UNIT - 3

ODE (Contd...)

Section: 1

SIMULTANEOUS DIFFERENTIAL EQUATIONS

In an ordinary differential equation involving only two variables one of which was taken as the independent and the other as the dependent variable. In the simultaneous differential equations involving more than two variables only one of which, however is the independent variable. A set of such equations is called as set of ordinary simultaneous equations.

Section: 2

Simultaneous equations of the first order and first degree.

Taking z as the independent variable and x, y as the pair of dependent variables, a pair of simultaneous differential equations of the first order and first degree may be written as

$$\left.\begin{aligned} P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0 \\ P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0 \end{aligned}\right\} \longrightarrow (1)$$

where $P_1, Q_1, P_2, Q_2, R_1, R_2$ are functions of x, y and z.

Equation (1) can be written as

$$\left.\begin{aligned} P_1\,dx + Q_1\,dy + R_1\,dz = 0 \\ P_2\,dx + Q_2\,dy + R_2\,dz = 0 \end{aligned}\right\} \longrightarrow (2)$$

The ratios of the differentials dx, dy, dz can be obtained as

$$\frac{dx}{\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix}} = \frac{-dy}{\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix}} = \frac{dz}{\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}}$$

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{-dy}{P_1R_2 - P_2R_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{P_2R_1 - P_1R_2} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \longrightarrow (3)$$

where P, Q, R are functions of x, y, z.

Equation ③ is the standard form for a pair of ordinary simultaneous equations of the first order and first degree

Two independent relations between the variables x, y and z, each involving an arbitrary constant, constitute the general solution of equations ② and so ③.

Section : 3

Solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow$ ①

if $u = c_1$ and $v = c_2$ are the general solution of equation ① where u, v are independent functions of x, y, z then $\phi(u, v) = c_3$ is also the part of general solution of ①.

The general solution of equation ① can be represented as $\phi(u, v) =$ constant (or)

$\phi(u,v)=0$, ϕ being an arbitrary function.

Section: 4

Methods for solving $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \longrightarrow (1)$

CASE (i):

When two of the ratios in equation (1) involve only two out of the three variables x, y and z.

CASE (ii):

A part or the whole of the general solution of equations (1) can be found by multipliers.

1. Solve: $\frac{dx}{yz}=\frac{dy}{xz}=\frac{dz}{xy}$

Solution:

Given: $\frac{dx}{yz}=\frac{dy}{xz}=\frac{dz}{xy} \longrightarrow (1)$

Taking first two equations, we get

$$\frac{dx}{yz}=\frac{dy}{xz}$$

$$\frac{dx}{y} = \frac{dy}{x}$$

$$x\,dx = y\,dy$$

$$x\,dx - y\,dy = 0$$

Integrating, we get,

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c_1}{2}$$

$$x^2 - y^2 = c_1$$

Taking the first and the last terms of the equations, we get

$$\frac{dx}{yz} = \frac{dz}{xy}$$

$$\frac{dx}{z} = \frac{dz}{x}$$

$$x\,dx = z\,dz$$

Integrating, we get

$$\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$x^2 - z^2 = c_2$$

The general solution is

$$\phi(x^2 - y^2, x^2 - z^2) = 0.$$

2. Solve: $\frac{dx}{-y^2-z^2} = \frac{dy}{xy} = \frac{dz}{xz}$

Solution:

given: $\frac{dx}{-y^2-z^2} = \frac{dy}{xy} = \frac{dz}{xz} \longrightarrow (1)$

Taking the last two ratios,

$$\frac{dy}{xy} = \frac{dz}{xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating we get.

$$\log y = \log z + \log c_1$$

$$\log\left(\frac{y}{z}\right) = \log c_1$$

$$y/z = c_1$$

$$\boxed{y = c_1 z}$$

Choosing (x, y, z) as multipliers,

each of the ratios in equation (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(-y^2-z^2) + y(xy) + z(xz)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{-xy^2 - xz^2 + xy^2 + xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$x^2 + y^2 + z^2 = c_2$$

$\therefore$ The general solution of (1) is

$\phi\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0$.

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3. Solve: $\frac{dx}{y-xz} = \frac{dy}{x+yz} = \frac{dz}{x^2+y^2}$

Solution:

Given:

$$\frac{dx}{y-xz} = \frac{dy}{x+yz} = \frac{dz}{x^2+y^2} \longrightarrow (1)$$

Choosing $(y, x, 0)$ as multipliers, each ratio of (1) is

$$= \frac{y\,dx + x\,dy + 0.dz}{y(y-xz) + x(x+yz) + 0(x^2+y^2)}$$

$$= \frac{y\,dx + x\,dy}{y^2 - xyz + x^2 + xyz}$$

$$= \frac{y\,dx + x\,dy}{y^2 + x^2}$$

Taking this and third ratio in ① we get

$$\frac{y\,dx + x\,dy}{x^2+y^2} = \frac{dz}{x^2+y^2}$$

$$y\,dx + x\,dy = dz$$

Integrating, we get

$$\int d(xy) = \int dz$$

$$xy = z + c_1$$

$$xy - z = c_1$$

Choosing $(x, -y, z)$ as multipliers, then each of the ratios in equation ①

$$= \frac{x\,dx - y\,dy + z\,dz}{x(y-xz) - y(x+yz) + z(x^2+y^2)}$$

$$= \frac{x\,dx - y\,dy + z\,dz}{xy - x^2z - xy - y^2z + x^2z + y^2z}$$

$$= \frac{x\,dx - y\,dy + z\,dz}{0}$$

$$\therefore x\,dx - y\,dy + z\,dz = 0$$

Integrating, we get

$$\frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$x^2 - y^2 + z^2 = c_2$$

