06/01/2020

UNIT - II

Linear Differential Equations with Constant Co-efficients.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X.$$

a, b, c are constants, X is a function of x Auxillary equation m.

$$am^2 + bm + c = 0$$

m satisfies the above equation.

Solution for A.E

Case (i) The A.E m has 2 real and distinct roots m_1 & m_2 The complementary function is,

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

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1. Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0.$

Auxillary equation is $m^2 - 5m + 4 = 0$

$\Rightarrow (m-4)(m-1) = 0$

$\Rightarrow m = 4, 1$

The solution is $y = Ae^{x} + Be^{4x}.$

2. Solve: $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 0.$

A:E is $m^2 - 6m + 8 = 0.$ $\quad$ $\overset{8}{\wedge}$ $-4 \; -2$

$\Rightarrow m = 4, 2$

The Solution is $y = Ae^{2x} + Be^{4x}.$

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3. Solve: $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 24y = 0.$

A.E is $m^2 + 2m - 24 = 0$ $\quad$ $\overset{-24}{\wedge}$ $6 \; -4$

$\Rightarrow (m+6)(m-4) = 0.$

$\Rightarrow m = -6, +4.$

The Solution is $y = Ae^{+4x} + Be^{-6x}.$

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4. Solve: $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0.$

Auxillary equation is

$m^3 + 2m^2 - m - 2 = 0.$

$$\begin{array}{c|cccc} 1 & 1 & 2 & -1 & -2 \\ & 0 & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

$\overset{2}{\wedge}$ $1 \; 2$

$m^2 + 3m + 2 = 0$

$(m-1)$ is a factor.

$(m+1)(m+2)$ are also factors.

$\Rightarrow (m-1)(m+1)(m+2) = 0.$

$m = +1, -1, -2.$

The solution is.

$$y = Ae^{x} + Be^{-x} + Ce^{-2x}$$

Case (ii)

The auxillary equation has two equal and real roots.

$$m = m_1 = m_2.$$

The solution is $y = (A + Bx)\,e^{m_1 x}$

(or) $y = (Ax + B)e^{m_1 x}$

1) Solve:. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0.$

Auxillary equation is $m^2 + 2m + 1 = 0.$

$\Rightarrow (m+1)(m+1) = 0.$

$\Rightarrow m = -1, -1$

1
^
1 1

The solution is.

$$y = (Ax + B)e^{-x}.$$

2) Solve: $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

A.E is $m^2 - 4m + 4 = 0$

$$m = \frac{4 \pm \sqrt{16-16}}{2}.$$

$$\boxed{m = -2}$$

Solution is $y = (Ax+B)e^{-2x}.$

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(3). Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$

A.E is $m^3 - 3m^2 + 3m - 1 = 0.$

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & 0 & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$\Rightarrow m^2 - 2m + 1 = 0.$

1

-1 -1

$\Rightarrow (m-1)(m-1) = 0$

$\Rightarrow m = 1$

The solution is $\cancel{y = (Ax+Bx)e^{x}}$

$y = (Ax^2 + Bx + C)e^{x}.$

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(4). Solve :- $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = 0.$

A.E is $m^3 + 0m^2 - 3m + 2 = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$(m-1)$ is a factor.

$\Rightarrow m^2 + m - 2 = 0$ $\quad -2$

$\Rightarrow (m-1)(m+2) = 0.$ $\quad -1 \quad 2$

$\Rightarrow (m-1)^2 (m+2) = 0.$

$\Rightarrow m = 1, 1, \quad m = -2$

$$y = (Ax + B)e^{x} + Ce^{-2x}.$$

Case (iii) The auxillary equation has imaginary roots & real roots

$$m = \alpha \pm i\beta$$

The solution is.

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

① Solve : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 0.$

A.E is $m^2 - 3m + 5 = 0$

$$m = \frac{3 \pm \sqrt{9 - 20}}{2}$$

$$m = \frac{3 \pm \sqrt{-11}}{2}$$

$$m = \frac{3 \pm \sqrt{11}\, i}{2}$$

$$m = \frac{3}{2} \pm \frac{\sqrt{11}}{2} i.$$

The solution is,

$$y = e^{3/2\,x}\left[A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x\right]$$

Solve:-

(2) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 0.$$

A.E is $m^2 + m + 4 = 0$.

$$m = \frac{-1 \pm \sqrt{1-16}}{2}.$$

$$= \frac{-1 \pm \sqrt{-15}}{2}.$$

$$= \frac{-1 \pm \sqrt{15}\, i}{2}.$$

$$m = \frac{-1}{2} \pm \frac{i\sqrt{15}}{2}.$$

The solution is,

$$y = e^{-1/2\,x}\left[A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right].$$

(3). Solve:- $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 8y = 0.$$

A.E is $m^2 + 3m - 8 = 0$.

$$m = \frac{-3 \pm \sqrt{9+32}}{2}.$$

$$m = \frac{-3 \pm \sqrt{41}}{2} = \frac{-3}{2} \pm \frac{\sqrt{41}}{2}$$ //.

Derivation :-

① Let $z = \frac{1}{(D-\alpha)} e^{\alpha x}$

$$(D-\alpha) z = e^{\alpha x}$$

$$\Rightarrow \frac{dz}{dx} - \alpha z = e^{\alpha x}$$

$$\Rightarrow z e^{\int(-\alpha) dx} = \int e^{\alpha x} e^{\int(-\alpha) dx} dx$$

$$z e^{-\alpha x} = \int e^{\alpha x} e^{-\alpha x} dx.$$

$$z e^{-\alpha x} = \int (1) dx.$$

$$z e^{-\alpha x} = x.$$

$$z = x e^{\alpha x}.$$

$\frac{dy}{dx} + Py = Q.$

$\Rightarrow y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$\frac{1}{(D-\alpha)} e^{\alpha x} = x e^{\alpha x}.$$

(2) To Find $\frac{1}{(D-\alpha)^2} e^{\alpha x}$.

$$\frac{1}{(D-\alpha)^2} e^{\alpha x} = \frac{1}{(D-\alpha)(D-\alpha)} e^{\alpha x}$$

$$= \frac{1}{(D-\alpha)} x e^{\alpha x} \quad \text{— (1)}.$$

Let $z = \frac{1}{D-\alpha} x e^{\alpha x}$

$(D-\alpha) z = x e^{\alpha x}$.

