UNIT - 4

PARTIAL DIFFERENTIAL EQUATIONS

Section: 1

Partial differential equations are those which involve one or more partial derivatives. The order of the partial differential equation is determined by the highest order of the partial ~~dif~~ derivative occuring in it.

In this chapter, we consider the partial differential equations involving one dependent variable 'z' and only 2 independent variables x and y. We shall denote

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial y^2} = t \quad \frac{\partial^2 z}{\partial x \cdot \partial y} = s.$$

Section : 2

Derivation of partial differential equations.

Partial Differential equations can be derived either by the elimination of :

i) arbitrary constants from a relation between x, y, z.

ii) of arbitrary functions of these variables.

Section : 2.1

By elimination of arbitrary constants.

Consider the function

$$f(x, y, z, a, b) = 0 \longrightarrow (1)$$

containing two in dependent arbitrary constants a and b. To eliminate two constants, we require three equations.

Differentiating equation (1) partially w. r. to x and y, we get

$x \Rightarrow$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \longrightarrow (2)$$

$y \Rightarrow$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0 \longrightarrow (3)$$

Eliminating a and b from equations (1) and (2) and (3), we get a partial differential equations of the first order of the form $F(x, y, z, p, q) = 0$.

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1. Eliminate a and b from $z = (x+a)(y+b)$.

Solution:

Given: $z = (x+a)(y+b) \longrightarrow (1)$

Diff eq. (1). partially w.r.to x and y,

$$\frac{\partial z}{\partial x} = p = (1+0)(y+b) = (y+b)$$

$$\frac{\partial z}{\partial y} = q = (x+a)(1+0) = (x+a)$$

Put $p = y+b$, $q = x+a$, in ①

$z = pq$ which is the required Partial differential equation.

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2. Obtain the P.D.E. of all spheres whose centres lie on the plane $z=0$ and whose radius is constant and equal to r.

Solution:

The cartesian eq. of all such spheres can be written in the form

$$(x-a)^2 + (y-b)^2 + (z-0)^2 = r^2$$

$$(x-a)^2 + (y-b)^2 + z^2 = r^2 \longrightarrow ①$$

where a, b are arbitrary constants and r is the fixed given constant.

Diff ① partially w.r.to x, y,

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$2(x-a) = -2zp$$

$$(x-a) = -zp \longrightarrow ②$$

$$2(y-b)+2z\cdot\frac{\partial z}{\partial y}=0$$

$$2(y-b)=-2z\frac{\partial z}{\partial y}$$

$$(y-b)=-zq \longrightarrow (3)$$

Sub (2), (3) in (1) we get

$$(-zp)^2+(-zq)^2+z^2=r^2$$

$$z^2p^2+z^2q^2+z^2=r^2$$

$$z^2(p^2+q^2+1)=r^2.$$

This is the required P.D.Eq.

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3. Find the differential equation of all spheres whose centres lie on the z-axis.

Solution:

The equation of spheres with centres on the z-axis is

$$x^2+y^2+(z-c)^2=r^2 \longrightarrow (1)$$

where c and r are arbitrary constants.

Diff eq. (1) partially w.r.to x, y,

$$2x+2(z-c)\cdot\frac{\partial z}{\partial x}=0$$

$$2x=-2(z-c)p$$

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$$\frac{-x}{p} = (z-c) \longrightarrow (2)$$

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$2y = -2(z-c)q$$

$$\frac{-y}{q} = (z-c) \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{-x/p}{-y/q} = \frac{z-c}{z-c}$$

$$-x/p = -y/q$$

$qx = yp$ which is the required P.D. Eq.

BY THE ELIMINATION OF ARBITRARY FUNCTIONS:

Let u and v be any two functions of x, y, z and be connected by an arbitrary relation.

$$\phi(u, v) = 0 \longrightarrow (1)$$

Diff eq. (1) partially w.r.to x, y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right)=0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)=-\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) \longrightarrow (2)$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=-\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}=\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}$$

$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)$$

$$\boxed{Pp+Qq=R} \longrightarrow (4)$$

where

$$P=\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}=\frac{\partial(u,v)}{\partial(y,z)}$$

$$Q=\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)}$$

$$R=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)}$$

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Eq (4) is known as lagrange's linear equation.

1. Eliminate the arbitrary function from $z = f(x^2+y^2)$

Sol: Given equation is

$$z = f(x^2+y^2) \longrightarrow (1)$$

Diff. eq. (1) partially w.r. to x, y

$$\frac{\partial z}{\partial x} = p = f(x^2+y^2) \cdot 2x \longrightarrow (2)$$

$$\frac{\partial z}{\partial y} = q = f'(x^2+y^2) \cdot 2y \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'(x^2+y^2)}{f'(x^2+y^2)} \cdot \frac{2x}{2y}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py = xq$$

2. Eliminate the arbitrary functions f and ϕ from the relation $z = f(x+ay) + \phi(x-ay)$.

Solution :

Given : $z = f(x+ay) + \phi(x-ay)$ $\longrightarrow (1)$

Diff . eq (1) partially w. r. to x, y.

$$\frac{\partial z}{\partial x} = p = f'(x+ay)(1) + \phi'(x-ay)1$$

$$\frac{\partial^2 z}{\partial x^2} = r = f''(x+ay)(1) + \phi''(x-ay)(1)$$

$$\frac{\partial z}{\partial y} = q = f'(x+ay).a + \phi'(x-ay)(-a)$$

$$= a\left[f'(x+ay) - \phi'(x-ay)\right]$$

$$\frac{\partial^2 z}{\partial y^2} = t = a\left[f''(x+ay).a - \phi''(x-ay)(-a)\right]$$

$$t = a^2\left[f''(x+ay) + \phi''(x-ay)\right]$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = a^2\left[f''(x+ay) + \phi''(x-ay)\right]$$

$$= a^2 \frac{\partial^2 z}{\partial x^2}$$

$$\therefore t = a^2 r.$$

SECTION : 3

DIFFERENT INTEGRALS OF PARTIAL DIFF. EQS.

Definition:

SOLUTION (OR) INTEGRAL

A solution or integral of a p.d.eq. is a relation b/w the dependent and the independent variables that satisfied the differential eq.

SECTION : 4

Solutions of P.D.E in some simple cases:

1. Solve: $\frac{\partial^2 z}{\partial x \partial y} = 0$

Sol: Given: $\frac{\partial^2 z}{\partial x \partial y} = 0 \longrightarrow (1)$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0$$

Integrating w.r.to 'x', we get

$$\frac{\partial z}{\partial y} = f(y) \longrightarrow (2)$$

Integrating (2), $z = \int f(y)dy + \phi(x)$

$$z = F(y) + \phi(x)$$

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where $F(y)$, $\phi(x)$ are arbitrary functions.

TYPE - I

1. Solve : $\frac{\partial^2 z}{\partial x^2} = a^2 z$ Given that when $x=0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$.

Solve:

1. $\frac{\partial^2 z}{\partial y^2} = \sin y$

Sol :-

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \sin y$$

Integrating,

$$\frac{\partial z}{\partial y} = \cos y + f(x)$$

Again integrating

$$z = \sin y + \int f(x)\, dx + g(x)$$

which is the solution of the given equation

TYPE - II

1. Solve: the system of equations $\frac{\partial z}{\partial x} = 4x - 2y$ and $\frac{\partial z}{\partial y} = -2x + 6y$.

Sol: $\frac{\partial z}{\partial y} = -2x + 6y \rightarrow (2)$

Integrating

$z = 2x^2 - 2xy + f(y) \rightarrow (3)$

Diff (3) partially w.r.to y

$\frac{\partial z}{\partial y} = -2x + f'(y) \rightarrow (4)$

Comparing eq (2) and (4), we get

$-2x + 6y = -2x + f'(y)$

$6y = f'(y) \rightarrow (5)$

Integrating (5),

$\int f'(y)\,dy = \int 6y\,dy$

$f(y) = 6\,y^2/2$

$f(y) = 3y^2 + c$

(3) $\Rightarrow z = 2x^2 - 2xy + 3y^2 + c$

TYPE - III

1. Solve the equation $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial z}{\partial x} + 3z = e^{3x}$

Sol:

$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial z}{\partial x} + 3z = e^{3x} \rightarrow (1)$

It can be written in the form

$$\left(\frac{\partial}{\partial x}-1\right)\left(\frac{\partial}{\partial x}-3\right)z = e^{2x}$$

Hence the complementary function.

$$z = f(y)e^{x} + \phi(y)e^{3z}$$

$$P.I = \frac{1}{(D-1)(D-3)} e^{3xy}$$

$$= \frac{1}{(2-1)(2-3)} e^{2x}$$

$$P.I = -e^{2x}$$

The general solution is

$$z = f(y)e^{x} + \phi(y)e^{2x} - e^{2x}.$$

STANDARD TYPES OF FIRST ORDER EQUATIONS

Many of the important equations of the order that occur in practice are one or the other of the following standard forms.

