$$

$$\boxed{y+c = \frac{1}{2}\left(p\sqrt{1+p^2} - \sinh^{-1} p\right)}$$

2. Solve : $xyp^2 + p(3x^2-2y^2) - 6xy = 0.$

Soln:

Given equation is,

$$xyp^2 + p(3x^2-2y^2) - 6xy = 0$$

This is homogeneous in x and y and solvable for p.

$$P = \frac{-(3x^2-2y^2) \pm \sqrt{(3x^2-2y^2)^2 - 4 \times xy \times -6xy}}{2xy}$$

$$= \frac{-(3x^2-2y^2) \pm \sqrt{9x^4 + 4y^4 - 12x^2y^2 + 24x^2y^2}}{2xy}$$

$$= \frac{-(3x^2-2y^2) \pm \sqrt{9x^4+4y^4+12x^2y^2}}{2xy}$$

$$= \frac{-(3x^2-2y^2) \pm \sqrt{(3x^2+2y^2)^2}}{2xy}$$

$$= \frac{-(3x^2-2y^2) \pm (3x^2+2y^2)}{2xy}$$

$$P = \frac{-3x^2+2y^2+3x^2+2y^2}{2xy}, \frac{-3x^2+2y^2-3x^2-2y^2}{2xy}$$

$$p = \frac{4y^2}{2xy}, \frac{-6x^2}{2xy}$$

$$p = \frac{2y}{x}, \frac{-3x}{y}$$

$$\therefore \frac{dy}{dx} = \frac{2y}{x} \qquad \frac{dy}{dx} = -\frac{3x}{y}$$

$$\frac{dy}{y} = 2\frac{dx}{x} \qquad y\,dy = -3x\,dx.$$

$$\log y = 2\log x + \log c$$

$$\log y - \log x^2 = \log c$$

$$\log\left(\frac{y}{x^2}\right) = \log c$$

$$\frac{y}{x^2} = c$$

$$y = cx^2$$

$$\frac{y^2}{2} = -\frac{3x^2}{2} + \frac{c}{2}$$

$$y^2 + 3x^2 = c$$

$$y^2 + 3x^2 = c$$

$\therefore$ The solution is,

$$\boxed{(y-cx^2)(y^2+3x^2-c) = 0.}$$

(3). Solve : $p^2y + p(x-y) - x = 0.$

Soln:

Given equation is,

$$p^2y + p(x-y) - x = 0.$$

$$p = \frac{-(x-y) \pm \sqrt{(x-y)^2 - 4\times(-x)\times y}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{x^2+y^2-2xy+4xy}}{2y}$$

$$p = \frac{-(x-y) \pm \sqrt{x^2+y^2+2xy}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{(x+y)^2}}{2y}$$

$$p = \frac{-(x-y) \pm (x+y)}{2y}$$

$$p = \frac{-(x-y)+(x+y)}{2y}, \quad \frac{-(x-y)-(x+y)}{2y}$$

$$= \frac{-x+y+x+y}{2y}, \quad \frac{-x+y-x-y}{2y}$$

$$p = \frac{2y}{2y}, \quad \frac{-2x}{2y}$$

$$\frac{dy}{dx} = 1, \; -x/y$$

$\frac{dy}{dx} = 1,$	$\frac{dy}{dx} = \frac{-x}{y}$
$dy = dx$	$y\,dy = -x\,dx$
$y = x + c$	$y^2/2 = -x^2/2 + c/2$
$\boxed{y - x = c}$	$\boxed{y^2 + x^2 = c.}$

∴ The solution is,

$$\boxed{(y-x-c)(y^2+x^2-c) = 0.}$$

(4). Solve: $y^2 = 1 + p^2$

Soln:

Given equation is,

$$y^2 = 1 + p^2 \quad \text{--- (1)}$$

$$y = \pm\sqrt{1+p^2} \quad \text{--- (2)}.$$

differentiating equation (2) w.r.to 'x',

$$\frac{dy}{dx} = \pm \frac{1}{2}(1+p^2)^{-1/2} \cdot 2p \cdot \frac{dp}{dx}$$

$$p = \pm \frac{p}{\sqrt{1+p^2}} \cdot \frac{dp}{dx}$$

$$dx = \pm \frac{dp}{\sqrt{1+p^2}}$$

integrating on both sides,

$$\boxed{x = \cos^{-1} p + c}$$ //