$\Rightarrow \frac{dz}{dx} - \alpha z = x e^{\alpha x}$.

$\Rightarrow z e^{\int(-\alpha)\,dx} = \int x e^{\alpha x} e^{\int(-\alpha)dx} dx$.

$\Rightarrow z e^{-\alpha x} = \int x e^{\alpha x} e^{-\alpha x} dx$.

$\Rightarrow z e^{-\alpha x} = \frac{x^2}{2}$

$\Rightarrow z = \frac{x^2}{2} e^{\alpha x}$.

$\Rightarrow \frac{1}{D-\alpha} x e^{\alpha x} = \frac{x^2 e^{\alpha x}}{2}$.

2. The operators D and D^{-1}

Let D stand for operator $\frac{d}{dx}$,

Let D^2 stand for the operator $\frac{d^2}{dx^2}$

& D^{-1} be the inverse of the operator D.

$\left(a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\right)=X$ can be written as

$(aD^2+bD+c)y=X \Rightarrow f(D)y=X$

$P.I=\frac{1}{f(D)}X$ $\left[\frac{1}{f(D)}\right.$ is the inverse of the operator $f(D)$ (ie) $\left.f(D)\frac{1}{f(D)}X=X\right]$

a) $X=e^{\alpha x}$, where α is a constant.

i) If $f(\alpha)\neq 0$, $\frac{1}{f(D)}e^{\alpha x}=\frac{1}{f(\alpha)}e^{\alpha x}$.

ii) If $f(\alpha)=0$ $f(D-\alpha)$ is a factor.

$$\frac{1}{f(D)}e^{\alpha x}=\frac{1}{a(D-\alpha)(D-m_1)}e^{\alpha x}=\frac{1}{a(\alpha-m_1)(D-\alpha)}e^{\alpha x}=\frac{1}{a(\alpha-m_1)}xe^{\alpha x}$$

iii) If $f(\alpha)=0$ & $(D-\alpha)^2$ is a factor.

$$\frac{1}{f(D)}e^{\alpha x}=\frac{1}{a(D-\alpha)^2}e^{\alpha x}=\frac{1}{a}\frac{x^2e^{\alpha x}}{2}.$$

b) $X=\cos\alpha x$, or $\sin\alpha x$, where α is a constant.

To find $\frac{1}{\phi(D^2)}X$.

i) If $\phi(-\alpha^2)\neq 0$,

$$\frac{1}{\phi(D^2)}\sin\alpha x=\frac{\sin\alpha x}{\phi(-\alpha^2)}$$

$$\frac{1}{\phi(D^2)}\cos\alpha x=\frac{\cos\alpha x}{\phi(-\alpha^2)}$$

(ii) If $\phi(-\alpha^2)=0$.

$D^2+\alpha^2$ is a factor of $\phi(D^2)$

$$\frac{1}{D^2+\alpha^2}\sin\alpha x=\frac{1}{D^2+\alpha^2}\left[I.P\text{ of }e^{i\alpha x}\right]$$

$$=I.P\text{ of }\frac{1}{(D-\alpha i)(D+\alpha i)}e^{i\alpha x}.$$

$$=I.P\text{ of }\frac{1}{(D-\alpha i)(2\alpha i)}e^{i\alpha x}.$$

$$=I.P\text{ of }\frac{xe^{i\alpha x}}{2\alpha i}$$

$$=I.P\text{ of }-\frac{xi}{2\alpha}(\cos\alpha x+i\sin\alpha x)$$

$$=-\frac{x\cos\alpha x}{2\alpha}.$$

Similarly,

$$\frac{1}{D^2+\alpha^2}\cos\alpha x=\frac{x\sin\alpha x}{2\alpha}.$$

① Solve : $(D^2+5D+6)y = e^x$.

Auxillary equation is

$$m^2+5m+6=0$$

6

2 3

$$(m+2)(m+3)=0$$

$$m=-2 \text{ or } -3$$

$$C.F = Ae^{-2x}+Be^{-3x} \quad \text{— ①.}$$

$$P.I = \frac{1}{D^2+5D+6}e^x$$

$$= \frac{1}{1+5+6}e^x$$

$$= \frac{1}{12}e^x$$

$\therefore$ Solution is $y = CF + PI$

$$y = Ae^{-2x}+Be^{-3x}+\frac{1}{12}e^x$$

② Solve : $(D^2-5D+6)y = e^{4x}$

Auxiliary equation is

6

-2 -3

$$m^2-5m+6=0$$

$$(m-2)(m-3)=0.$$

$$m=2 \text{ or } m=3$$

$$C.F = Ae^{2x}+Be^{3x} \quad \text{— ①.}$$

$$P.I = \frac{1}{D^2-5D+6}e^{4x}$$

$$= \frac{1}{16-20+6}e^{4x}$$

$$P.I = \frac{e^{4x}}{2}.$$

Solution is $y = C.F + P.I$

$$y = Ae^{2x} + Be^{3x} + \frac{e^{4x}}{2}$$

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③. Solve: $(3D^2 + D - 14)y = 13e^{2x}$.

A.E is

$$3m^2 + m - 14 = 0.$$

$$-42$$

$$-6 \quad 7$$

$$\frac{-6}{3} \quad \frac{7}{3}$$

$$(m-2)(3m+7) = 0$$

$$m = 2 \quad \text{or} \quad m = -7/3.$$

$$C.F = Ae^{2x} + Be^{-7/3 x}.$$

$$P.I = \frac{1}{3D^2 + D - 14} 13e^{2x}$$

$$= \frac{1}{3(D-2)(D+7/3)} 13e^{2x}$$

$$= \frac{\cancel{13}}{\cancel{3} \times (\cancel{13}/\cancel{3})(D-2)} e^{2x}$$

$$= xe^{2x}.$$

Solution is $y = C.F + P.I.$

$$y = Ae^{2x} + Be^{-7/3 x} + xe^{2x}$$

(4). Solve: $(D^2+6D+8)\,y = e^{-2x}$.