The general solution is

$$\phi(xy - z,\ x^2 - y^2 + z^2) = 0.$$

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4. Solve: $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz - 2x)}$

Solution:

Given:

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz-2x)} \longrightarrow (1)$$

Taking the first two ratios in (1), we get

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\log x = \log y + \log c_1$$

$$\log (x/y) = \log c_1$$

$$x/y = c_1$$

$$\boxed{x = yc_1}$$

Taking the last two ratios in (1),

we get $\frac{dy}{y^2} = \frac{dz}{x(yz-2x)}$

Put $x = yc_1$

$$\frac{dy}{y^2} = \frac{dz}{yc_1(yz-2yc_1)}$$

$$\frac{dy}{y^2} = \frac{dz}{y^2c_1(z-2c_1)}$$

$$c_1 dy = \frac{dz}{z-2c_1}$$

Integrating, we get

$$c_1 y = \log(z-2c_1) + \log c_2$$

$$c_1 y = \log c_2(z-2c_1)$$

$$c_2(z-2c_1) = e^{c_1 y}$$

$$c_2\left(z-2\frac{x}{y}\right) = e^{x/y \cdot y}$$

$$c_2\left(z-2x/y\right) = e^{x}$$

$$c_2\left(\frac{yz-2x}{y}\right) = e^{x}$$

$$\frac{ye^{x}}{yz-2x} = c_2$$

$\therefore$ The general solution is

$$\phi\left(\frac{x}{y}, \frac{ye^x}{yz-2x}\right) = 0.$$

EXERCISE:

1. Solve: $\dfrac{dx}{x^2+y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{(x+y)z}$

Sol:

Given:

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z} \longrightarrow (1)$$

Choosing $(1,-1,0)$ as multipliers, each fraction of

$$(1) \Rightarrow \frac{dx-dy}{x^2+y^2-2xy} = \frac{dx-dy}{(x-y)^2} \longrightarrow (3)$$

From (2), (3) we get

$$\frac{d(x+y)}{(x+y)^2} = \frac{d(x-y)}{(x-y)^2}$$

$$(x+y)^{-2}\, d(x+y) = (x-y)^{-2}\, d(x-y)$$

Integrating, we get

$$\frac{(x+y)^{-1}}{-1} = \frac{(x-y)^{-1}}{-1} + c_1$$

$$(x-y)^{-1} - (x+y)^{-1} = c_1$$

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$$\frac{1}{x-y} - \frac{1}{x+y} = c_1$$

From last term of ① and from ②,

$$\frac{d(x+y)}{(x+y)^2} = \frac{dz}{(x+y)z}$$

$$\frac{d(x+y)}{(x+y)} = \frac{dz}{z}$$

Integrating,

$$\log(x+y) = \log z + \log c_2$$

$$\log(x+y) - \log z = \log c_2$$

$$\log\left(\frac{x+y}{z}\right) = \log c_2$$

$$\frac{x+y}{z} = c_2$$

The general solution is

$$\phi\left(\frac{1}{x-y} - \frac{1}{x+y}, \frac{x+y}{z}\right) = 0$$

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2. Solve:

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Solution:

Given: $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \rightarrow ①$$

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Choosing $(1,-1,0)$ and $(0,1,-1)$ as multipliers, each fraction of ①

$$= \frac{dx-dy}{y+z-z-x} = \frac{dy-dz}{z-y}$$

$$\frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, we get

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) - \log(y-z) = \log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

Choosing $(1,1,1)$ as multipliers, each given fraction of ①

$$= \frac{dx+dy+dz}{y+z+x+z+x+y}$$

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$$= \frac{d(x+y+z)}{2(x+y+z)}$$

Consider

$$\frac{dx-dy}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)}$$

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)}$$

Integrating,

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) + \log c_2$$

$$\log(x+y+z)^{1/2} + \log(x-y) + \log c_2 = 0$$

$$\log(x+y+z)^{1/2}(x-y) = \log c_2$$

$$(x-y)(x+y+z)^{1/2} = c_2$$

The general solution of 1 is

$$\phi\left(\frac{x-y}{y-z}, (x-y)(x+y+z)^{1/2}\right) = 0$$

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3. Solve: $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$

Solution:

Given: $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2} \longrightarrow (1)$

Choosing $(x, -y, -z)$ as multipliers each fraction of (1)

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)}$$

$$= \frac{x\,dx - y\,dy - z\,dz}{x^2z + xyz - xyz + y^2z - x^2z - y^2z}$$

$$= \frac{x\,dx - y\,dy - z\,dz}{0}$$

$\therefore x\,dx - y\,dy - z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_1}{2}$$

$$x^2 - y^2 - z^2 = c_1$$

Choosing $(y, x, -z)$ as multipliers

each fraction of (1)

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$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)}$$

$$= \frac{y\,dx + x\,dy - z\,dz}{xyz + y^2z + x^2z - xyz - x^2z - y^2z}$$

$$0 = y\,dx + x\,dy - z\,dz$$

Integrating, we get

$$d(xy) - z\,dz = 0$$

$$xy - \frac{z^2}{2} = c_2$$

$$2xy - z^2 = c_2$$

$\therefore$ The general solution of (1) is $\phi(x^2 - y^2 - z^2, \; 2xy - z^2) = 0$.