STANDARD TYPE 1.

The variables x, y, z do not occur explicitly.

Such equations are of the form $f(p, q) = 0 \longrightarrow (1)$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

The solution of equation (1) is of the form

$$z = ax + by + c \longrightarrow (2)$$

$(1) \Rightarrow f(a, b) = 0 \longrightarrow (3)$

Solving (3) for b, $b = f(a)$

$$z = ax + F(a)y + c \longrightarrow (4)$$

The complete integral is

obtained by eliminating a and c between

$$z = ax + y(a) + c$$

$$\frac{\partial z}{\partial a} = 0 \Rightarrow 0 = x + y F(a) \rightarrow (5)$$

$$\frac{\partial z}{\partial b} = 0 \Rightarrow 0 = 1 \rightarrow (6)$$

In equation (6), there is no singular integral in this case.

To obtain the general integral we assume an arbitrary solution

$$c = \phi(a)$$

$$(4) \Rightarrow z = ax + y F(a) + \phi(a) \rightarrow (7)$$

Differentiating equation (7) partially with respect to a, we get

$$0 = x + F'(a)y + \phi'(a)$$

The elimination between equations (7) and (8) we get the general integral.

EXAMPLE 1 :

Solve : $p^2+q^2=npq$

Solution :

Given equation is

$$p^2+q^2=npq \longrightarrow (1)$$

This equation is of the form $f(p,q)=0$.

The solution of equation (1) is

$$z=ax+by+c \longrightarrow (2)$$

Substitute $p=a$, $q=b$ in equation (1), we get

$$a^2+b^2=nab$$

$$b^2-nab+a^2=0$$

$$b=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$b=\frac{na\pm\sqrt{n^2a^2-4a^2}}{2}$$

$$b=\frac{-a\left[n\pm\sqrt{n^2-4}\right]}{2}y+c \longrightarrow (3)$$

Claim singular integral

Differentiating equation (3) partially with respect to 'c', we get

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$0 = 1$

Equation is absurd.

There is no singular integral for given equation ①.

Claim general integral

Put $c = f(a)$ in equation ③, we get

$$z = ax + \frac{ay}{2}\left[n \pm \sqrt{n^2-4}\right] + f(a) \longrightarrow ④$$

Differentiating equation ② partially with respect to 'a', we get

$$0 = x + \frac{y}{2}\left[n \pm \sqrt{n^2-4}\right] + f'(a) \longrightarrow ⑤$$

Eliminating a between equations ③, ④, ⑤ we get

The general integral.

EXERCISES 23

Solve the following equations.

1. $p^2 + q^2 = 4$

Solution:

Given equation $p^2 + q^2 = 4 \longrightarrow ①$

This equation is of the form $f(p,q)=0$

The solution of equation (1) is

$z = ax + by + c \longrightarrow (2)$

Substitute $p=a$, $q=b$ in equation (1), we get

$$a^2+b^2=4$$

$$b^2=4-a^2$$

$$b=\pm\sqrt{4-a^2}$$

The complete integral is (2),

$$z = ax \pm y\sqrt{4-a^2} + c \longrightarrow (3)$$

Claim singular integral

Differentiating equation (3) partially with respect to 'c', we get $0=1$.

This equation is absurd.

There is no singular integral for the given equation (1).

Claim : general integral

Put $c=f(a)$ in equation (3), we get

$$z = ax \pm y\sqrt{4-a^2} + f(a)$$

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2. $p = q^2$

Solution:

Given equation is

$$p = q^2 \rightarrow (1)$$

The given equation is of the form $f(p,q) = 0$.

The solution of the equation (1) is

$$z = ax + by + c \longrightarrow (2)$$

Put $p = a$, $q = b$ in equation (1), we get

$$a = b^2$$

$$b = \pm\sqrt{a}$$

The complete integral is

$$z = b^2 x + by + c \longrightarrow (3)$$

Claim: Singular integral

Differentiating equation (3) partially with respect to 'c', we get

$$0 = 1$$

The equation is absurd.

There is no singular integral for this equation.

Claim: General integral

Put $c = f(b)$ in equation ③,

we get
$$z = b^2x + by + f(b) \longrightarrow ④$$

Differentiating equation ④ with respect to 'b', we get
$$0 = 2bx + y + f'(b) \longrightarrow ⑤$$

Eliminating 'b' between ③, ④ and ⑤ we get the general integral.

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3. $pq = 1$.

Solution:

Given equation is
$$pq = 1 \longrightarrow ①$$

This equation is of the form
$$f(p,q) = 0$$

The solution of the equation ① is
$$z = ax + by + c \longrightarrow ②$$

Put $p = a$, $q = b$ in equation ①,

we get
$$ab = 1$$
$$b = 1/a$$

The complete integral is

$$z = ax + y\left(\frac{1}{a}\right) + c \longrightarrow (3)$$

$$az = a^2x + y + ac$$

Claim: Singular integral

Differentiating equation (2) partially with respect to 'c', we get

$$0 = 1.$$

This equation is absurd.

There is no singular integral for the given equation (1).

Claim: General integral

Put $c = f(a)$ in equation (3), we get

$$z = ax + \frac{1}{a}y + f(a) \longrightarrow (4)$$

Differentiating equation (4) partially with respect to 'a', we get

$$0 = x - \frac{y}{a^2} + f'(a) \longrightarrow (5)$$

Eliminating 'a' between (3), (4) and (5), we get the general integral.

4. $pq + p + q = 0$

Solution:

Given equation is

$$pq + p + q = 0 \longrightarrow (1)$$

This equation is of the form $f(p, q) = 0$.

$\therefore$ The solution of the equation (1) is

$$z = ax + by + c \longrightarrow (2)$$

Put $p = a$, $q = b$ in equation (1), we get

$$ab + a + b = 0$$

$$ab + b = -a$$

$$b(1 + a) = -a$$

$$b = \frac{-a}{1+a}$$

$\therefore$ The complete integral is

$$z = ax - \frac{ay}{1+a} + c$$

$$z = ax - a(1+a)^{-1} y + c \longrightarrow (3)$$

Claim: Singular integral

Differentiating (3) partially with respect to 'c', we get

$0 = 1.$

This equation is absurd.

There is no singular integral for this equation.

Claim : General integral

Put $c = f(a)$ in equation ③, we get

$$z = ax + a(1+a)^{-1}y + f(a) \rightarrow ④$$

Differentiating equation ④ partially with respect to 'a', we get

$$0 = x - (1+a)^{-1}y + f'(a) \rightarrow ⑤$$

Eliminating 'a' between equations ③, ④, ⑤ we get the general integral.

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5. $q^2 - 3q + p = 2.$

Solution:

Given equation is

$$q^2 - 3q + p = 2 \longrightarrow ①$$

This equation is of the form $f(p, q) = 0.$

$\therefore$ The solution of the equation ① is

$$z = ax + by + c \longrightarrow ②$$

Put $p = a$ and $q = b$ in equation ① we get

$$b^2 - 3b + a = 2$$

$$b^2 - 3b = 2 - a$$

$$b(b-3) = 2 - a$$

$$b - 3 = \frac{2-a}{b}$$

$$b = \frac{2-a}{b} + 3$$

$$\therefore b = \frac{2-a+3b}{b}$$

The complete integral is

$$z = ax + \left(\frac{2-a+3b}{b}\right) y + c \longrightarrow ③$$

<u>Claim: Singular integral</u>

Differentiating equation ③ partially with respect to "c", we get $0 = 1$.

This equation is absurd.

There is no singular integral for this equation.

Claim: General integral

Put $c = f(a)$ in equation (3), we get

$$z = ax + \left(\frac{2-a+3b}{b}\right)y + f(a) \rightarrow (4)$$

Differentiating equation (4) partially with respect to 'a', we get

$$z = x + \left(\frac{2+3b}{b}\right)y + f'(a).$$

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6. $3p^2 - 2q^2 = 4pq$.