A.E is
(Auxiliary equation).

$\Rightarrow m^2+6m+8 = 0$

$\Rightarrow (m+4)(m+2)=0$.

$\Rightarrow m=-4$ or $m=-2$.

8
^
4 2

$C.F = Ae^{-2x} + Be^{-4x}$.

$$P.I = \frac{1}{(D^2+6D+8)} e^{-2x}$$

$$= \frac{1}{(D+4)(D+2)} e^{-2x}.$$

$$= \frac{1}{2} x e^{-2x}.$$

The solution is,

$y = C.F + P.I$.

$$y = Ae^{-2x} + Be^{-4x} + \frac{xe^{-2x}}{2}.$$

(5). solve: $(D^2-2mD+m^2)\,y = e^{mx}$

Auxiliary equation is,

$k^2-2mk+m^2=0$

$(k-m)^2=0$

$k=m, m$.

$$C.F = (Ax+B)e^{mx}.$$

$$P.I = \frac{1}{D^2 - 2mD + m^2} e^{mx}$$

$$= \frac{1}{(D-m)^2} e^{mx}.$$

$$= \frac{x^2 e^{mx}}{2}.$$

Solution is $y = C.F + P.I$

$$y = (Ax+B)e^{mx} + \frac{x^2}{2} e^{mx}.$$

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(6) Solve: $(D^2+2D+1)y = 2e^{3x}$.

Auxiliary equation is,

$$\Rightarrow m^2+2m+1 = 0.$$

$$\Rightarrow (m+1)(m+1) = 0.$$

$$\Rightarrow m = -1, -1$$

$$C.F = (Ax+B)e^{-x}.$$

$$P.I = \frac{1}{(D^2+2D+1)} 2e^{3x}.$$

$$= \frac{2e^{3x}}{(D+1)(D+1)}$$

$$= \frac{\cancel{2}e^{3x}}{\cancel{16}\ 8}$$

$$= \frac{e^{3x}}{8}.$$

The solution is. $y = C.F + P.I.$

$$y = (Ax+B)e^{-x} + \frac{e^{3x}}{8}.$$

7. Solve: $(D^2 - 6D + 9)y = 2e^{-x} + e^{3x}$.

Auxiliary equation is,

$$m^2 - 6m + 9 = 0.$$

$$9$$
$$-3 \quad -3$$

$$(m-3)(m-3) = 0$$

$$m = 3, 3.$$

$$C.F = (Ax+B)e^{3x}.$$

$$P.I = 2\frac{1}{D^2-6D+9}e^{-x} + \frac{1}{D^2-6D+9}e^{3x}$$

$$= \frac{2e^{-x}}{(D-3)(D-3)} + \frac{e^{3x}}{(D-3)(D-3)}.$$

$$= \frac{\cancel{2}e^{-x}}{\cancel{16}\ 8} + \frac{x^2}{2}e^{3x}.$$

$$= \frac{e^{-x}}{8} + \frac{x^2}{2}e^{3x}.$$

$\therefore$ The solution is,

$$y = C.F + P.I.$$

$$y = (Ax+B)e^{3x} + \frac{e^{-x}}{8} + \frac{x^2}{2}e^{3x}.$$

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(8). Solve : $(D^2 - 3D + 2)y = \sin 3x$.

Solution :-

Auxiliary equation.

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0.$$

$$m = 1 \text{ or } m = 2.$$

$$\boxed{C.F = Ae^{x} + Be^{2x}.}$$

$$P.I = \frac{1}{D^2 - 3D + 2}\sin 3x.$$

$$= \frac{1}{-9-3D+2}\sin 3x \qquad \left[\frac{1}{\phi(D^2)}\sin \alpha x = \frac{1}{\phi(-\alpha)^2}\sin \alpha x\right]$$

$$= \frac{-1}{(3D+7)}\sin 3x$$

$$= \frac{-(3D-7)}{(3D+7)(3D-7)}\sin 3x$$

$$= \frac{-(3D-7)}{9D^2 - 49}\sin 3x.$$

$$= -\frac{(3D-7)}{9(-9)-49} \sin 3x.$$

$$= \frac{1}{130}(3D-7)\sin 3x.$$

$$= \frac{1}{130}\left[3.3\cos 3x - 7\sin 3x\right].$$

$$P.I = \frac{1}{130}\left[9\cos 3x - 7\sin 3x\right].$$

Solution is $y = C.F + P.I.$

$$y = Ae^{x} + Be^{2x} + \frac{1}{130}\left[9\cos 3x - 7\sin 3x\right].$$

(9) $(D^2+16)y = 2e^{-3x} + \cos 4x.$

Auxiliary equation.

$m^2+16 = 0. \Rightarrow m = \sqrt{-16}.$

$m = \pm 4i$

$C.F = A\cos 4x + B\sin 4x.$

$$P.I = \frac{1}{D^2+16}\left[2e^{-3x} + \cos 4x\right]$$

$$= \frac{1}{D^2+16}\left[2e^{-3x}\right] + \frac{1}{D^2+16}\left[\cos 4x\right]$$

$$= \frac{2}{9+16}\left[e^{-3x}\right] + \frac{1}{(D-4i)(D+4i)}\left[R.P \text{ of } e^{i4x}\right]$$

$$= \frac{2}{25}e^{-3x} + R.P \text{ of } \frac{1}{(D-4i)(D+4i)} e^{i4x}.$$

$$= \frac{2}{25}e^{-3x} + R.P \text{ of } \frac{1}{8i(D-4i)} e^{i4x}.$$

$\times$ (i) and $\div$ (i)

$$= R.P \text{ of } \frac{-i}{8} x e^{i4x}.$$

$$\therefore \left[\frac{1}{i} = -i \quad \frac{-1}{i} = i \right]$$

$$= R.P \text{ of } \frac{-i}{8} x(\cos 4x + i\sin 4x)$$

$$= R.P \text{ of } \frac{-ix}{8} \cos 4x + \frac{x \sin 4x}{8}$$

$$= \frac{x \sin 4x}{8}.$$

$$\boxed{P.I = \frac{2}{25} e^{-3x} + \frac{x \sin 4x}{8}}$$

$\therefore$ The Solution is $y = C.F + P.I.$

$$y = A\cos 4x + B\sin 4x + \frac{2}{25} e^{-3x} + \frac{x \sin 4x}{8}.$$

3) $(D^2 - 4D + 3)y = \sin 3x \cos 2x.$

Solution:-

Auxiliary equation.