4. Solve: $\dfrac{dx}{mz - ny} = \dfrac{dy}{mx - lz} = \dfrac{dz}{ly - mx}$

Solution:

Given: $\dfrac{dx}{mz - ny} = \dfrac{dy}{mx - lz} = \dfrac{dz}{ly - mx}$ $\longrightarrow$ (1)

Choosing (l, m, n) as multipliers, each fraction of (1)

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$$\Rightarrow \frac{l\,dx + m\,dy + n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - mlz + nly - mnx}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$\therefore l\,dx + m\,dy + n\,dz = 0$

Integrating, we get

$$lx + my + nz = c.$$

Choosing (x, y, z) as multipliers, each fraction of (1)

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - ylz + ylz - mzx}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore x\,dx + y\,dy + z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$x^2 + y^2 + z^2 = c_2$$

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The general solution of 1 is

$\phi(lx+my+nz,\ x^2+y^2+z^2)=0$.

5. $$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z}$$

Solution:

Given: $$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z} \longrightarrow (1)$$

Taking first two ratios from 1.

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

$$x^{-2}dx=y^{-2}dy$$

Integrating, we get

$$-x^{-1}=-y^{-1}+c_1$$

$$\frac{1}{y}-\frac{1}{x}=c_1$$

Choosing $(1,-1,0)$ as multipliers, each ratio of 1

$$=\frac{dx-dy}{x^2-y^2} \longrightarrow (2)$$

Taking last ratio of 1 and from 2,

$$\frac{dx-dy}{x^2-y^2}=\frac{dz}{(x+y)z}$$

$$\frac{dx-dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{(x-y)} = \frac{dz}{z}$$

Integrating, we get

$$\log(x-y) = \log z + \log c_2$$

$$\log(x-y) - \log z = \log c_2$$

$$\frac{x-y}{z} = c_2$$

The general solution is

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0$$

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6. Solve: $\dfrac{dx}{x^2-yz} = \dfrac{dy}{y^2-zx} = \dfrac{dz}{z^2-xy}$

Solution:

given: $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \longrightarrow (1)$$

Choosing (x, y, z) as $(1,1,1)$ as multipliers, each fraction of (1).

$$\Rightarrow \frac{xdx+ydy+zdz}{x^3-xyz+y^3-xyz+z^3-xyz}$$

$$= \frac{dx + dy + dz}{x^2 - yz + y^2 - zx + z^2 - xy}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}.$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{dx + dy + dz}{\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$\frac{d(x+y+z)}{(x+y+z)} = d(x+y+z)$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)\,d(x+y+z)$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c_1$$

$$x^2 + y^2 + z^2 = (x+y+z)^2 + c_1$$

$$x^2 + y^2 + z^2 - (x+y+z)^2 = c_1$$

$$x^2 + y^2 + z^2 - x^2 - y^2 - z^2 - 2xy - 2yz - 2zx = c_1$$

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$\therefore\ 2xy + 2yz + 2zx = -c_2$

$xy + yz + zx = c_2$

Choosing $(1,-1,0)$ and $(0,1,-1)$ as multipliers each fraction of ①,

$$\Rightarrow \frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy}$$

$$\frac{d(x-y)}{(x^2-y^2)+z(x-y)} = \frac{d(y-z)}{(y^2-z^2)+x(y-z)}$$

$$\frac{d(x-y)}{(x+y)(x-y)+z(x-y)} = \frac{d(y-z)}{(y-z)(y+z)+x(y-z)}$$

$$\frac{d(x-y)}{(x-y)[\cancel{x+y+z}]} = \frac{d(y-z)}{(y-z)[\cancel{x+y+z}]}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, we get

$$\log(x-y) = \log(y-z) + \log c_2$$

$$\log(x-y) - \log(y-z) = \log c_2$$

$$\frac{x-y}{y-z} = c_2$$

The general solution of ① is

$$\phi\left(xy+yz+zx,\ \frac{x-y}{y-z}\right)=0$$

7. Solve: $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

Solution:

Given: $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)} \longrightarrow (1)$

Choosing $(1,1,1)$ as multipliers, each of the fraction (1)

$$= \frac{dx+dy+dz}{xy-xz+yz-xy+xz-yz}$$

$$= \frac{dx+dy+dz}{0}$$

$\therefore dx+dy+dz=0$

Integrating, we get,

$x+y+z = C_1$

Choosing $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ on multipliers each fraction (1)

$$= \frac{dx/x + dy/y + dz/z}{y-z+z-x+x-y}$$

$$= \frac{dx/x + dy/y + dz/z}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

The general solution of (1) is

$$\phi(x+y+z,\ xyz) = 0.$$

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8. Solve : $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{y(z^2-x^2)} = \dfrac{dz}{z(x^2-y^2)}$

Solution :

Given :

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \longrightarrow (1)$$

Choosing (x, y, z) as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$x^2+y^2+z^2=c_1$$