Solution:

Given equation is

$$3p^2 - 2q^2 = 4pq \rightarrow (1)$$

This equation is of the form $f(p,q) = 0$.

The solution of the equation (1) is

$$z = ax + by + c \rightarrow (2)$$

Put $p = a$, $q = b$ in equation (1), we get

$$3a^2 - 2b^2 = 4ab$$

$$-2b^2 = 4ab - 3a^2$$

$$-b^2 = \frac{4ab - 3a^2}{2}$$

$$b^2 = \frac{3a^2 - 4ab}{2}$$

$$b = \sqrt{\frac{3a^2 - 4ab}{2}}$$

$\therefore$ The complete integral is

$$z = ax + \sqrt{\frac{3a^2 - 4ab}{2}}\, y + c \longrightarrow (3)$$

Claim: Singular integral

Differentiating equation (3) partially with respect to 'c', we get $0 = 1$.

This equation is absurd.

There is no singular integral for this equation.

Claim: General integral

Put $c = f(a)$ in equation (3) we get

$$z = ax + \left(\frac{2 - a + 3b}{b}\right) y + f(a) \longrightarrow (4)$$

Differentiating equation (4) partially with respect to 'a', we get

$$z = x + \left(\frac{2+3b}{b}\right)y + f'(a).$$

7. $p^2 - q^3 = 0.$

Solution:

given equation is

$$p^3 - q^3 = 0 \longrightarrow (1)$$

This equation is of the form $f(p,q) = 0$.

The solution of equation (1) is

$$z = ax + by + c \longrightarrow (2)$$

Put $p = a$ and $q = b$ in equation (1) we get

$$a^3 - b^3 = 0$$

$$-b^3 = -a^3$$

$$b^3 = a^3$$

$$b = (a^3)^{1/3}$$

$\therefore$ The complete integral is

$$z = ax + (a^3)^{1/3}y + c \longrightarrow (3)$$

Claim: Singular integral

Differentiating equation (3)

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partially with respect to 'a', we get

$$0 = x + (3a^2)^{1/3} y + f'(a)$$

Eliminating 'a' between (3), (4) and (5), we get the general integral.

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8. $p + q = \frac{3}{a}$

<u>Solution:</u>

Given equation is

$$p + q = \frac{3}{a} \longrightarrow (1)$$

This equation is of the form $f(p,q) = 0$.

The solution of this equation (1) is

$$z = ax + by + c \longrightarrow (2)$$

Put $p = a$, $q = b$ in equation (1), we get

$$a + b = \frac{3}{a}$$

$$b = \frac{3}{a} - A$$

The complete integral is

$$z = ax + \left(\frac{3}{a} - \frac{A}{y} + c\right) \longrightarrow (3)$$

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Claim: Singular integral

Differentiating equation (3), with respect to 'c', we get

$$0 = 1.$$

There is no singular integral in this equation.

Claim: General integral

Put $c = f(a)$ in equation (3), we get

$$z = ax + \left(\frac{3}{a} - A\right)y + f(a) \longrightarrow (4)$$

Differentiating equation (4) partially with respect to 'a', we get

$$0 = x + (-3a^{-2})y + f'(a)$$

Eliminating 'c' between (3), (4) and (5) we get the general integral.

STANDARD TYPE 2.

Only one of the variables x, y, z occurs explicitly, such equations can be written in one of the forms.

i) $F(x, p, q) = 0$; ii) $F(y, p, q) = 0$; iii) $F(z, p, q) = 0$.

Type (i)

Consider the form $F(x, p, q) = 0 \longrightarrow (1)$

since 'z' is a function of x and y.

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

$$dz = p\,dx + q\,dy \longrightarrow (2)$$

Let us assume that $q = a$.

Equation (1) becomes $F(x, p, a) = 0$

Solving this for 'p' we get

$p = \phi(x, a)$

$\therefore$ (2) $\Rightarrow$ $dz = \phi(x, a)\,dx + a\,dy$

Integrating on both sides, we get $z = \phi(x, a)\,dx + ay + b$

This equation consists of two arbitrary constants a and b.

Hence, it is a complete integral of equation (1).

EXAMPLE

Solve: $q = xp + p^2$

Solution:

Given equation is

$$q = xp + p^2 \longrightarrow (1)$$

The equation is of the form $F(x, p, q) = 0$.

Put $q = a$ in equation (1)

$$a = xp + p^2$$

$$p^2 + xp - a = 0$$

$$p = \frac{-x \pm \sqrt{x^2+4a}}{2}$$

$$\therefore dz = p\,dx + q\,dy$$

$$dz = \left[\frac{-x \pm \sqrt{x^2+4a}}{2}\right] dx + a\,dy$$

Integrating, we get

$$\int dz = \int \left[\frac{-x \pm \sqrt{x^2+4a}}{2}\right] dx + a\,dy$$

$$z = \frac{-x^2}{4} \pm \left[\frac{x}{4}\sqrt{4a+x^2} + a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right)\right] + ay + b$$

which is the complete integral of the given equation.

EXERCISES 24

Solve the following equations :

1. $p = 2qx$.

Sol: Given equation is

$$p = 2qx \longrightarrow (1)$$

This equation is of the form

$F(x, p, q) = 0$.

Put $q = a$ in equation (1),

$$p = 2ax$$

$$dz = p\,dx + q\,dy$$

$$dz = (2ax)\,dx + a\,dy$$

Integrating both sides, we get

$$\int dz = \int 2ax\,dx + \int a\,dy$$

$$z = ax^2 + ay + c$$

which is the complete integral of the given equation.

Type (ii)

Consider the form $F(y, p, q) = 0$

Let us assume that $p = a$.

$\therefore F(y, a, q) = 0$

$\therefore q = \phi(y, a)$

Hence $dz = a\,dx + \phi(y, a)\,dy$

$\therefore z = ax + \int \phi(y, a)\,dy + b$,

which is a complete integral.

EXAMPLE

1. Solve : $p = y^2 q^2$

Solution :

Given equation is $p = y^2 q^2 \longrightarrow (1)$

This equation is of the form $F(y, p, q) = 0$

Let $p = a$ in equation (1), we get

$$a = y^2 q^2$$

$$q^2 = \frac{a}{y^2}$$

$$q = \pm \frac{\sqrt{a}}{y}$$

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx \pm \frac{\sqrt{a}}{y}\,dy$$

Integrating on both sides, we get

$$z = ax \pm \sqrt{a}\log y + b$$

which is the complete integral of equation.

EXERCISES 24:

Solve:

2. $q = 2yp^2$

Solution:

Given equation is

$$q = 2yp^2 \longrightarrow (1)$$

This equation is of the form $F(y, p, q) = 0$

Put $p = a$ in equation (1), we get

$$q = 2ya^2$$

Hence

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + 2ya^2\,dy$$

Integrating on both sides, we get

$$\int dz = \int a\,dx + \int 2ya^2\,dy$$

$$z = ax + y^2a^2 + b$$

which is the solution of the given equation.

---

3. $\sqrt{p} + \sqrt{q} = \sqrt{y}$.

<u>Solution</u>:

Given equation is

$$\sqrt{p} + \sqrt{q} = \sqrt{y} \longrightarrow (1)$$

This equation is of the form

$F(y, p, q) = 0$

Put $p = a$ in equation (1)

$$\sqrt{a} + \sqrt{q} = \sqrt{y}$$

$$\sqrt{q} = \sqrt{y} - \sqrt{a}$$

$$q = (\sqrt{y} - \sqrt{a})^2$$

Hence $dz = p\,dx + q\,dy$

$$dz = a\,dx + (y^2 + a^2 - 2\sqrt{ay})\,dy$$

Integrating both sides,

$$z = ax + \frac{y^3}{3} + a^2y - 4\sqrt{a} \cdot \frac{y^{3/2}}{3} + c$$

$$z = ax + \frac{y^3}{3} + a^2y - \frac{4\sqrt{a}}{3}\, y^{3/2} + c.$$

4. $q^2 = yp^4$

Solution:

given equation is

$$q^2 = yp^4 \longrightarrow (1)$$

This equation is of the form

$F(y, p, q) = 0$

Put $p = a$ in equation (1)

$$q^2 = ya^4$$

$$q = \sqrt{a^4 y}$$

$$q = a^2\sqrt{y}$$

Hence $dz = p\,dx + q\,dy$

$$dz = a\,dx + a^2\sqrt{y}\,dy$$

Integrating on both sides, we get

$$z = ax + \frac{2a^2 y^{3/2}}{3} + c$$

which is the complete integral of given solution.

