UNIT-V

Laplace Transforms

Definitions:

If a function $f(t)$ is defined for all positive values of the variable t then the laplace transform of $f(t)$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

The operator L that transforms $f(t)$ into $F(s)$ is called the laplace transform operator

NOTE: $\lim_{s \to \infty} F(s) = \int_0^\infty e^{-st} f(t)\, dt = 0 \quad [\because e^{-\infty} = 0]$

Piecewise continuous (or) continuity:

A function $f(t)$ is said to be piecewise continuous in a closed interval $[a,b]$, if it is defined on that interval and is such that the interval can be broken up into a finite number of subinterval in each of which $f(t)$ is continuous. Also $f(t)$ can have only ordinary finite discontinuous in the interval.

Example:

A function $f(t)$ is said to be piecewise continuous in the closed interval $[0,3]$

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 2t & 1 < t < 2 \\ (t-2)^2 & 2 < t < 3 \end{cases}$$

is a piecewise continuous on $[0,3]$

Exponential order:

A function $f(t)$ is said to be of exponential order, if

$$\lim_{s \to \infty} e^{-st} . f(t) = 0$$

(or) if for some number. So, the product $|f(t)| < M$, for $t > T$. i.e, $e^{-st} |f(t)|$ is bounded for large values of t, say for $t > T$.

Sufficient condition for the Laplace Transform :-

(i). $f(t)$ is continuous (or) piecewise continuous in the closed interval $[a,b]$ where $a>0$

(ii) $f(t)$ is of exponential order.

(iii). $t^n f(t)$ is bounded near $t=0$ for some number $n>1$.

Laplace transform of Elementary function :-

* Laplace transform of 1: $[L\{1\}]$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

here $f(t) = 1$

$$L\{1\} = \int_0^\infty e^{-st} (1).\,dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s}[-1]$$

$$L\{1\} = \frac{1}{s}$$

*Laplace Transform of $L\{t\}$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

here $f(t) = t$

$$L\{t\} = \int_0^\infty e^{-st}\, t\,dt$$

let $u = t$ $\quad dv = e^{-st}dt$

$du = dt$ $\quad v = \frac{e^{-st}}{-s}$

$$L\{t\} = \left[\frac{-t\,e^{-st}}{s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

$$= \frac{-1}{s}\left[t e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s^2}\left[e^{-\infty} - e^{-0}\right]$$

$$= \frac{-1}{s^2}[-1]$$

$$L\{1\} = \frac{1}{s^2}, \quad s > 0$$

*. Laplace Transform of t^2 : $L\{t^2\}$

we know that

$$L f\{t\} = \int_0^\infty e^{-st} f(t) = F(s)$$

let $f(t) = t^2$

$$L\{t^2\} = \int_0^\infty e^{-st}\, t^2\,dt$$

let $u = t^2$ $\quad dv = e^{-st}dt$

$u' = 2t$ $\quad v = \frac{e^{-st}}{-s}$

$u'' = 2$ $\quad v_1 = \frac{1}{s^2} e^{-st}$

$u''' = 0$

$v_2 = -\frac{1}{s^3} e^{-st}$

By Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$L\{t^2\} = \left[t^2 \frac{e^{-st}}{-s}\right]_0^\infty - \left[2t \frac{1}{s^2} e^{-st}\right]_0^\infty +$$

$$\left[2\left(\frac{-1}{s^3}\right) e^{-st}\right]_0^\infty$$

$$L\{t^2\} = \frac{-2}{s^3}[-1]$$

$$L\{t^2\} = \frac{2}{s^3}, \boxed{s>0}$$

In general $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}, s>0$

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* Laplace Transform of e^{-at} : $L\{e^{-at}\}$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

here $f(t) = \int_0^\infty e^{-at}$

$$L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{(s+a)}[0-1]$$

$$L\{e^{-at}\} = \frac{1}{s+a}, \ s+a>0$$

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* Laplace Transform of e^{at}, $L\{e^{at}\}$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

here $f(t) = e^{at}$

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \frac{-1}{(s-a)} [0 - 1]$$

$$L\{e^{at}\} = \frac{1}{s-a}, \quad s-a > 0 \Rightarrow s = a$$

* Laplace transform of $t^{1/2}$: $L\{t^{1/2}\}$

we know that $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \rightarrow (1)$

put $n = \frac{1}{2}$ in (1)

$$L\{t^{1/2}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}}$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$L\{t^{1/2}\} = \frac{\sqrt{\pi}}{2 s^{3/2}}$$