Choosing $(1/x, 1/y, 1/z)$ as multipliers, each fraction of (1)

$$=\frac{dx/x+dy/y+dz/z}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

Integrating,

$$\log x+\log y+\log z=\log c_2$$

$$\log(xyz)=\log c_2$$

$$xyz=c_2$$

The general solution of (1) is

$$\phi(x^2+y^2+z^2, xyz)=0$$

<u>SECTION: 6.</u>

Simultaneous Linear Differential Equations with constant coefficients:

Consider a pair of simultaneous linear differential equations of the form

$$f_1(D)x + \phi_1(D)y = T_1$$
$$f_2(D)x + \phi_2(D)y = T_2$$

where f_1, f_2, ϕ_1, ϕ_2 are rational integral functions of D with constant coefficients, T_1, T_2 explicit functions of t and $D = \frac{d}{dt}$

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1. Solve: $$2\frac{dx}{dt} + x + \frac{dy}{dt} = \cos t$$
$$\frac{dx}{dt} + 2\frac{dy}{dt} + y = 0$$

Solution:

Let $\frac{d}{dt} = D$ then the given equation can be written as

$$2Dx + x + Dy = \cos t$$
$$Dx + 2Dy + y = 0$$
$$\therefore (2D+1)x + Dy = \cos t \rightarrow (1)$$
$$Dx + (2D+1)y = 0 \rightarrow (2)$$

$(1) \times D \Rightarrow (2D+1)Dx + D^2y = D\cos x$
$= -\sin t$

$(2) \times (2D+1) \Rightarrow \underset{(-)}{(2D+1)Dx} + \underset{(-)}{(2D+1)^2 y} = \underset{(-)}{0}$

$$\left[D^2 - (2D+1)^2\right] y = -\sin t$$

$$\left[D^2 - (4D^2 + 1 + 4D)\right] y = -\sin t$$

$$\left[-3D^2 - 4D - 1\right] y = -\sin t$$

$$-\left[3D^2 + 4D + 1\right] y = -\sin t$$

$$(3D^2 + 4D + 1) y = \sin t \longrightarrow (3)$$

Claim: Solve equation (3)

The auxiliary equation is

$$3m^2 + 4m + 1 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4(3)1}}{2(3)}$$

$$m = \frac{-4 \pm \sqrt{16-12}}{6}$$

$$m = \frac{-4 \pm 2}{6}$$

$$m = \frac{-2}{6}, \frac{-6}{6}$$

$$\boxed{m = -\tfrac{1}{3}, -1}$$

Complimentary function is

$$y = Ae^{-t} + Be^{-\frac{1}{3}t}$$

$$P.I = \frac{1}{3D^2 + 4D + 1} \sin t$$

$$= \frac{1}{-3+4D+1} \sin t$$

$\because D^2 = -1$

$$= \frac{1}{4D-2} \sin t$$

$$= \frac{4D+2}{(4D-2)(4D+2)} \sin t$$

$$= \frac{4D(\sin t) + 2\sin t}{16D^2 - 4}$$

$$= \frac{4\cos t + 2\sin t}{16(-1) - 4}$$

$$= \frac{2(2\cos t + \sin t)}{-20}$$

$$P.I = \frac{-1}{10}\left[2\cos t + \sin t\right]$$

$\therefore y = y_c + y_p$

$$y = Ae^{-t} + Be^{-\frac{1}{3}t} - \frac{1}{10}\left[2\cos t + \sin t\right]$$

Diff. equation (4), we get

$$Dy = -Ae^{-t} - \frac{B}{3}e^{-\frac{1}{3}t} - \frac{1}{10}\left[2(-\sin t) + \cos t\right]$$

$$Dy = -Ae^{-t} - \frac{B}{3}e^{-\frac{1}{3}t} + \frac{1}{5}\sin t - \frac{1}{10}\cos t$$

$① - 2 \times ② \Rightarrow$

$$(2D+1)x + Dy = \cos t$$

$$2Dx + 2(2D+1)y = 0$$

$$(2D+1-2D)x + (D-4D-2)y = \cos t$$

$$x + [-3D-2]y = \cos t$$

$$x = \cos t + (3D+2)y$$

$$x = \cos t + (3D+2)\left\{Ae^{-t} + Be^{-t/3} - \frac{1}{10}(\sin t + 2\cos t)\right\}$$

$$x = \cos t + 3A(-1)e^{-t} + 2Ae^{-t} + 3B\left(-\frac{1}{3}\right)e^{-t/3} + 2Be^{-t/3} - \frac{3}{10}\cos t - \frac{2}{10}\sin t - \frac{6}{10}(-\sin t) - \frac{4}{10}\cos t$$

$$x = \cos t - 3Ae^{-t} + 2Ae^{-t} - Be^{-t/3} + 2Be^{-t/3} - \frac{3}{10}\cos t - \frac{1}{5}\sin t + \frac{6}{10}\sin t - \frac{4}{10}\cos t$$

$$= -Ae^{-t} + Be^{-t/3} + \left[\cos t - \frac{3}{10}\cos t - \frac{4}{10}\cos t\right] - \left[\frac{1}{5}\sin t - \frac{6}{10}\sin t\right]$$

$$= -Ae^{-t} + Be^{-t/3} + \left[\frac{10-3-4}{10}\right]\cos t$$

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$-\sin t \left[\frac{2-6}{10}\right]$

$= -Ae^{-t} + Be^{-t/3} + \left[\frac{3}{10}\cos t\right] - \sin t \left[\frac{-4}{10}\right]$

$x = -Ae^{-t} + Be^{-t/3} + \frac{1}{10}\left[3\cos t + 4\sin t\right]$

and

$y = Ae^{-t} + Be^{-t/3} - \frac{1}{10}\left[\sin t + 2\cos t\right]$

---

5. $\frac{dx}{dt} + 5x - 2y = t$; $\frac{dy}{dt} + 2x + y = 0$

Given : $x = 0, y = 0$ when $t = 0$.