$$m^2 - 4m + 3 = 0$$

$$(m-1)(m-3) = 0.$$

$$m = 1 \text{ (or) } m = 3$$

3, −1, −3

$$\boxed{C.F = Ae^{x} + Be^{3x}}$$

$$P.I = \frac{1}{D^2 - 4D + 3} [\sin 3x \cos 2x]$$

[$2\sin A \sin B = \sin(A+B) + \sin(A-B)$]

$$= \frac{1}{D^2 - 4D + 3} \left[\frac{\sin 5x + \sin x}{2} \right]$$

$$P.I_1 = \frac{1}{2}\left[\frac{1}{D^2-4D+3}(\sin 5x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}(\sin 5x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}(\sin 5x)\right].$$

$$= \frac{1}{2}\left[\frac{-1}{(4D+22)}(\sin 5x)\right]$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{(4D+22)(4D-22)}(\sin 5x)\right].$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{16D^2-484}(\sin 5x)\right].$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{-400-484}(\sin 5x)\right].$$

$$= \frac{1}{2}\left[\frac{4D-22}{-884}(\sin 5x)\right].$$

$$= \frac{1}{168}\left[4\cdot 5\cos 5x - 22\sin 5x\right]$$

$$= \frac{1}{168}\left[20\cos 5x - 22\sin 5x\right].$$

$$P.I_2 = \frac{1}{2}\left[\frac{1}{D^2-4D+3}(\sin x)\right].$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)(D-3)}\ \text{I.P of } e^{ix}\right]$$

$$= \frac{1}{2}\left[I.P \text{ of } \frac{1}{(D-1)(D-3)} e^{ix} \right]$$

$$= \frac{1}{2}\left[I.P \text{ of } \frac{1}{(-2)(D-1)} e^{ix} \right].$$

$$= \frac{1}{2}\left[I.P \text{ of } \left(\frac{-1}{2}\right) x e^{ix} \right]$$

$$= \frac{1}{2}\left[I.P \text{ of } \left(\frac{-1}{2}\right)(x)(\cos x + i\sin x) \right].$$

$$= \frac{-x}{4}\left[I.P \text{ of } (\cos x + i\sin x) \right]$$

$$= -\frac{x}{4}\cos x + \frac{x}{4} i\sin x.$$

$$P.I_2 = -\frac{x}{4}\sin x.$$

$$P.I = P.I_1 + P.I_2$$

$$= \frac{1}{168}\left[20\cos 5x - 22\sin 5x \right] + \frac{x}{4}\sin x$$

The solution is

$$\boxed{y = C.F + P.I}$$

$$y = (Ae^{x} + Be^{3x}) + \frac{1}{168}\left[20\cos 5x - 22\sin 5x \right] - \frac{x}{4}\sin x.$$

10/01/2020

Model : iii

To Find the particular Integral of.

$X = x^m$

Example :-

1) $(D^2 + D + 1)y = x^2$

Auxiliary equation.

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

$$C.F = e^{-x/2}\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

$$P.I = \frac{1}{(D^2+D+1)} x^2 \qquad (1+x)^{-1} = 1 - x + x^2 \ldots$$

$$= \frac{1}{1+D+D^2} x^2$$

$$= \left[1 + (D + D^2)\right]^{-1} x^2$$

$$= \left[1 - (D+D^2) + (D+D^2)^2 \ldots\right] x^2$$

$$= \left[1 - D - D^2 + D^2 + \ldots\right] x^2$$

$= x^2 - 2x.$

Solution is,

$y = C.F + P.I.$

$y = e^{-x/2}\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + x^2 - 2x.$

H.W

1) $(D^2+4D+5)y = e^x + x^3 + \cos 2x.$

[Ans :- $e^{-2x}[A\cos x + B\sin x] + \frac{1}{5}\left(x^3 - \frac{12}{5}x^2 + \frac{66}{25}x - \frac{144}{125}\right) + (\cos 2x + 8\sin 2x + 65]$

Model - IV

X is of the form.

$e^{ax}V$, where V is a function of x.

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} X.$$

①. Solve $(D^2 - 4D + 3)y = e^{-x} \sin x.$

Auxiliary equation is,

$m^2 - 4m + 3 = 0.$

$(m-1)(m-3) = 0.$

$m = 1$ or $m = 3.$

$CF = Ae^{x} + Be^{3x}.$

$$P.I = \frac{1}{D^2 - 4D + 3}(e^{-x}\sin x).$$

$$= e^{-x}\frac{1}{(D-1)^2 - 4(D-1)+3}\sin x.$$

$$= e^{-x}\frac{1}{D^2 - 2D + 1 - 4D + 4 + 3}\sin x.$$

$$= e^{-x}\frac{1}{D^2 - 6D + 8}\sin x.$$

$$= e^{-x}\frac{1}{-1 - 6D + 8}\sin x.$$

$$= e^{-x}\frac{1}{-6D + 7}\sin x$$

$$= e^{-x}\frac{1}{-(6D - 7)}\frac{(6D+7)}{(6D+7)}\sin x.$$

$$= -e^{-x}\frac{(6D+7)}{36D^2 - 49}\sin x.$$

$$= -e^{-x}\frac{(6D+7)}{-36 - 49}\sin x.$$

$$= \frac{e^{-x}}{85}(6\cos x + 7\sin x)$$

① $(D^2 + 2D + 5)y = xe^x$ $\left[\text{Ans:- } e^{-x}(a\cos 2x + B\sin 2x) + \frac{e^x}{9}(x - \frac{1}{2})\right.$..

(2). Show that the solution of

$$\frac{d^2y}{dt^2} + 4y = A\sin pt \quad y=0 \text{ and } \frac{dy}{dt}=0$$

when $t=0$ is $y = A\dfrac{(\sin pt - \frac{1}{2}p\sin 2t)}{4-p^2}$

if $p \neq 2$ of $p=2$, S.T $y = \dfrac{A(\sin 2t - 2t\cos 2t)}{8}$

Solution:-

Auxiliary equation.