Type (iii)

Let us consider the equation

$F(z, p, q) = 0 \longrightarrow (1)$

Let us assume that

$$q = ap.$$

Then equation ① becomes

$F(z, p, ap) = 0$

$$p = \phi(z, a)$$

$$dz = p\,dx + q\,dy$$

$$dz = \phi(z,a)\,dx + ap\,dy$$

$$dz = \phi(z,a)\,dx + a\,\phi(z,a)\,dy$$

$$\frac{dz}{\phi(z,a)} = dx + a\,dy$$

Integrating on both sides, we get

$$\int \frac{dz}{\phi(z,a)} = x + ay + b$$

which is the complete integral of the given equation.

EXAMPLE

1. Solve: $p(1+q^2) = q(z-1)$

Solution:

Given equation is

$$p(1+q^2) = q(z-1) \rightarrow ①$$

This equation is of the form

$F(z, p, q) = 0$

Put $q = ap$ in equation ①.

$$p(1+a^2p^2) = a\,p(z-1)$$

$$1+a^2p^2 = az-a$$

$$a^2p^2 = az-a-1$$

$$p^2 = \frac{az-a-1}{a^2}$$

$$p = \frac{\sqrt{az-a-1}}{a}$$

$$q = ap$$

$$q = a\left[\frac{\sqrt{az-a-1}}{a}\right]$$

$$q = \sqrt{az-a-1}$$

$$\therefore dz = p\,dx + q\,dy$$

$$dz = \frac{\sqrt{az-a-1}}{a}dx + \sqrt{az-a-1}\,dy$$

$$\frac{dz}{\sqrt{az-a-1}} = \frac{dx}{a} + dy$$

Integrating on both sides

$$\frac{2}{a}\sqrt{az-a-1} = \frac{x}{a} + y + c$$

$$2\sqrt{az-a-1} = x + ay + c$$

which is the complete integral of the given equation.

EXERCISE

Solve the following:

2. $9(p^2 z + p^2) = 4$

Solution:

given equation is

$$9(p^2 z + p^2) = 4 \longrightarrow (1)$$

This equation is of the form $f(z, p, q) = 0$

Put $q = ap$ in equation (1) we get

$$9(p^2 x + a^2 p^2) = 4$$

$$9p^2(z + a^2) = 4$$

$$p^2 = \frac{4}{9(z+a^2)}$$

$$p = \frac{\pm 2}{3(z+a^2)^{1/2}}$$

$$q = ap$$

$$q = a \cdot \left(\pm \frac{2}{3}(z+a^2)^{-1/2}\right)$$

$$q = \pm \frac{2a}{3}(z+a)^{-1/2}$$

$$\therefore dz = p\,dx + q\,dy$$

$$dz = \pm \frac{2}{3}(z+a^2)^{-1/2} dx \pm \frac{2a}{3}(z+a^2)^{-1/2} dy$$

$$(z+a^2)^{1/2} dz = \pm \frac{2}{3} dx \pm \frac{2a}{3} dy$$

Integrating on both sides, we get

$$\int (z+a^2)^{1/2} dz = \pm \frac{2}{3}\left[\int dx + \int a\,dy\right]$$

$$\frac{2(z+a^2)^{3/2}}{3} = \pm \frac{2}{3}\left[(x+ay)+c\right]$$

$$2(z+a^2)^{3/2} = \pm 2(x+ay)+c$$

$$(z+a^2)^{3/2} = (x+ay+c)$$

which is the complete integral of the given equation.

---

3. $p(1+q) = qz$

Solution:

given equation is

$$p(1+q) = qz \longrightarrow (1)$$

This equation is of the form $F(z,p,q)=0$.

Put $q = ap$ in equation (1), we get.

$$(1) \Rightarrow p(1+ap) = qz$$

$p + ap^2 = qz$

$ap^2 + p - qz = 0$

$$p = \frac{-1 \pm \sqrt{1+3baz}}{2a}$$

$q = ap$

$$q = a\left[\frac{-1 \pm \sqrt{1+3baz}}{2a}\right]$$

$$q = \frac{-1 \pm \sqrt{1+3baz}}{2}$$

$dz = p\,dx + q\,dy$

$$dz = \left(\frac{-1 \pm \sqrt{1+3baz}}{2a}\right)dx + \left(\frac{-1 \pm \sqrt{1+3baz}}{2}\right)dy$$

$$\frac{2\,dz}{-1 \pm \sqrt{1+3baz}} = \frac{1}{a}\,dx + dy$$

Integrating on both sides, we get

$$\int \frac{2\,dz}{-1 \pm \sqrt{1+3baz}} = \int \frac{1}{a}\,dx + \int dy$$

$$2\int \left(-1 \pm \sqrt{1+3baz}\right)^{-1} dz = \int \frac{1}{a}\,dx + \int dy$$

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$$\frac{2\log\left(-1 \pm \sqrt{1+3baz}\right)}{3b\frac{a}{2}\sqrt{1+3baz}} = \frac{x}{a}y + c$$

$$\frac{2(2)\sqrt{1+3baz}\,\log\left(-1 \pm \sqrt{1+3baz}\right)}{3ba} = \frac{x}{a} + y + c$$

$$\frac{\sqrt{1+3baz}}{a}\log\left(-1 \pm \sqrt{1+3baz}\right) = \frac{x}{a} + y + c$$

$$a\sqrt{1+3baz}\,\log\left(-1 \pm \sqrt{1+3baz}\right) = 9x + 9ay + 9ac$$

which is the complete integral of the given equation.

---

4. $z^2(p^2+q^2+1) = a^2$

Solution:

Given equation is

$$z^2(p^2+q^2+1) = a^2 \longrightarrow (1)$$

This equation is of the form

$F(z, p, q) = 0$

Put $ap = q$ in equation (1), we get

$$① \Rightarrow \quad z^2(p^2+q^2+1) = a^2$$

$$p^2 + p^2a^2 + 1 = a^2/z^2$$

$$(1+a^2)p^2 = \frac{a^2}{z^2} - 1$$

$$(1+a^2)p^2 = \frac{a^2-z^2}{z^2}$$

$$p^2 = \frac{a^2-z^2}{z^2(1+a^2)}$$

$$p = \frac{\pm\sqrt{a^2-z^2}}{z\sqrt{1+a^2}}$$

$$p = \frac{\pm\sqrt{a^2-z^2}\,(1+a^2)^{-1/2}}{z}$$

$$q = ap = \pm\frac{a\sqrt{a^2-z^2}\,(1+a^2)^{-1/2}}{z}$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{\pm\sqrt{a^2-z^2}(1+a^2)^{-1/2}}{z}dx \pm \frac{a\sqrt{a^2-z^2}(1+a^2)^{-1/2}}{z}dy$$

$$\frac{z\,dz}{\sqrt{a^2-z^2}\sqrt{1+a^2}} = \pm\, dx + a\,dy$$

Integrating on both sides, we get

$$\frac{1}{\sqrt{1+a^2}}\int\frac{z.dz}{\sqrt{a^2-z^2}} = \pm\left[\int dx+\int a\,dy\right] \longrightarrow ②$$

Let $t = a^2 - z^2$

$dt = -2z\,dz$

$\frac{-dt}{2} = z\,dz$

Put, $t = a^2 - z^2$ in equation ②

$$② \Rightarrow \frac{1}{\sqrt{1+a^2}}\int\frac{1}{\sqrt{t}}\left(\frac{-1}{2}\right)dt = \pm\int dx+\int a\,dy$$

$$\frac{-1}{2\sqrt{1+a^2}}\cdot\frac{2t^{3/2}}{3} = \pm x \pm ay + c$$

$$\frac{-(a^2-z^2)^{3/2}}{3\sqrt{1+a^2}} = x + ay + c$$

$$-(a^2-z^2)^{3/2} = 3\left[\sqrt{1+a^2}\,(x+ay+c)\right]$$

which is the complete integral.