Solution :

Let $\frac{d}{dt} = D$.

$Dx + 5x - 2y = t$

$Dy + 2x + y = 0$

$(D+5)x - 2y = t \longrightarrow (1)$

$2x + (D+2)y = 0 \longrightarrow (2)$

Solving (1) & (2),

$\left[4 - (D+2)(D+5)\right] y = 2t$

$\left[4 - (D^2 + 2D + 5D + 10)\right] y = 2t$

$$\left[-D^2 - 7D - 10 - 4\right] y = 2t$$

$$\left[D^2 + 7D + 14\right] y = -2t$$

The auxiliary equation is

$$m^2 + 7m + 14 = 0$$

$$m = \frac{-7 \pm \sqrt{49 - 4(14)1}}{2}$$

$$= \frac{-7 \pm \sqrt{49-56}}{2}$$

$$= \frac{-7 \pm \sqrt{-7}}{2}$$

$$= \frac{-7 \pm i\sqrt{7}}{2}$$

$$m = \frac{-7}{2} \pm \frac{i\sqrt{7}}{2}$$

$$y_c = e^{-7/2\, t}\left[A \cos \frac{\sqrt{7}}{2} t + B \sin \frac{\sqrt{7}}{2} t\right]$$

$$P.I = y_p = \frac{1}{D^2 + 7D + 14} (-2t)$$

$$= \frac{-2}{14\left(1 + \frac{D^2 + 7D}{14}\right)} (t)$$

$$= \frac{-1}{7}\left[1 + \frac{D^2 + 7D}{14}\right]^{-1} (t)$$

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$$= \frac{-1}{7}\left[1 - \left(\frac{D^2+7D}{14}\right) + \frac{(D^2+7D)^2}{14}\right](t)$$

$$= \left(\frac{-1}{7}\right)\left[t - \frac{1}{14}(7)\right]$$

$$= \left(\frac{-1}{7}\right)\left[t - \frac{1}{2}\right]$$

$$y_p = \frac{1}{14} - \frac{t}{7}$$

$$\therefore\ y = y_c + y_p$$

$$y(t) = e^{-7/2\, t}\left[A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right] + \left(\frac{1}{14} - \frac{t}{7}\right) \longrightarrow (3)$$

If $t = 0 \Rightarrow x = 0,\ y = 0$

$$0 = A + \frac{1}{14}$$

$$A = -\frac{1}{14}$$

(2) $\Rightarrow 2x = -y - Dy$

$$2x = \left\{-e^{-7/2\, t}\left[A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right] + \left(\frac{1}{14} - \frac{t}{7}\right)\right\} -$$

$$\left\{\left(-\frac{7}{2}\right)e^{-7/2\, t}\left[A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right] + e^{-7/2\, t}\left[\frac{-\sqrt{7}}{2}A\sin\frac{\sqrt{7}}{2}t + B\frac{\sqrt{7}}{2}\cos\frac{\sqrt{7}}{2}t\right] - \frac{1}{7}\right\}$$

$$2x = \left[A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right]\left[-1-\tfrac{7}{2}\right] + \frac{1}{14} - \frac{t}{7} - \frac{1}{7} \quad e^{-7/2\,t}\left[-\frac{\sqrt{7}}{2}A\sin\frac{\sqrt{7}}{2}t + \frac{\sqrt{7}}{2}B\cos\frac{\sqrt{7}}{2}t\right]$$

$$2x = \left(-\frac{9}{2}\right)\left[A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right] + e^{-7/2\,t}\left[\left(-\frac{\sqrt{7}}{2}\right)A\sin\frac{\sqrt{7}}{2}t + \frac{\sqrt{7}}{2}B\cos\frac{\sqrt{7}}{2}t\right] - \left(t/7 + 1/7\right)$$

$$x = \frac{1}{2}\left\{\left(-9/2\right)\left[A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right] + e^{-7/2\,t}\left[\left(-\frac{\sqrt{7}}{2}\right)A\sin\frac{\sqrt{7}}{2}t + \frac{\sqrt{7}}{2}B\cos\frac{\sqrt{7}}{2}t - \left(t/7 + 1/7\right)\right]\right\} \longrightarrow (4)$$

If $t = 0 \Rightarrow x = 0$

$$0 = \frac{1}{2}\left\{\left(-9/2\right)A + \frac{\sqrt{7}}{2}B - \frac{1}{7}\right\}$$

$$0 = \frac{1}{2}\left\{\left(-9/2\right)\left(-1/14\right) + \left(\frac{\sqrt{7}}{2}\right)B - \frac{1}{7}\right\}$$

$$\left(\frac{1}{2}\right)\left(\frac{\sqrt{7}}{2}\right)B = -\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{1}{14} + \frac{1}{2}\cdot\frac{1}{7}$$

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$$= \frac{1}{2}\left[\frac{1}{7} - \frac{9}{28}\right]$$

$$\frac{1}{2}\left(\frac{\sqrt{7}}{2}\right)B = \frac{1}{2}\left[\frac{4-9}{28}\right]$$

$$\therefore B = \frac{-5}{14\sqrt{7}}$$

Substitute the values of A, B in (3), (4), we get $x(t), y(t)$.