$\Rightarrow m^2+4 = 0$

$m^2 = -4$.

$m = \pm 2i$.

$C.F = [A\cos 2t + B\sin 2t]$

$$P.I = \frac{1}{(D^2+4)} A\sin pt.$$

Case (i) $p \neq 2$

$$P.I = \frac{1}{D^2+4} A\sin pt.$$

$$= \frac{1}{-p^2+4} A\sin pt.$$

Solution is $y = C.F + P.I$

$$y = B\cos 2t + C\sin 2t + \frac{1}{4-p^2} A\sin pt \quad \text{—(1)}.$$

where B & C are Orbitrary Constants.

Given $y=0$ when $t=0$.

Sub in ①,

$$0 = B + 0 + \frac{1}{4-p^2}(0)$$

$$\boxed{B=0}$$

$$① \Rightarrow \frac{dy}{dt} = -2B\sin 2t + 2C\cos 2t + \frac{A}{4-p^2}\, p\cos pt.$$

Given $\frac{dy}{dt}=0$, when $t=0$.

$$0 = 0 + 2C + \frac{A}{4-p^2}\, p$$

$$2C = \frac{-pA}{4-p^2}$$

$$C = -\frac{pA}{2(4-p^2)}$$

Sub in ①

$$y = \frac{-pA}{2(4-p^2)}\sin 2t + \frac{A}{4-p^2}\sin pt.$$

$$y = \frac{A}{4-p^2}\left[\frac{-p}{2}\sin 2t + \sin pt\right]$$

$$y = \frac{A\left(\sin pt - \frac{1}{2}p\sin 2t\right)}{4-p^2}$$

Hence proved.

Case (ii) $P = 2$.

$$P.I = \frac{1}{D^2+4} A\sin 2t.$$

$$= \frac{A}{(D-2i)(D+2i)} [I.P \text{ of } e^{i2t}]$$

$$= \frac{A}{4i(D-2i)} e^{i2t}$$

$$= I.P \text{ of } A\left(\frac{-i}{4} te^{i2t}\right)$$

$$= I.P \text{ of } A\left(\frac{-i}{4} t(\cos 2t + i\sin 2t)\right)$$

$$= I.P \text{ of } A\left(\frac{-i}{4} t\cos 2t + \frac{\sin 2t}{4}\right)$$

$$P.I = \frac{-A}{4} t\cos 2t.$$

Solution is $\boxed{y = C.F + P.I}$

$$y = [B\cos 2t + C\sin 2t] - \frac{A}{4} t\cos 2t \quad (2)$$

B & C are arbitrary constants,

Given $y=0$, when $t=0$

$\therefore \boxed{B=0}$

(2) $\Rightarrow \frac{dy}{dt} = -2B\sin 2t + 2C\cos 2t$

$$- \frac{A}{4}[t(-2\sin 2t) + \cos 2t]$$

$$\frac{dy}{dt} = 0 \quad \text{and} \quad t = 0$$

$$0 = 2C - \frac{A}{4}$$

$$2C = \frac{A}{4}$$

$$\boxed{C = \frac{A}{8}}$$

Sub in (2) we get

$$y = \frac{A \sin 2t}{8} - \frac{A}{4} t \cos 2t$$

$$= \frac{A \sin 2t - 2At \cos 2t}{8}$$

$$y = \frac{A (\sin 2t - 2t \cos 2t)}{8}$$ //

Linear Equation with Variable co-efficients.

Consider the homogenous linear Equation of the second order of the form.

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = X.$$

where a, b, c are constants and 'X' is a f(x)

Method (1)

Put $z = \log x$ (or) $x = e^{z}$

$$\frac{dz}{dx} = \frac{1}{x} \Rightarrow x = \frac{dx}{dz} \Rightarrow \frac{x}{dx} = \frac{1}{dz}$$

Introduce an operator $\theta = x\frac{d}{dx}$.

$$\theta y = x\frac{dy}{dx} = \frac{dy}{dz} = Dy.$$

Then.

(i) $x\frac{dy}{dx} = Dy.$ where $D = \frac{d}{dz}.$

(ii) $x^2\frac{d^2y}{dx^2} = D(D-1)y.$

III^{ly}

(iii) $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y.$

and so on.

Example.

1) Solve $3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x.$ — ①.

Solution :-

put $z = \log x$ and $D = \frac{d}{dz}$ in ①

(ie) $x\frac{dy}{dx} = Dy.$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y.$$

$$x = e^{z}. \quad [z = \log x].$$

Sub in ①,

$$3D(D-1)y + Dy + y = e^{z}$$

$$(3D^2 - 3D + D + 1)y = e^{z}$$

$$(3D^2 - 2D + 1)y = e^{z}$$

Auxiliary equation.

$$3m^2 - 2m + 1 = 0.$$

$$m = \frac{2 \pm \sqrt{4-12}}{6}$$

$$= \frac{2 \pm \sqrt{-8}}{6}$$

$$= \frac{2 \pm \sqrt{4 \times (-2)}}{6}.$$

$$= \frac{\cancel{2}(1 \pm i\sqrt{2})}{\cancel{6}\,3}$$

$$m = \frac{1 \pm i\sqrt{2}}{3} = \frac{1}{3} \pm \frac{i\sqrt{2}}{3}$$

$$C.F = e^{z/3}\left[A\cos\frac{\sqrt{2}}{3}z + B\sin\frac{\sqrt{2}}{3}z\right]$$

$$P.I = \frac{1}{3D^2 - 2D + 1}e^{z}$$

$$= \frac{1}{3-2+1}e^{z}$$

$$= \frac{1}{2}e^{z}$$

The solution is,

$$y = C.F + P.I.$$

$$y = e^{z/3}\left[A\cos\left(\frac{\sqrt{2}}{3}z\right)+B\sin\left(\frac{\sqrt{2}}{3}z\right)\right]+\frac{1}{2}e^{z}$$

$\therefore$ The solution of given equation (1) is.

$$y = e^{\frac{1}{3}\log x}\left[A\cos\left(\frac{\sqrt{2}}{3}\log x\right)+B\sin\left(\frac{\sqrt{2}}{3}\log x\right)\right]+\frac{1}{2}e^{\log x}$$

$$y = x^{1/3}\left[A\cos\left(\frac{\sqrt{2}}{3}\log x\right)+B\sin\left(\frac{\sqrt{2}}{3}\log x\right)\right]+\frac{1}{2}x.$$

---

(2). Solve : $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = \log x.$ — (1)

Solution :-

Put $z = \log x$ and $D = \frac{d}{dz}$ in (1).