Evaluate : $L\{t^{-1/2}\}$

we know that

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \rightarrow (1)$$

put $n = -\frac{1}{2}$ in (1)

$$L\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}$$

$$L\{t^{-1/2}\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$L\{t^3\}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$f(t) = t^3$

$$L\{t^3\} = \int_0^\infty e^{-st} t^3\, dt$$

$u = t^3$ $\quad dv = e^{-st} dt$

$u' = 3t^2$ $\quad v = \frac{e^{-st}}{-s}$

$u'' = 6t$ $\quad v_1 = \frac{e^{-st}}{s^2}$ $\quad v_3 = \frac{e^{-st}}{s^4}$

$u''' = 6$ $\quad v_2 = \frac{e^{-st}}{-s^3}$

$u'''' = 0$

$\boxed{\text{By Bernouli's theorem, } \int u\,dv = uv - uv_1 + u''v_2 \ldots\ldots}$

$$\int_0^\infty e^{-st} t^3 dt = \left(t^3\left(\frac{e^{-st}}{-s}\right)\right)_0^\infty - \left(3t^2\left(\frac{e^{-st}}{s^2}\right)\right)_0^\infty + \left(6t\ \frac{e^{-st}}{-s^3}\right)_0^\infty - \left(6\left(\frac{e^{-st}}{s^4}\right)\right)_0^\infty$$

$$= \frac{-6}{s^4}\left(e^{-\infty} - e^0\right)$$

$$= \frac{-6}{s^4}(-1)$$

$$\boxed{\int_0^\infty e^{-st} t^3\, dt = \frac{6}{s^4}}$$

$L\{t^4\}$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{t^4\} = \int_0^\infty e^{-st} t^4\, dt$$

$u = t^4$, $u' = 4t^3$, $u'' = 12t^2$, $u''' = 24t$, $u'''' = 24$

$dv = e^{-st} dt$, $v = \frac{e^{-st}}{-s}$, $v_1 = \frac{e^{-st}}{s^2}$, $v_2 = \frac{e^{-st}}{-s^3}$, $v_3 = \frac{e^{-st}}{s^4}$

$\boxed{\text{By Bernouli's theorem, } \int u\,dv = uv - u'v_1 + u''v_2 + \ldots\ldots}$

$$\int_0^\infty e^{-st} t^4 \, dt = \left[t^4\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \left[4t\left(\frac{e^{-st}}{s^2}\right)\right]_0^\infty + \left[24\left(\frac{e^{-st}}{-s^5}\right)\right]_0^\infty$$

$$= \frac{-24}{s^5}\left(e^{-\infty} - e^{0}\right)$$

$$\boxed{\int_0^\infty e^{-st} t^4 \, dt = \frac{24}{s^5}}$$

① Laplace Transform of sinat :- [L{sin at}]

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

here $f(t) = \sin at$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt \quad - ①$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos x\right]$$

comparing with ①

$a = -s$, $t = x$, $b = a$

$$L\{\sin at\} = \left\{\frac{e^{-st}}{s^2+a^2}\left[-s \sin at - a \cos at\right]\right.$$

$$= \frac{-1}{s^2+a^2}\left[-a\right]$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}, \quad s > 0$$

②. Laplace Transform of cosat :- L{cos at}

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

here

$f(t) =$ ~~sinat~~ cos at

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt \rightarrow ①$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[b \sin bx + a \cos bx\right]$$

Comparing equation ①

$a = -s$, $t = x$, $b = a$

$$L\{\cos at\} = \frac{e^{-st}}{a^2+b^2}\left[a \sin at - s\cos at\right]$$

$$= \frac{-1}{s^2+a^2}(-s)$$

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

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③ Laplace Theorem of Sin at :- $[L\{\sin at\}]$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

here $f(t) = \sin h\, at$

$$L\{\sinh at\} = \int_0^\infty e^{-st} \sin at\, dt$$

we know that $\sinh x = \frac{e^x - e^{-x}}{2}$

$$L\{\sinh at\} = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-st}e^{at} - e^{-st}e^{-at}\right] dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t} - e^{-(s+a)t}\right) dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{2}\left[\frac{-1}{(s-a)} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a) - (s-a)}{(s^2-a^2)}\right]$$

$$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right) = \frac{a}{s^2-a^2}$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2-a^2}}$$

Laplace Transform of coshat :- $L\{\cosh at\}$

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

here $f(t) = \cosh at$

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at\, dt$$

we know that

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$= \int_0^\infty e^{-st}\left(\frac{e^x + e^{-x}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty (e^{-st} e^{at} + e^{-st} e^{-at})\, dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} + e^{-(s+a)t}\right] dt$$

$$= \frac{1}{2}\left[\frac{1}{-(s-a)} - \frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{2s}{2(s^2-a^2)}$$

$$\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$$

Properties of Laplace Transforms :-

Linearity property :-

If $f(t)$ and $\phi(t)$ are two functions of t defined for positive values of t and c is any channel.