---

5. $z = p^2 + q^2$

Solution:

Given equation is

$$z = p^2 + q^2 \longrightarrow ①$$

This equation is of the form

$F(z, p, q) = 0$

Put $ap = q$ in equation ①, we get

$$① \Rightarrow z = p^2 + q^2$$

$$z = p^2 + a^2 p^2$$

$$z = p^2(1+a^2)$$

$$p^2 = \frac{z}{(1+a^2)}$$

$$p = \pm \frac{\sqrt{z}}{\sqrt{(1+a^2)}}$$

$$q = ap = a\left(\pm \frac{\sqrt{z}}{\sqrt{1+a^2}}\right)$$

$$q = \pm \frac{a\sqrt{z}}{\sqrt{1+a^2}}$$

$$dz = p\,dx + q\,dy$$

$$dz = \pm \frac{\sqrt{z}}{\sqrt{1+a^2}}\,dx \pm \frac{a\sqrt{z}}{\sqrt{1+a^2}}\,dy$$

$$\frac{\sqrt{1+a^2}}{\sqrt{z}}\,dz = \pm\,dx \pm a\,dy$$

Integrating on both sides, we get

$$\sqrt{1+a^2}\int \frac{1}{\sqrt{z}}\,dz = \pm\int dx \pm \int a\,dy$$

$$2\sqrt{1+a^2}\cdot\sqrt{z} = \pm(x+ay) + c$$

$$2\sqrt{z(1+a^2)} = \pm(x+ay)+c$$

which is the complete integral of the given equation.

STANDARD TYPE 3

Let us consider the form

$$f_1(x,p) = f_2(y,q) \longrightarrow (1)$$

In this form the equation is of the first order and the variables are separable.

In this equation 'z' does not appear.

Let us assume that

$$f_1(x,p) = a \Rightarrow f_2(y,a) = a.$$

$$\therefore p = \phi_1(x,a) \text{ and } q = \phi_2(a,y)$$

Hence $dz = p\,dx + q\,dy$

$$dz = \phi_1(a,x)\,dx + \phi_2(a,y)\,dy$$

Integrating on both sides, we get

$$\therefore z = \int \phi_1(a,x)\,dx + \int \phi_2(a,y)\,dy + b$$

which is the complete integral of the given equation.

EXAMPLE

1. Solve: $p+q=x+y$

Solution:

given equation is

$$p+q=x+y \longrightarrow (1)$$

This equation is of the form

$f_1(x,p)=f_2(y,q)$.

$(1) \Rightarrow p-x=q-y$

Let $b-x=y-q=a$

$p-x=a$ and $y-q=a$

$p=x+a$ and $q=y-a$

Hence $dz=pdx+qdy$

$$dz=(x+a)dy+(y-a)dy$$

Integrating on both sides, we get

$$z=\frac{x^2}{2}+ax+\frac{y^2}{2}-ay+c$$

$$z=\frac{(x+a)^2}{2}+\frac{(y-a)^2}{2}+b$$

$$2z=(x+a)^2+(y-a)^2+2b$$

which is the complete integral.

EXERCISES 25

Solve the following equations:-

2. $q-p+x-y=0$

Solution:

given equation is

$$q-p+x-y=0 \longrightarrow (1)$$

$$p-x=q-y$$

This equation is of the form

$f_1(x,p)=f_2(y,q)$.

$(1) \Rightarrow p-x=q-y$

Let $p-x=a$ and $q-y=a$

$p=x+a$; $q=a+y$

Hence,

$$dz=p\,dx+q\,dy$$

$$dz=(x+a)\,dx+(a+y)\,dy$$

Integrating on both sides,

$$z=\frac{x^2}{2}+ax+\frac{y^2}{2}+ay+c$$

which is the complete integral.

3. $p^2+q^2=x+y$

Solution:

given equation is

$$p^2+q^2=x+y \longrightarrow (1)$$

This equation is of the form

$f_1(x,p) = f_2(y,q)$.

$① \Rightarrow p^2+q^2 = x+y$

$$p^2 - x = y - q^2$$

$$p^2 - x = a \quad \text{and} \quad y - q^2 = a$$

$$p^2 = a + x \qquad q^2 = y - a$$

$$p = \pm\sqrt{a+x} \qquad q = \pm\sqrt{y-a}$$

Hence, $dz = p\,dx + q\,dy$

$$dz = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$$

Integrating on both sides, we get

$$z = \pm\frac{2}{3}(x+a)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c$$

$$3z = \pm 2(x+a)^{3/2} \pm (y-a)^{3/2} + 3c$$

which is the complete integral of given equation.

---

4. $p^2+q^2 = x^2+y^2$

Solution:

Given equation is

$$p^2+q^2 = x^2+y^2 \longrightarrow ①$$

This equation of the form

$f_1(x,p) = f_2(y,q)$

$① \Rightarrow p^2 - x^2 = y^2 - q^2$

Let $p^2 - x^2 = a$ $\quad$ $y^2 - q^2 = a$

$$p^2 = a + x^2 \qquad q^2 = y^2 - a$$

$$p = \pm\sqrt{a+x^2} \qquad q = \pm\sqrt{y^2-a}$$

Hence, $dz = p\,dx + q\,dy$

$$dz = \sqrt{x^2+a}\,dx + \sqrt{y^2-a}\,dy$$

Integrating on both sides, we get

$$z = \frac{2(x^2+a)^{3/2}}{3(2x)} + \frac{2(y^2-a)^{3/2}}{3(2y)} + c$$

$$z = \frac{(x+a)^{3/2}}{3x} + \frac{(y^2-a)^{3/2}}{3y} + c$$

$$3xyz = y(x^2+a^2)^{3/2} + x(y^2-a)^{3/2} + 3xyc$$

which is the complete integral of the given equation.

---

5. $\sqrt{p} + \sqrt{q} = 2x$

Solution:

Given equation is

$$\sqrt{p} + \sqrt{q} = 2x \longrightarrow (1)$$

This equation is of the form

$f_1(x,p) = f_2(y,q)$.

$$\sqrt{q} = 2x - \sqrt{p}$$

Let $2x-\sqrt{p}=a$; $\sqrt{q}=a$

$\sqrt{p}=2x-a$; $q=a^2$.

$p=(2x-a)^2$

Hence, $dz=p\,dx+q\,dy$

$dz=(2x-a)^2dx+a^2dy$

Integrating on both sides, we get

$$z=\frac{(2x-a)^3}{3(2)}+a^2y+c$$

$$6z=(2x-a)^3+6a^2y+6C$$

which is the complete integral of the equation.

---

6. $\sqrt{p}+\sqrt{q}=\sqrt{x}$

Solution: Given equation is

$$\sqrt{p}+\sqrt{q}=\sqrt{x} \longrightarrow ①$$

This equation is of the form $f_1(x,p)=f_2(y,q)$.

$① \Rightarrow \sqrt{p}+\sqrt{q}=\sqrt{x}$

$\sqrt{q}=\sqrt{x}-\sqrt{p}$

Let $\sqrt{x}-\sqrt{p}=a$; $\sqrt{q}=a$

$\sqrt{p}=\sqrt{x}-a$; $q=a^2$

$p = (\sqrt{x} - a)^2$

Hence, $dz = p\,dx + q\,dy$

$$dz = (\sqrt{x} - a)^2 dx + a^2 dy$$

Integrating on both sides, we get

$$z = \frac{x^2}{2} + a^2 x - \frac{4ax^{3/2}}{3} + a^2 y + c$$

which is the complete integral of the given equation.

---

7. $p + q = \sin x + \sin y$

Solution:

Given equation is

$$p + q = \sin x + \sin y \rightarrow (1)$$

This equation is of the form

$f_1(x, p) = f_2(y, q)$

$(1) \Rightarrow \quad p - \sin x = \sin y - q$

Let $p - \sin x = a \;;\quad \sin y - q = a$

$p = a + \sin x \;;\quad q = \sin y - a$

Hence $dz = p\,dx + q\,dy$

$$dz = (\sin x + a)\,dx + (\sin y - a)\,dy$$

Integrating on both sides, we get

$$z = -\cos x + ax - \cos y - ay + c$$

$z = ax - \cos x - \cos y - ay + c$

which is the complete integral of the equation.

8. $pq = xy$

Solution:

given equation is

$$pq = xy \rightarrow (1)$$

This equation is of the form

$f_1(x, p) = f_2(y, q)$.

$(1) \Rightarrow p/x = y/q$

Let $p/x = a$; $y/q = a$

$p = xa$; $q = y/a$

Hence, $dz = p\,dx + q\,dy$

$$dz = ax\,dx + y/a\,dy$$

Integrating on both sides,

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + c$$

$$2z = ax^2 + \frac{y^2}{a} + 2c$$

which is the complete integral of the equation.