TOTAL DIFFERENTIAL EQUATIONS

In a total differential equation, we have the differential coefficients of several dependent variables with reference to a single independent variable. Such an equation in three variables is represented by

$$Pdx + Qdy + Rdz = 0 \longrightarrow (1)$$

where P, Q, R are functions of x, y, z.

Let (1) have an integral

$$u(x, y, z) = c \longrightarrow (2)$$

where c is an arbitrary constant.

Then diff. ② totally,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \rightarrow ③$$

comparing ① and ③,

$$\frac{\partial u}{\partial x} = \mu P, \quad \frac{\partial u}{\partial y} = \mu Q, \quad \frac{\partial u}{\partial z} = \mu R$$

where μ is a function of x, y, z.

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(\mu Q)$$

$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \quad \rightarrow ④$$

Similarly,

$$\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \quad \rightarrow ⑤$$

and

$$\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x} \quad \rightarrow ⑥$$

Multiplying equations ④, ⑤, ⑥, by R, P and Q, respectively, and adding, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \longrightarrow (7)$$

The condition (7) is the integrability of eq. (1). Also eq. (7) is the necessary and sufficient condition for the existence of the integral of eq. (1).

PROBLEMS:

1. $(y^2+yz)dx+(xz+z^2)dy+(y^2-xy)dz=0$

Solution:

Given:

$$(y^2+yz)dx+(xz+z^2)dy+(y^2-xy)dz=0 \longrightarrow (1)$$

Claim: Condition of integrability

$P=y^2+yz \quad Q=xz+z^2 \quad R=y^2-xy$

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

$\frac{\partial P}{\partial x} = 0 \qquad \frac{\partial Q}{\partial x} = z \qquad \frac{\partial R}{\partial x} = -y$

$\frac{\partial P}{\partial y} = 2y + z \qquad \frac{\partial Q}{\partial y} = 0 \qquad \frac{\partial R}{\partial y} = 2y - x$

$\frac{\partial P}{\partial z} = y \qquad \frac{\partial Q}{\partial z} = x + 2z \qquad \frac{\partial R}{\partial z} = 0$

$(y^2+yz)\;[(x+2z)-(2y-x)] +$

$(xz+z^2)\,[(-y)-(y)] + (y^2-xy)[(2y+z)-z]$

$= (y^2+yz)\;[2x-2y+2z] + (xz+z^2)(-2y) +$

$(y^2 - xy)[2y]$

$= 2xy^2 + 2xyz - 2y^3 - 2y^2z + 2y^2z +$

$2yz^2 - 2xyz - 2yz^2 + 2y^3 - 2xy^2$

$= 0$

The condition of integrability is satisfied.

Let us assume $z =$ constant.

$dz = 0$

$\therefore$ Neglecting $(y^2 - xy)dz = 0$

① $\Rightarrow (y^2+yz)dx + (xz+z^2)dy = 0$

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$$y(y+z)\,dx = -z(x+z)\,dy$$

$$\frac{dx}{x+z} = \frac{-z\,dy}{y(y+z)}$$

$$\frac{dx}{x+z} + \frac{z\,dy}{y(y+z)} = 0$$

Consider,

$$\frac{1}{y(y+z)} = \frac{A}{y} + \frac{B}{y+z} = \frac{A(y+z)+By}{y(y+z)}$$

$$1 = A(y+z) + By$$

$$y = 0 \Rightarrow Az = 1$$
$$A = 1/z$$

$$y = -z \Rightarrow 1 = -Bz$$
$$B = -1/z$$

$$\frac{1}{y(y+z)} = \frac{1/z}{y} + \frac{(-1/z)}{y+z}$$

$$= \frac{1}{zy} - \frac{1}{z(y+z)}$$

$$\frac{dx}{x+z} + z\left(\frac{1}{zy} - \frac{1}{z(y+z)}\right) dy = 0$$

$$\frac{dx}{x+z} + \left(\frac{1}{y} - \frac{1}{y+z}\right) dy = 0$$

($\because$ z is a constant).

Integrating, we get

$$\log(x+z) + \log y - \log(y+z) = \log c$$

$$\log(x+z) + \log\left(\frac{y}{y+z}\right) = \log c$$

$$\log(x+z)\left(\frac{y}{y+z}\right) = \log c$$

$$(x+z)\left(\frac{y}{y+z}\right) = c = f(z) \rightarrow (2)$$

Diff eq (2) totally w.r.to x, y, z, we get

$$\frac{(y+z)\left[(x+z)dy + y(dx+dz)\right] - y(x+z)\left[dy+dz\right]}{(y+z)^2} = f'(z)dz$$

$$\left.\begin{array}{l}(y+z)(x+z)dy + y(y+z)(dx+dz) - \\ y(x+z)(dy+dz)\end{array}\right\} = (y+z)^2 f'(z)dz$$

$$(y+z)(x+z)dy + y(y+z)dx + y(y+z)dz$$
$$- y(x+z)dy - y(x+z)dz - (y+z)^2 f'(z)dz = 0$$

$$y(y+z)dx + \left[(y+z)(x+z) - y(x+z)\right]dy$$
$$+ \left[y(y+z) - y(x+z) - (y+z)^2 f'(z)\right]dz = 0$$