(ie) $x\frac{dy}{dx} = Dy$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y.$$

$$x = e^{z}$$

Sub in ①,

$$D(D-1)y + Dy + y = z$$

$$(D^2 - D + D + 1)y = z.$$

$$(D^2+1)y = z.$$

Auxiliary equation.

$$m^2 = -1$$

$$m = \sqrt{-1}$$

$$m = \pm i$$

$$C.F = A\cos z + B\sin z.$$

$$P.I = \frac{1}{D^2+1} z.$$

$$= \frac{1}{1+D^2} z. \qquad (1+x)^{-1} = 1 - x + x^2$$

$$= (1+D^2)^{-1} z.$$

$$= (1 - D^2 + D^4 \ldots) z.$$

$$P.I = z, \ z = \log x$$

The solution is.

$$y = C.F + P.I.$$

$$y = A\cos z + B\sin z + z.$$

Method : 2.

(a). To Find $\frac{1}{\theta-\alpha}X$.

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$$u = \frac{1}{\theta-\alpha}X$$

$$(\theta-\alpha)u = X$$

$$\theta u - \alpha u = X.$$

$$x\frac{du}{dx} - \alpha u = X$$

$$\frac{du}{dx} - \frac{\alpha}{x}u = \frac{X}{x}$$

$$\frac{dy}{dx} + Py = Q.$$

$$u e^{\int(-\frac{\alpha}{x})dx} = \int \frac{X}{x} e^{\int(-\frac{\alpha}{x})dx} dx.$$

$$u e^{-\alpha \log x} = \int \frac{X}{x} e^{-\alpha \log x} dx$$

$$u e^{\log x^{-\alpha}} = \int \frac{X}{x} e^{\log x^{-\alpha}} dx.$$

$$u x^{-\alpha} = \int \frac{X}{x} x^{-\alpha} dx.$$

$$u = x^{\alpha}\int \frac{X}{x} x^{-\alpha} dx$$

$$\boxed{\frac{1}{\theta-\alpha}X = x^{\alpha}\int \frac{X}{x} x^{-\alpha} dx}$$

Method : 2.

$$P.I = \frac{1}{f(\theta)} X.$$

$$= \frac{1}{a(\theta-\alpha_1)(\theta-\alpha_2)} X$$

where $\frac{1}{\theta-\alpha} X = x^{\alpha}\int \cancel{X}\, x^{-\alpha-1}\,dx.$

Example :-

(1) Solve :- $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$ — (1).

Solution :-

Put $z = \log x$ & $\frac{xd}{dx} = \frac{d}{dz} = D$ in (1).

then $\frac{x^2d^2}{dx^2} = D(D-1)$

$$(D(D-1)+3D+1)\,y = \frac{1}{(1-x)^2}$$

$$(D^2-D+3D+1)y = \frac{1}{(1-x)^2}$$

$$(D^2+2D+1)y = \frac{1}{(1-x)^2} \quad \text{—(2)}$$

Auxiliary equation is

$$m^2+2m+1=0.$$

$$(m+1)(m+1)=0$$

$$m=-1,-1$$

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$$C.F = (Ax+B)e^{-x}.$$

To Find P.I

① can be Written as.

$$(\theta^2+2\theta+1)y = \frac{1}{1-x^2}$$

where $\theta = x\frac{d}{dx} = D$

$$(\theta+1)^2 y = \frac{1}{1-x^2}$$

$$P.I = \frac{1}{(\theta+1)^2} \cdot \frac{1}{(1-x)^2}$$

$$= \frac{1}{(\theta+1)}\left[\frac{1}{\theta+1} \cdot \frac{1}{(1-x)^2}\right].$$

$$= \frac{1}{(\theta+1)}\left[x^{-1}\int \frac{1}{(1-x)^2} dx\right] \quad ③.$$

$$\frac{1}{\theta-\alpha} X = x^{\alpha}\int X x^{-\alpha-1} dx.$$

$\int \frac{1}{x^2}dx = \frac{-1}{x}$.

$$= \frac{1}{(\theta+1)}\left[\frac{1}{x}\int \frac{1}{(1-x)^2} dx\right]$$

$$= \frac{1}{(\theta+1)}\left[\frac{1}{x} \cdot \frac{1}{(1-x)} \cdot \frac{1}{(1)}\right]$$

$$= \frac{1}{(\theta+1)}\left[\frac{1}{x(1-x)}\right]$$

$$= x^{-1}\int \frac{1}{x(1-x)} x^{0} dx.$$

$$= x^{-1}\int \frac{1}{x(1-x)}\,dx.$$

$$= x^{-1}\int \left[\frac{1}{x}+\frac{1}{1-x}\right] dx.$$

$$= x^{-1}\left[\log x + \frac{\log(1-x)}{(-1)}\right]$$

$$= x^{-1}\left[\log x - \log(1-x)\right].$$

$y = C.F + P.I.$

20101120. $y = (Ax+B)e^{-x} + x^{-1}[\log x - \log(1-x)]$

1. $(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x$ — (1)

$$z = 5+2x \Rightarrow x = \frac{z-5}{2}.$$

$$\frac{dz}{dx} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = 2\frac{dy}{dz}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d}{dz}\left(2\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= 2\frac{d^2y}{dz^2}\cdot 2. = 4\frac{d^2y}{dz^2}$$

Sub in (1), we get.

$$z^2\, 4\frac{d^2y}{dz^2} - 6z\, 2\frac{dy}{dz} + 8y = \cancel{6}^{3}\left(\frac{z-5}{\cancel{2}}\right)$$

Let $u = \log z$; $z = e^u$

Then $\theta = z\frac{d}{dz} = \frac{d}{du} = D.$

& $x^2 \frac{d^2}{dx^2} = D(D-1)$

Sub in ② we get,

$(4D(D-1) - 12D + 8)y = 3(e^u - 5)$

$(4D^2 - 16D + 8)y = 3e^u - 15$. — ③

Auxiliary equation is,

$4m^2 - 16m + 8 = 0$.