9. $2yp^2 = q$

Solution:

Given equation is

$$2yp^2 = q \longrightarrow (1)$$

This equation is of the form

$f_1(x, p) = f_2(y, q)$

$(1) \Rightarrow \quad 2p^2 = q/y$

Let $2p^2 = a$; $\quad \frac{q}{y} = a$

$p = \sqrt{\frac{a}{2}}$; $\quad q = ay$

Hence, $dz = p\,dx + q\,dy$

$$dz = \sqrt{\frac{a}{2}}\,dx + ay\,dy$$

Integrating on both sides, we get

$$z = x\sqrt{\frac{a}{2}} + \frac{ay^2}{2} + c$$

which is the complete integral of the equation.

---

10. $xp - y^2p^2 = 1$

Solution:

Given equation is

$$xp - y^2p^2 = 1 \longrightarrow (1)$$

This equation is of the form

$f_1(x, p) = f_2(y, q)$

$① \Rightarrow \quad xp - y^2q^2 = 1$

$$xp = 1 + y^2q^2$$

Let $xp = a$; $\quad 1 + y^2q^2 = a$

$$p = \frac{a}{x}; \quad y^2q^2 = a - 1$$

$$q = \frac{\pm\sqrt{a-1}}{y}$$

Hence, $dz = p\,dx + q\,dy$

$$dz = \frac{a}{x}dx \pm \frac{\sqrt{a-1}}{y}dy$$

Integrating on both sides, we get

$$z = a\log x \pm (\sqrt{a-1})\log y + c$$

which is the complete integral of the equation.

---

11. $p^2y(1+x^2) = qx^2$

Solution:

given equation is

$$p^2y(1+x^2) = qx^2 \longrightarrow ①$$

This equation is of the form $f_1(x,p) = f_2(y,q)$.

$① \Rightarrow \quad p^2y(1+x^2) = qx^2$

$$p^2\left(\frac{1+x^2}{x^2}\right) = \frac{q}{y}$$

Let

$$p^2\left(\frac{1+x^2}{x^2}\right)=a\ ; \quad \frac{q}{y}=a$$

$$p^2=\frac{ax^2}{1+x^2}\ ; \quad q=ay$$

$$p=\pm\frac{x\sqrt{a}}{\sqrt{1+x^2}}\ ; \ q=ay$$

Hence,
$$dz=p\,dx+q\,dy$$

$$dz=\pm\frac{x\sqrt{a}}{\sqrt{1+x^2}}dx+ay\,dy$$

Integrating on both sides, we get

$$\int dz=\pm\sqrt{a}\int\frac{x}{\sqrt{1+x^2}}dx+a\int y\,dy \longrightarrow (2)$$

Let $t=1+x^2$

$$dt=2x\,dx$$

$$\frac{dt}{2}=x\,dx$$

Substitute in (2), we get

$$(2)\Rightarrow \int dz=\pm\frac{\sqrt{a}}{2}\int\frac{1}{\sqrt{t}}dt+a\int y\,dy$$

$$z=\pm\frac{\sqrt{a}}{2}\cdot 2\sqrt{t}+a\frac{y^2}{2}+c$$

$$z=\pm\sqrt{a}\cdot\sqrt{1+x^2}+\frac{a}{2}y^2+c$$

which is the complete integral of the given equation.

12. $py + qx = pq$

Solution:

Given equation is

$$py + qx = pq \longrightarrow (1)$$

The equation is of the form

$f_1(x, p) = f_2(y, q)$

$(1) \Rightarrow py + qx = pq$

$$py = pq - qx$$

$$py = q(p-x)$$

$$\frac{p}{p-x} = \frac{q}{y}$$

Let $\frac{p}{p-x} = a$; $\frac{q}{y} = a$

$$p = a(p-x) ; \; q = ay$$

$$p = ap - ax$$

$$ap - p = ax$$

$$p(a-1) = ax$$

$$p = \frac{ax}{a-1} ; \; q = ay$$

Hence, $dz = pdx + qdy$

$$dz = \frac{ax}{a-1} dx + ay\, dy$$

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Integrating on both sides, we get

$$z = \frac{ax^2}{2(a-1)} + \frac{ay^2}{2} + c$$

which is the complete integral of the given equation.

STANDARD TYPE 4

Clairaut's form

This equation is of the form

$$z = px + qy + f(p,q).$$

The solution of the equation is

$$z = ax + by + f(a,b)$$ for $p = a$ and $q = b$ can easily be verified to satisfy the given equations.

EXAMPLE:

Solve: $z = px + qy + \sqrt{1+p^2+q^2}$

Solution:

Given equation is

$$z = px + qy + \sqrt{1+p^2+q^2} \rightarrow (1)$$

This equation is of the form

$$z = px + qy + f(p,q)$$

The solution of equation (1) is

$$z = ax + by + \sqrt{1+a^2+b^2} \rightarrow (2)$$

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which is the complete integral of the given equation.

Claim: Singular integral

Differentiating equation (2) partially with respect to a and b, we get

$$0 = x + \frac{1}{2}(1+a^2+b^2)^{-1/2}(2a)$$

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}} \longrightarrow (3)$$

$$0 = y + \frac{1}{2}(1+a^2+b^2)^{-1/2}(2b)$$

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}} \longrightarrow (4)$$

Eliminating a ~~and~~ b from the equations (3) and (4) we get the singular integral.

LAGRANGE'S EQUATION

If we eliminate the arbitrary function F from the relation $F(u,v)=0$ where u and v are functions of x, y, z, we get the relation

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}$$

This can be expressed in the form

$$Pp + Qq = R$$

where P, Q, R are functions of x, y, z.

This P.D.E is known as Lagrange's equation.

Now, to find the solution of the equation ①, the general solution of the partial differential equation.

$Pp+Qq=R$ is $F(u,v)=0$

where 'F' is an arbitrary function and $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$

From two independent solutions of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

EXERCISES 28.

Solve : given $p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$.

1. $xp + yq = z$

Solution:

Given equation is

$$xp + yq = z \rightarrow (1)$$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = x$, $Q = y$, $R = z$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \rightarrow (2)$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating on both sides, we get

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1$$

Taking last two ratios in equation ②,

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides, we get $\log y = \log z + \log c_2$

$$\log y - \log z = \log c_2$$

$$\log\left(\frac{y}{z}\right) = \log c_2$$

$$\frac{y}{z} = c_2$$

The general solution of the equation ① is,

$$F\left(\frac{x}{y}, \frac{y}{z}\right) = 0 \quad (\text{or})$$

$$\frac{x}{y} = \phi\left(\frac{y}{z}\right) \quad (\text{or})$$

$$\frac{y}{z} = f\left(\frac{x}{y}\right)$$

---

2. $xp - yp = xy$

Solution:

Given equation is

$$xp - yp = xy \longrightarrow ①$$

This auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = x,\ Q = -y,\ R = xy$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{xy} \longrightarrow (2)$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating on both sides, we get

$$\log x = -\log y + \log c_1$$

$$\log x + \log y = \log c_1$$

$$\log(xy) = \log c_1$$

$$xy = c_1$$

Taking last two ratios in equation (2), we get

$$\frac{dy}{-y} = \frac{dz}{xy}$$

Integrating on both sides, we get

$$\frac{dy}{-y} = \frac{dz}{xy}$$

$$\frac{-dy}{y} = \frac{dz}{c_1}$$

$$-\log y = \frac{1}{c_1} z + c_2$$

$$\log y^{-1} = \frac{z}{xy} + c_2$$

$$\log\left(\frac{1}{y}\right) = \frac{z}{xy} + c_2$$

$$\log\left(\frac{1}{y}\right) - \frac{z}{xy} = c_2$$

The general solution is

$$F\left[xy\left(\log(1/y), \frac{z}{xy}\right)\right] = 0$$

---

3. $ap + bq + cz = 0$

Solution :

Given equation is

$$ap + bq + cz = 0$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e $P = a$, $Q = b$, $R = cz$.