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$(y^2+yz)\,dx + \left[(x+z)(y+z-y)\right] dy +$

$\left[y^2+yz-yx-yz-(y^2+z^2+2yz)\right.$

$\left. f'(z)\right] dz = 0$

$(y^2+yz)dx + \left[z(x+z)\right] dy + \left[y^2-xy-f'(z)\right.$

$\left.(y+z)^2\right] dz = 0 \longrightarrow (3)$

Comparing (1) & (3),

$y^2-xy-f'(z)(y+z)^2 = (y^2-xy)$

$-f'(z)\,(y+z)^2 = 0$

$f'(z) = 0$

$f(z) = c$

(2) $\Rightarrow$ $\dfrac{y(x+z)}{y+z} = c$

The integral of (1) is

$y(x+z) = c(y+z)$

---

2. Show that the equation

$(x^2y-y^3-y^2z)\,dx + (xy^2-x^2z-x^3)$

$dy + (xy^2+x^2y)\,dz = 0.$

Solution:

Condition of integrability is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+$$
$$R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$$

$P=x^2y-y^3-y^2z \qquad Q=xy^2-x^2z-x^3$

$\frac{\partial P}{\partial x}=2xy \qquad \frac{\partial Q}{\partial x}=y^2-2xz-3x^2$

$\frac{\partial P}{\partial y}=x^2-3y^2-2yz \qquad \frac{\partial Q}{\partial y}=2xy$

$\frac{\partial P}{\partial z}=-y^2 \qquad \frac{\partial Q}{\partial z}=-x^2$

$R=xy^2+x^2y$

$\frac{\partial R}{\partial x}=y^2+2xy$

$\frac{\partial R}{\partial y}=2xy+x^2$

$\frac{\partial R}{\partial z}=0$

$(x^2y-y^3-y^2z)\left[-x^2-2xy-x^2\right]+$

$(xy^2-x^2z-x^3)\left(y^2+2xy+y^2\right)+$

$(xy^2+x^2y)\left[x^2-3y^2-2yz-y^2+2xz+3x^2\right]$

$=(x^2y-y^3-y^2z)\left[-2x^2-2xy\right]+$

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$$(xy^2 - x^2z - x^3)\,[2y^2 + 2xy] + (xy^2 + x^2y)\,[4x^2 - 4y^2 - 2yz + 2xz]$$

$$= -2x^4y + 2x^2y^3 + 2x^2y^2z - 2x^3y^2 + 2xy^4 + 2xy^3z + 2xy^4 - 2x^2y^2z - 2x^3y^3 + 2x^2y^3 - 2x^3yz - 2x^4y + 4x^3y^2 + 4x^4y - 4xy^4 - 4x^2y^3 - 2xy^3z - 2x^2y^2z + 2x^2y^2z + 2x^3yz$$

$$= 0$$

The condition of integrability is satisfied for eq. (1).

Let us assume $y =$ constant

$dy = 0$

(1) $\Rightarrow$

$$(x^2y - y^3 - y^2z)\,dx + (xy^2 + x^2y)\,dz = 0$$

$$(x^2y - y^3 - y^2z)\,dx = -(xy^2 + x^2y)\,dz$$

$$y\,(x^2 - y^2 - yz)\,dx = -y\,(xy + x^2)\,dz$$

$$\frac{-dz}{dx} = \frac{x^2 - y^2 - yz}{x(x+y)}$$

$$= \frac{x^2 - y^2}{x(x+y)} - \frac{yz}{x(x+y)}$$

$$= \frac{(x-y)(x+y)}{x(x+y)} - \frac{yz}{x(x+y)}$$

$$-\frac{dz}{dx} = \frac{x-y}{x} - \frac{yz}{x(x+y)}$$

$$\frac{dz}{dx} = \frac{yz}{x(x+y)} - \frac{x-y}{x}$$

$$\frac{dz}{dx} - \frac{yz}{x(x+y)} = \frac{y-x}{x}$$

This is linear in z.

$$\left[\because \frac{dy}{dx} + Py = Q \right.$$

$$\left. y e^{\int P dx} = \int Q e^{\int P dx} dx + c \right]$$

$$\therefore z e^{\int \frac{-y}{x(x+y)} dx} = \int \frac{y-x}{x} e^{\int -\frac{y}{x(x+y)} dx} dx + c$$

Consider,

$$\frac{y}{x(x+y)} = \frac{A}{x} + \frac{B}{x+y}$$

$$y = A(x+y) + Bx$$

$x = -y$, $\quad y = B(-y)$

$$\therefore \boxed{B = -1}$$

$x = 0 \Rightarrow \quad y = Ay$

$$\boxed{A = 1}$$

$$\frac{y}{x(x+y)} = \frac{1}{x} - \frac{1}{x+y}$$

$$\therefore Ze^{-\int\left(\frac{1}{x} - \frac{1}{x+y}\right)dx} = \int \frac{y-x}{x} e^{-\int\left(\frac{1}{x} - \frac{1}{x+y}\right)dx} dx + c$$

$$Ze^{-\log x + \log(x+y)} = \int\left(\frac{y-x}{x}\right) e^{-\log x + \log(x+y)} dx + c$$