$\div 4$, $m^2 - 4m + 2 = 0$

$m = \frac{4 \pm \sqrt{16 - 8}}{2}$

$= \frac{4 \pm \sqrt{8}}{2}$.

$= \frac{4 \pm 2\sqrt{2}}{2}$.

$m = 2 \pm \sqrt{2}$

$C.F = e^{2u}[A\cos\sqrt{2}u + B\sin\sqrt{2}u]$.

$C.F = Ae^{(2+\sqrt{2})u} + Be^{(2-\sqrt{2})u}$.

$P.I_1 = \frac{1}{4D^2 - 16D + 8} 3e^u$

$= \frac{3}{4 - 16 + 8} e^u$.

$= \frac{-3}{4} e^u$

$$P.I_2 = \frac{1}{4D^2-16D+8}(15)e^0.$$

$$P.I_2. = \frac{15}{8}.$$

$\therefore$ The solution is.

$$P.I = \frac{-3}{4} e^{\log z} - \frac{15}{8}.$$

$$= \frac{-3}{4} z - \frac{15}{8}$$

$$= \frac{-3}{4}(5+2x) - \frac{15}{8}$$

$$= -\frac{15}{4} - \frac{6x}{4} - \frac{15}{8}.$$

$4 \,|\, 4, 8$
$1, 2$

$$= \frac{-30-12x-15}{8}$$

$$= \frac{-12x-45}{8}.$$

$$P.I = -\frac{3x}{2} - \frac{45}{8} \quad \text{——} \quad (4).$$

$$C.F = Ae^{2u}.e^{\sqrt{2}u} + Be^{2u}.e^{-\sqrt{2}u}$$

$$= e^{2u}\left[Ae^{\sqrt{2}u} + Be^{-\sqrt{2}u}\right]$$

$$= e^{2\log z}\left[Ae^{\sqrt{2}\log z} + Be^{-\sqrt{2}\log z}\right]$$

$$= z^2\left[Az^{\sqrt{2}} + Bz^{-\sqrt{2}}\right]$$

$$= (5+2x)^2\left[A(5+2x)^{\sqrt{2}} + B(5+2x)^{-\sqrt{2}}\right] \quad (6)$$

Solution is.

$$y = C.F + P.I$$

$$y = (5+2x)^2 \left[A(5+2x)^{\sqrt{2}} + B(5+2x)^{-\sqrt{2}} \right] - \frac{3x}{2} - \frac{45}{8}$$

Variation of Parameters.

1. Solve :- $\frac{d^2y}{dx^2} + y = \sec x$ — ①

Solution:-

$$m^2 + 1 = 0.$$

$$m = \pm i$$

$$C.F = c_1 \cos x + c_2 \sin x.$$

Assume that,

$$y = V_1 \cos x + V_2 \sin x \quad \text{—— ②}$$

be the solution of ①, where V_1 & V_2 are the function of x.

$$y' = \frac{dy}{dx} = V_1(-\sin x) + V_1' \cos x + V_2 \cos x + V_2' \sin x \quad \text{— ③}$$

Impose the condition,

$$V_1' \cos x + V_2' \sin x = 0 \quad \text{—— ④}.$$

∴ ③ becomes.

$$y' = -V_1 \sin x + V_2 \cos x \quad \text{—— ⑤}$$

$y'' = -v_1 \cos x - v_1' \sin x + v_2(-\sin x)$
$+ v_2' \cos x$ —(6).

Sub (5) & (6) in (1), we get.

$-\cancel{v_1 \cos x} - v_1' \sin x - \cancel{v_2 \sin x} + v_2' \cos x$
$+ \cancel{v_1 \cos x} + \cancel{v_2 \sin x} = \sec x$

$-v_1' \sin x + v_2' \cos x = \sec x$ —(7).

(4) $\Rightarrow$ $v_1' \cos x + v_2' \sin x = 0$.

(7) $\Rightarrow$ $-v_1' \sin x + v_2' \cos x = \sec x$.

(4) $\times \sin x$.

$\cancel{v_1' \sin x \cos x} + v_2' \sin^2 x = 0$.

(5) $\times \cos x \Rightarrow \cancel{-v_1' \sin x \cos x} + v_2' \cos^2 x = 1$.

$v_2'(\sin^2 x + \cos^2 x) = 1$

$\boxed{v_2' = 1}$

$\Rightarrow$ $v_2 = \int dx$

$\boxed{v_2 = x + c_2}$

(4) $\times \cos x$ — (7) $\times \sin x$

$\Rightarrow v_1' \cos^2 x + \cancel{v_2' \cos x \sin x} + v_1' \sin^2 x$
$- \cancel{v_2' \sin x \cos x} = -\sec x \sin x$

$v_1'(1) = \frac{-1}{\cos x} (\sin x)$

$v_1' = -\tan x$.

$$\Rightarrow \boxed{V_1 = \log(\sec x) + c_1}$$

$\therefore$ The solution is. Sub in (2).

$$y = (\log(\sec x) + c_1)\cos x + (x + c_2)\sin x$$

$$y = c_1\cos x + c_2\sin x + \cos\log(\sec x) + x\sin x)$$

---

(2) Solve :- $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$. — (1).