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{-cz} \longrightarrow (2)$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{a} = \frac{dy}{b}$$

Integrating on both sides,

$$\int \frac{dx}{a} = \int \frac{dy}{b}$$

$$\frac{x}{a} = \frac{y}{b} + c_1$$

$$\frac{x}{a} - \frac{y}{b} = c_1$$

Taking last two ratios in equation (2), we get

$$\frac{dy}{b} = \frac{dz}{-cz}$$

$$\frac{dy}{b} = -\frac{1}{c}\frac{dz}{z}$$

Integrating on both sides,

$$\int \frac{dy}{b} = \frac{-1}{c}\int \frac{dz}{z}$$

$$\frac{y}{b} = \frac{-1}{c}\log z + c_2$$

$$\frac{y}{b} + \frac{1}{c}\log z = c_2$$

The general solution of equation (1) is

$$F\left(\frac{x}{a} - \frac{y}{b}, \frac{y}{b} + \frac{\log z}{c}\right) = 0$$

---

4. i) $p\tan x + q\tan y = \tan z$

Solution:

Given equation is

$$p\tan x + q\tan y = \tan z \longrightarrow (1)$$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $p = \tan x$, $Q = \tan y$, $R = \tan z$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \longrightarrow ②$$

Taking first two ratios in equation ②, we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating on both sides, we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\cot x \, dx = \cot y \, dy$$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{\cos y}{\sin y} dy$$

$$\log \sin x = \log \sin y + \log c_1$$

$$\log \sin x - \log \sin y = \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$$

Taking last two ratios from equation ②, we get

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Integrating on both sides, we get

$$\cot y \, dy = \int \frac{\cos z}{\sin z} dz$$

$$\log \sin y = \log \sin z + \log c_2$$

$$\log \sin y - \log \sin z = \log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$\therefore$ The general solution is

$$F\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

---

ii) $p \cot x + q \cot y = \cot z$

Solution:

given equation is

$$p \cot x + q \cot y = \cot z \longrightarrow (1)$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = \cot x,\ Q = \cot y,\ R = \cot z$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{\cot x} = \frac{dy}{\cot y}$$

Integrating on both sides, we get

$$\int \frac{dx}{\cot x} = \int \frac{dy}{\cot y}$$

$$\int \tan x \, dy = \int \tan y \, dy$$

$$\sec^2 x = \sec^2 y + c_1$$

$$\sec^2 x - \sec^2 y = c_1$$

Taking last two ratios in equation ②, we get

$$\frac{dy}{\cot y} = \frac{dz}{\cot z}$$

Integrating on both sides, we get

$$\int \frac{dy}{\cot y} = \int \frac{dz}{\cot z}$$

$$\int \tan y \, dy = \int \tan z \, dz$$

$$\sec^2 y = \sec^2 z + c_2$$

$$\sec^2 y - \sec^2 z = c_2$$

∴ The general solution of the equation ① is

$$F(\sec^2 x - \sec^2 y, \sec^2 y - \sec^2 z) = 0$$

5. $(a-x)p+(b-y)q=c-z$

Solution:

Given equation is

$$(a-x)p+(b-y)q=c-z \longrightarrow (1)$$

The auxiliary equation is,

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

where $P=a-x$, $Q=b-y$, $R=c-z$

$$\frac{dx}{a-x}=\frac{dy}{b-y}=\frac{dz}{c-z}$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{a-x}=\frac{dy}{b-y}$$

Let $a-x=t$; $b-y=s$

Differentiating with respect to x and y respectively,

$$-dx=dt \quad ; \quad -dy=ds$$

$$dx=-dt \quad ; \quad dy=-ds$$

Integrating on both sides, we get

$$\int\frac{-dt}{t}=\int-\frac{ds}{s}$$

$$-\log t = -\log s + \log a$$

$$\log(b-y) - \log(a-x) = \log c_1$$

$$\log\left(\frac{b-y}{a-x}\right) = \log c_1$$

$$\therefore c_1 = \frac{b-y}{a-x}$$

Taking last two ratios in equation (2), we get

$$\frac{dy}{b-y} = \frac{dz}{c-z}$$

Integrating on both sides, we get

Let $b-y=t$; $c-z=s$

$dt=-dy$; $-dz=ds$

$\therefore dy=-dt$; $ds=-dz$

$$\int \frac{dt}{t} = \int \frac{-ds}{s}$$

$$-\log t = -\log s + \log c_2$$

$$\log(c-z) - \log(b-y) = \log c_2$$

$$\log \frac{c-z}{b-y} = \log c_2$$

$$\frac{c-z}{b-y} = c_2$$

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The general solution is

$$F\left(\frac{b-y}{a-x}, \frac{c-z}{b-y}\right)=0$$

---

6. $y^2zp - x^2zq = x^2y$

<u>Solution:</u>

Given equation is

$$y^2zp - x^2zq = x^2y.$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where,

$$P=y^2z, \quad Q=-x^2z, \quad R=x^2y$$

$$\frac{dx}{y^2z} = \frac{dy}{-x^2z} = \frac{dz}{x^2y} \longrightarrow (2)$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{y^2z} = \frac{dy}{-x^2z}$$

$$\frac{dx}{y^2} = \frac{dy}{-x^2}$$

$$+x^2dx = -y^2dy$$

Integrating on both sides, we get

$$\frac{+x^3}{3} = \frac{-y^3}{3} + c_1$$

$$\frac{+x^3+y^3}{3} = c_1$$

$$\cancel{x}\ \cancel{x^3}\cancel{_3}+\cancel{y^3} = \cancel{x}_1$$

Taking last two ratios in equation ②, we get

$$\frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$$

$$y\,dy = -z\,dx$$

Integrating on both sides, we get

$$\frac{y^2}{2} = \frac{-z^2}{2} + c_2$$

$$\frac{y^2+z^2}{2} = c_2$$

The general solution is

$$F\left(\frac{x^3+y^3}{3}, \frac{y^2+z^2}{2}\right) = 0$$

---

7. $(y-z)p + (z-x)q = x-y$

Solution:

Given equation is

$$(y-z)p + (z-x)q = x-y \longrightarrow ①$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = y-z$, $Q = z-x$, $R = x-y$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} \longrightarrow (2)$$

~~Taking~~ first choosing (x, y, z) as multipliers,

$$\frac{x\,dx}{x(y-z)} = \frac{y\,dy}{y(z-x)} = \frac{z\,dz}{z(x-y)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{xy - xz + yz - xy + xz - yz} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

Integrating, we ~~have~~ get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\therefore\ x^2 + y^2 + z^2 = c_1$$

Choosing $(1, 1, 1)$ as multipliers, we get

$$\frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\therefore\ x^2 + y^2 + z^2 = c_1$$

$x+y+z = c_2$

$\therefore$ The general solution is

$F(x^2+y^2+z^2, x+y+z) = 0$

8. $x(y-z)p + y(z-x)q = z(x-y)$

Solution:

Given equation is

$x(y-z)p + y(z-x)q = z(x-y) \longrightarrow (1)$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = x(y-z)$, $Q = y(z-x)$, $R = z(x-y)$.

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \longrightarrow (2)$$

Choosing $(1,1,1)$ as multipliers, we get

$dx+dy+dz = 0$

Integrating on both sides, we get $\int dx + \int dy + \int dz = 0$

$x+y+z = \cancel{0}\ c_1$

$\therefore c_1 = x+y+z$

From ②, we get

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{dz/z}{x-y}$$

Integrating, we get

$$\int\frac{dx}{x} + \int\frac{dy}{y} + \int\frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$c_2 = xyz$$

∴ The general solution is

$$F(x+y+z\,;\ xyz) = 0$$

---

9. $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

Solution:

Given equation is

$$x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2) \rightarrow ①$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = x(y^2-z^2)$, $Q = y(z^2-x^2)$, $R = z(x^2-y^2)$.

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2} = \frac{dz}{z(x^2-y^2)} \longrightarrow (2)$$

Choosing (x,y,z) as multipliers, we get

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2y^2 - x^2z^2 + y^2z^2 - x^2y^2 + x^2z^2 - y^2z^2}$$

$$= \frac{x\,dx + y\,dy + z\,dx}{0}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$\therefore x^2 + y^2 + z^2 = c_1$$

From (2), we get

$$\frac{dx/x}{y^2-z^2} = \frac{dy/y}{z^2-x^2} = \frac{dz/z}{x^2-y^2} \longrightarrow (3)$$

Choosing $(1,1,1)$ as multipliers in (3)

$$= \frac{dx/x + dy/y + dz/z}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$= \frac{dx/x + dy/y + dz/z}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$\therefore$ The general solution is

$$F(x^2+y^2+z^2, xyz) = 0.$$

---

11. $xzp + yzq = xy$

Solution :

Given equation is

$$xzp + yzq = xy \longrightarrow (1)$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where

$$P = xz, \; Q = yz, \; R = xy$$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \longrightarrow (2)$$