$$Z e^{\log\left(\frac{x+y}{x}\right)} = \int\left(\frac{y-x}{x}\right) e^{\int \log\left(\frac{x+y}{x}\right)} dx + c$$

$$z\left(\frac{x+y}{x}\right) = \int\left(\frac{y-x}{x}\right) \cdot \left(\frac{x+y}{x}\right) dx + c$$

$$= \int \frac{y^2 - x^2}{x^2} dx + c$$

$$= \int\left(\frac{y^2}{x^2} - 1\right) dx + c$$

$$= y^2 \int x^{-2} dx - \int dx + c$$

$$= y^2 \left(\frac{x^{-1}}{-1}\right) - x + c$$

$$z\left(\frac{x+y}{x}\right) = \frac{-y^2}{x} - x + c$$

Let $c = f(y)$

$$z\left(\frac{x+y}{x}\right) = -x - \frac{y^2}{x} + f(y)$$

$$zx + zy = -x^2 - y^2 + x f(y)$$

$$zx + zy + x^2 + y^2 = x f(y) \longrightarrow (2)$$

Diff. eq (2) totally w.r.to x, y, z, we get

$$x\,dz + z\,dx + y\,dz + z\,dy + 2x\,dx + 2y\,dy = x f'(y)\,dy + f(y)\,dx$$

$$x\,dz + z\,dx + y\,dz + z\,dy + 2x\,dx + 2y\,dy - x f'(y)\,dy - f(y)\,dx = 0$$

$$[z + 2x - f(y)]\,dx + [z + 2y - x f'(y)]\,dy + (x+y)\,dz = 0 \longrightarrow (3)$$

(3) $\times\, xy \Rightarrow$

$$[xyz + 2x^2y - xy f(y)]\,dx + [xyz + 2xy^2 - x^2y f'(y)]\,dy + (x^2y + xy^2)\,dz = 0 \longrightarrow (4)$$

(1) $-$ (4) $\Rightarrow$

$$[x^2y - y^3 - y^2z - xyz - 2x^2y + xy f(y)]\,dx$$

$+ \left[xy^2 - x^2z - x^3 - xyz - 2xy^2 + x^2yf'(y)\right]dy + \left(xy^2 + x^2y - x^2y + xy^2\right)$
$dz = 0$

$\left[x^2y - xyz - y^3 - y^2z + xyf(y)\right]dx +$
$\left[-xy^2 - x^2z - x^3 - xyz + x^2yf'(y)\right]dy$
$+ (0)dz = 0$

$\left[x^2y + xyz + y^3 + y^2z - xy\,f(y)\right]dx +$
$\left[xy^2 + x^2z + x^3 + xyz - x^2y\,f'(y)\right]dy = 0$

$\left[y\left[(x^2 + xz + y^2 + yz) - xf(y)\right]\right]dx +$
$\left[x\left(y^2 + xz + x^2 + yz - xy\,f'(y)\right)\right]dy = 0$

Using equation 2, we get

$\left[y\left(xf(y) - xf(y)\right)\right]dx +$
$\left[x\left(xf(y) - xyf'(y)\right)\right]dy = 0$

$\left[x^2f(y) - x^2yf'(y)\right]dy = 0$

$dy \neq 0 \Rightarrow x^2f(y) - x^2y\,f'(y) = 0$

$x^2f(y) = x^2y\,f'(y)$

$$\frac{f'(y)}{f(y)} = \frac{1}{y}$$

$$\frac{\frac{d f(y)}{dy}}{f(y)} = \frac{1}{y}$$

$$\frac{df}{f} = \frac{dy}{y}$$

Integrating on both sides, we get

$\log f = \log y + \log c$ { c is an arbitrary constant).

$\log f = \log yc$

$f = yc$

(2) $\Rightarrow zx + zy + x^2 + y^2 = x(yc)$

$\therefore$ The solution of given equation (1) is

$$zx + zy + x^2 + y^2 = xyc$$

---

3. Solve:

$$z\,dz + (x-a)\,dx = \{h^2 - z^2 - (x-a)^2\}^{1/2}\,dy \rightarrow (1)$$

Solution:

$$\frac{z\,dz + (x-a)\,dx}{\{h^2 - z^2 - (x-a)^2\}^{1/2}} = dy \longrightarrow (2)$$

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Let $h^2 - z^2 - (x-a)^2 = u$

$-2z\,dz - 2(x-a)\,dx = du$

$-2\left[z\,dz + (x-a)\,dx\right] = du$

$z\,dz + (x-a)\,dx = -\dfrac{du}{2}$

$\therefore$ (2) $\Rightarrow$ $\dfrac{\left(-du/2\right)}{u^{1/2}} = dy$

$\left(-\frac{1}{2}\right) u^{-1/2}\,du = dy$

Integrating on both sides, we get

$\left(-\frac{1}{2}\right) \dfrac{u^{(-1/2+1)}}{(-1/2+1)} = y + c$

$\left(-1/2\right) \dfrac{u^{1/2}}{1/2} = y + c$

$-u^{1/2} = y + c$

$-\left\{h^2 - z^2 - (x-a)^2\right\}^{1/2} = y + c$

Squaring on both sides, we get

$h^2 - z^2 - (x-a)^2 = (y+c)^2$

$(y+c)^2 + z^2 + (x-a)^2 = h^2$

$\therefore$ Solution of (1) is

$(y+c)^2 + (x-a)^2 + z^2 = h^2$

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