Solution:-

$$m^2 + 1 = 0.$$

$$m = \pm i$$

$$C.F = c_1\cos x + c_2\sin x.$$

Assume that,

$$y = V_1\cos x + V_2\sin x \quad \text{—— (2)}.$$

be the solution of (1), where V_1 & V_2 are the function of x.

$$y' = -V_1\sin x + V_1'\cos x + V_2\cos x + V_2'\sin x.$$

Let $V_1'\cos x + V_2'\sin x = 0$ —— (3).

$$\therefore y' = -V_1\sin x + V_2\cos x.$$

$$y'' = -V_1\cos x - V_1'\sin x - V_2\sin x + V_2'\cos x.$$

Equation (1) becomes,

$$-V_1'\sin x + V_2'\cos x = \operatorname{cosec} x \quad \text{— (4)}.$$

Solve ② & ③ we get,

③ $\times \sin x$ + ④ $\times \cos x$.

$$V_1' \sin x \cos x + V_2' \sin^2 x = 0.$$

$$-V_1' \sin x \cos x + V_2' \cos^2 x = \frac{\cos x}{\sin x}.$$

$$V_2'(1) = \cot x.$$

$$\boxed{V_2 = \log(\sin x) + C_2}.$$

Sub in ③ we get,

$$V_1' \cos x + \cot x \sin x = 0.$$

$$V_1' \cos x + \frac{\cos x}{\cancel{\sin x}} \cdot \cancel{\sin x} = 0$$

$$V_1' \cos x = -\cos x.$$

$$V_1' = -1$$

$$\boxed{V_1 = -x + C_1}$$

∴ The solution is,

$$y = (-x + C_1)\cos x + [\log(\sin x) + C_2]\sin x.$$

$$= -x\cos x + \sin x \log(\sin x) + C_1 \cos x + C_2 \sin x.$$

21/01/2020

③ Solve the following equations:-

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$$

given that x and e^x are the P.I of the equation without the right hand member.

x & e^x are the Solutions of of the equation.

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad \text{—— (2)}.$$

Hence Assume that.

$$y = V_1 x + V_2 e^x \quad \text{—— (3)}.$$

is the Solution of (1), where V_1 & V_2 are functions of x.

$$y' = V_1 + xV_1' + V_2 e^x + V_2' e^x \quad \text{—— (4)}.$$

Impose the condition.

$$V_1' x + V_2' e^x = 0 \quad \text{—— (5)}.$$

we get,

$$y' = V_1 + V_2 e^x \quad \text{—— (6)}.$$

$$y'' = V_1' + V_2 e^x + V_2' e^x \quad \text{—— (7)}.$$

Sub in (1) we get,

$$(x-1)(V_1' + V_2 e^x + V_2' e^x) -$$

$$x(V_1 + V_2 e^x) + V_1 x + V_2 e^x = (x-1)^2$$

$$V_1' x + \cancel{x V_2 e^x} + x V_2' e^x - V_1' - \cancel{V_2 e^x}$$

$$- V_2' e^x - \cancel{x V_1} - \cancel{x V_2 e^x} + \cancel{V_1 x} + \cancel{V_2 e^x} = (x-1)^2$$

$$V_1'(x-1) + V_2'(xe^x - e^x) = (x-1)^2$$

$$V_1'(x-1) + V_2' e^x (x-1) = (x-1)^2$$

$\div (x-1)$

$$V_1' + V_2' e^x = (x-1) \quad \text{——— (8)}.$$

Solving (5) & (8) we get,

$$V_1' x + V_2' e^x = 0.$$

$$\underset{(-)}{V_1'} \overset{(-)}{+} V_2' e^x \overset{(-)}{=} (x-1)$$

$$V_1'(x-1) = (1-x).$$

$$-V_1' = \frac{(1-x)}{(1-x)}$$

$$V_1' = -1$$

$$\boxed{V_1 = -x + c_1} \quad \text{— (9)}.$$

Sub the above value in (5).

$$-x + V_2' e^x = 0.$$

$$V_2' e^x = x.$$

$$V_2' = x/e^x$$

$$V_2' = x e^{-x}.$$

$$V_2 = \int x e^{-x}\, dx.$$

$$V_2 = x(-e^{-x}) + \int e^{-x} dx.$$

$$V_2 = -x e^{-x} - e^{-x} + c_2.$$

└ (10)

Sub (9) and (10) in (3).

$\therefore$ The solution is,

$$y = x(-x + c_1) + e^x(-x e^{-x} - e^{-x} + c_2)$$

$$y = x^2 + x + 1 - x c_1 - e^x c_2$$

4. Solve the following equation, by the method of Variation of Parameters.

$$\frac{d^2y}{dx^2} + n^2 y = \sec nx \quad \text{--- (1)}.$$

Auxiliary equation is,

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm ni.$$

$$C.F = c_1 \cos nx + c_2 \sin nx.$$

$$y = v_1 \cos nx + v_2 \sin nx \quad \text{--- (2)}$$

$$y' = -v_1 n \sin nx + v_1' \cos nx + n v_2 \cos nx + v_2' \sin nx.$$

Impose the condition,

$$v_1' \cos nx + v_2' \sin nx = 0 \quad \text{--- (3)}$$

$$y' = n v_2 \cos nx - v_1 n \sin nx.$$

$$y'' = -n^2 v_2 \sin nx + n v_2' \cos nx - n^2 v_1 \cos nx - v_1' n \sin nx.$$

Sub in (1) we get,

$$\cancel{-n^2 v_2 \sin nx} + n v_2' \cos nx \cancel{- n^2 v_1 \cos nx} - n v_1' \sin nx \cancel{+ n^2 v_1 \cos nx} \cancel{+ n^2 v_2 \sin nx} = \sec nx.$$

$$n v_2' \cos nx - n v_1' \sin nx = \sec nx \rightarrow (4)$$

Solve ③ & ④.
③ × $n\sin nx$ & ④ × $\cos nx$.

$$nv_1' \sin nx \cos nx + nv_2' \sin^2 nx = 0.$$
$$-nv_1' \sin nx \cos nx + nv_2' \cos^2 nx = 1.$$
$$nv_2'(1) = 1$$
$$v_2' = 1/n$$
$$\boxed{v_2 = x/n + c_2.}$$

Sub above value in ③.

$$v_1' \cos nx + \frac{1}{n} \sin nx = 0.$$
$$v_1' \cos nx = \frac{-1}{n} \sin nx.$$
$$v_1' = \frac{-1}{n} \tan nx.$$
$$v_1 = \frac{-1}{n^2} \log(\sec nx) + c_1$$

∴ The solution is,

$$y = \left[\frac{1}{n^2} \log(\cos nx)\right] \cos nx + \left(x/n + c_2\right) \sin nx\Big].$$