Taking first two ratios in

equation ②, we get

$$\frac{dx}{xz} = \frac{dy}{yz}$$

Integrating on both sides, we get

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1$$

Taking last two ratios in equation ②, we get

$$\frac{dy}{yz} = \frac{dz}{xy}$$

$$x\,dy = z\,dz$$

Integrating on both sides, we get

$$xy = \frac{z^2}{2} + c_2$$

$$xy - \frac{z^2}{2} = c_2$$

The general solution of the given equation is

$$F\left(x/y,\ xy - \frac{z^2}{2}\right) = 0$$

12. $yzp - xzq = xy$

Solution:

Given equation is

$$yzp - xzq = xy \longrightarrow (1)$$

The auxiliary equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where

$$P = yz, \quad Q = -xz, \quad R = xy$$

$$\frac{dy}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

Taking first two ratios in equation (2), we get

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

Integrating on both sides we get

$$x\,dx = -y\,dy$$

$$\frac{x^2}{2} = \frac{-y^2}{2} + c_1$$

$$\frac{x^2+y^2}{2} = c_1$$

Taking last two ratios in equation (2), weget

$$\frac{dy}{-xz} = \frac{dz}{xy}$$

$$-z\,dz = y\,dy$$

Integrating on both sides, we get

$$\frac{-z^2}{2} = \frac{y^2}{2} + c_2$$

$$\frac{y^2+z^2}{2} = c_2$$

The solution of the given equation is $F\left(\frac{x^2+y^2}{2}, \frac{y^2+z^2}{2}\right) = 0$

EXAMPLE 1:

Solve $(y^2+z^2)p - xyq = -xz$

Solution:

The auxiliary equations are

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

Taking the last two equations, we get $\frac{dy}{y} = \frac{dz}{z}$

Integrating, we get $\log y = \log z + c$

$$\therefore \frac{y}{z} = c_1$$

Each of the equations ① is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) - xy^2 - xz^2}$$

i.e., $\dfrac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) - xy^2 - xz^2}$

$\therefore x\,dx + y\,dy + z\,dz = 0$

Hence after integration this reduces to $x^2+y^2+z^2 = c_2$

Hence the general solution of the equation is

$$F\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0.$$

EXAMPLE 2.

Find the general solution of

$(y+z)p + (z+x)q = x+y$

Solution:

Given equation is

$(y+z)p + (z+x)q = x+y \rightarrow (1)$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where

$P = y+z, \quad Q = z+x, \quad R = x+y$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Choosing $(1,1,1)$, $(1,-1,0)$ and $(0,1,-1)$ as multipliers in equation ①, we get

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)} \longrightarrow ②$$

Taking first two ratios in equation ②,

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}$$

Integrating on both sides, we get

$$\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log c_1$$

$$\log(x+y+z) + 2\log(x-y) = \log c_1$$

$$\log\left[(x+y+z)(x-y)^2\right] = \log c_1$$

$$(x+y+z)(x-y)^2 = 100$$

Taking last two ratios in equation ②, we get

$$-\log(x-y) = -\log(y-z) + \log c_2$$

$$\log(y-x) = \log(z-y) + \log c_2$$

$$\log(y-x) - \log(z-y) = \log c_2$$

$$\log(y-z) - \log(x-y) = \log c_2$$

$$\frac{y-z}{x-y} = c_2$$

Hence the general solution of the equation is

$$F\left[(x+y+z)(y-x)^2, \frac{z-y}{y-x}\right] = 0.$$

EXAMPLE 3.

Solve $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$

Solution:

Given equation is

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z \longrightarrow (1)$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where, $P = x^2$, $Q = y^2$, $R = (x+y)^2$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)^2} \longrightarrow (2)$$

Taking first two ratios in equation (2), we get

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

Integrating on both sides, we get

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$\int x^{-2}\,dx = \int y^{-2}\,dy$$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1$$

Choosing (1,-1,0) as multipliers in equation ②, we get

$$\frac{dx-dy}{x^2-y^2} = \frac{dx-dy}{(x+y)(x-y)} \longrightarrow ③$$

Taking equation ③ and last ratios in equation ②, we get

$$\frac{dx-dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\frac{dx-dy}{x-y} = \frac{dz}{z}$$

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Integrating on both sides, we get

$$\log(x-y) = \log z + \log c_2$$

$$\log(x-y) - \log z = \log c_2$$

$$\log\left(\frac{x-y}{z}\right) = \log c_2$$

$$\frac{x-y}{z} = c_2$$

The general solution is

$$F\left(\frac{1}{y}-\frac{1}{x}, \frac{x-y}{z}\right) = 0$$

EXAMPLE 4.

Find the equation of the cone satisfing the equation $xp+yq=z$ and passing through the circle $x^2+y^2+z^2=4$, $x+y+z=2$.

Solution:

Given equation is

$$xp+yq=z \longrightarrow (1)$$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

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where $P = x$, $Q = y$, $R = z$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \longrightarrow (2)$$

Taking first two ~~equations~~ ratios in equation (2), we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating on both sides, we get $\log x = \log y + \log c_1$

$$\log x - \log y = \log c_1$$

$$\frac{x}{y} = c_1 \Rightarrow x = y c_1 \longrightarrow (3)$$

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Taking last two ratios in ②, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides, we get

$$\log y = \log x + \log c_2$$

$$\log(y/z) = \log c_2$$

$$y/z = c_2$$

$$y = z c_2 \longrightarrow ④$$

The general equation is

$$F(x/y, y/z) = 0$$

Now, we have to find a functional relation between x/y and y/z such that they also satisfies the equation

$$x^2 + y^2 + z^2 = 4 \longrightarrow ⑤$$

$$x + y + z = 2 \longrightarrow ⑥$$

Substitute equation ③ + ④ in equation ⑤ + ⑥, we get

$$x = y c_1 \quad ; \quad y = z c_2$$

$z = z c_1 c_2$

$(5) \Rightarrow x^2 + y^2 + z^2 = 4$

$z^2 c_1^2 c_2^2 + z^2 c_2^2 + z^2 = 4$

$z^2 (c_1^2 c_2^2 + c_2^2 + 1) = 4$

$(6) \Rightarrow x + y + z = 2$

$z c_1 c_2 + z c_1 c_2 + z = 2$

$z (c_1 c_2 + c_2 + 1) = 2$

$z^2 (c_1 c_2 + c_2 + 1)^2 = 4$

$\therefore z^2 (c_1^2 c_2^2 + c_2^2 + 1)$

$= z^2 (c_1 c_2 + c_2 + 1)^2$

$c_1^2 c_2^2 + c_2^2 + 1 = c_1^2 c_2^2 + c_2^2 + 1 +$
$2 c_1 c_2^2 + 2 c_2 + 2 c_1 c_2$

$2 c_1 c_2^2 + 2 c_2 + 2 c_1 c_2 = 0$

$2 c_2 (c_1 c_2 + c_1 + 1) = 0$

$c_1 c_2 + c_1 + 1 = 0$

$\left(\frac{x}{y} \cdot \frac{y}{z} + \frac{x}{y} + 1\right) = 0$

$\frac{x}{z} + \frac{x}{y} + 1 = 0$

We get the required surface

$\therefore xy + xz + yz = 0.$

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EXAMPLE 5.

Solve $(x^2-yz)p+(y^2-2x)q = z^2-xy$.

Solution:

Given equation is

$$(x^2-yz)p+(y^2-2x)q = z^2-xy$$

The auxiliary equation is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

where $P=x^2-yz$, $Q=y^2-zx$, $R=z^2-xy$.

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \longrightarrow (2)$$

Choosing $(1,-1,0)$ and $(0,+1,-1)$ as multiples, we get

$$\frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2-xy}$$

$$\frac{dx-dy}{(x-y)(x+y)+z}=\frac{dy-dz}{(y-z)(x+y+z)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$

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$$\frac{dx-dy}{x-y} = \frac{dy-dz}{(y-z)}$$

Integrating on both sides, we get

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) - \log(y-z) = \log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

Choosing (x, y, z) and $(1, 1, 1)$ as multiplier, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-yz) + y(y^2-zx) + z(z^2-xy)} = \frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3-xyz+y^3-xyz+z^3-xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)} = \frac{d(x+y+z)}{x^2+y^2+z^2-xy-yz-xz}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x+y+z} = d(x+y+z)$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)\,d(x+y+z)$$

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2}$$

$$x^2 + y^2 + z^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz + c_2$$

$$2(xy + yz + zx) = c_2$$

$$\therefore (xy + yz + zx) = c_2$$

The general solution of the given equation is

$$F\left(\frac{x-y}{y-z}, xy + yz + zx\right) = 0$$

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