Then,

i) $L\{f(t) + \phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$

ii) $L\{c.f(t)\} = c L\{f(t)\}$

Proof:

we know that:

i) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t)+\phi(t)\} = \int_0^\infty e^{-st} (f(t)+\phi(t))\,dt$$

$$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} \phi(t)\,dt$$

$$L\{f(t)+\phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$$

ii) $$L\{c f(t)\} = \int_0^\infty e^{-st} (c f(t))\,dt$$

$$= c\int_0^\infty e^{-st} f(t)\,dt$$

$$= c\,(L\{f(t)\})$$

$$L\{c\,f(t)\} = c\,L\{f(t)\}$$

Shifting property:

If $$L\{f(t)\} = F(s)$$

Then

i) $L\{e^{-at} f(t)\} = F(s+a)$

ii) $L\{e^{at} f(t)\} = F(s-a)$

Proof:

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s) \quad -①$$

i) here $f(t) = e^{-at} f(t)$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st} e^{-at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt \quad -②$$

comparing ①, ②

$$L\{e^{-at} f(t)\} = F(s+a)$$

ii) here $f(t) = e^{at} f(t)$

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt \quad -③$$

comparing ①, ③

$$L\{e^{at} f(t)\} = F(s-a)$$

④ change of scale property:

If

$$L\{f(t)\} = F(s)$$

then

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

<u>Proof:</u>

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s) \quad - ①$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

let $at = x$ $t = 0 \Rightarrow x = 0$

$a\,dt = dx$ $t = \infty \Rightarrow x = \infty$

$dx = \frac{dx}{a}$

$$L\{f(at)\} = \int_0^\infty e^{-\infty(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} f(x)\,dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x)\,dx$$

$$L\{f(at)\} = \frac{1}{a} F(s/a) \quad (\text{From } ①)$$

Problems

1. If $L\{\cos t\} = F(s) = \frac{s}{s^2+1}$

Then find $L\{\cos a\}$ (by using change of scale property)

Solution

given $L\{\cos t\} = \frac{s}{s^2+1}$

by change of scale property

"If $L\{f(t)\} = F(s)$

Then

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad -①$$

From ①

$$L\{\cos at\} = \frac{1}{a}\left(\frac{s/a}{(s/a)^2+1}\right)$$

$$= \frac{1}{a^2}\left(\frac{s}{\frac{s^2+a^2}{a^2}}\right)$$

$$= \frac{1}{a^2}\left(\frac{sa^2}{s^2+a^2}\right)$$

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

2. If $L\{\sinh at\} = F(x) = \frac{1}{s^2-1}$. Then find $L\{\sinh at\}$ (by using change of scale property)

Solution:-

given $L\{\sinh t\} = \frac{1}{s^2-1}$

by change of scale property

If $L\{f(t)\} = F(s)$

Then

$$L'\{f(t)\} = \frac{1}{a} F\left(\frac{s}{a}\right) - ①$$

From (1)

$$L\{\sinh at\} = \frac{1}{a}\left(\frac{1/a}{(s/a)^2-1}\right)$$

$$= \frac{1}{a}\left(\frac{a^2}{s^2-a^2}\right)$$

$$\boxed{L\{\sinh at\} = \frac{1}{s^2-a^2}}$$

Laplace Transform of derivatives :-

If $L\{f(t)\} = F(s)$ then

i) $L\{f'(t)\} = sL\{f(t)\} - f(0) = sF(s) - f(0)$

ii) $L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - s f(0)$

$= s^2 F(s) - s f(0) - s f'(0)$

Proof:

i) We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$u = e^{-st}$ $\quad$ $dv = f'(t)\, dt$

$du = -s e^{-st}$ $\quad$ $v = f(t)$

$$L\{f(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)\, e^{-st} f(t)\, dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= sL\{f(t)\} - f(0)$$

$$L\{f'(t)\} = sF(s) - f(0)$$

ii)

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\, dt$$

$u = e^{-st}$, $u' = -s e^{-st}$, $u'' = s^2 e^{-st}$

$dv = f''(t)\, dt$, $v = f'(t)$, $v_1 = f(t)$, $v_2 = \int_0^\infty f(t)\, dt$

By Bernouli's formula,

$$\int u\, dv = uv - u'v_1 + u''v_2 \ldots$$

$$L\{f''(t)\} = \left[e^{-st} f(t)\right]_0^\infty + \left[s e^{-st} f(t)\right]_0^\infty + \int_0^\infty s^2 e^{-st} f(t)\, dt$$

$$= -f'(0) - s f(0) + s^2 \int_0^\infty e^{-st} f(t)\, dt$$

$$= s^2 L\{f(t)\} - s f(0) - f'(0)$$

$$L\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

Laplace Transform of n^{th} derivative:-

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0)\, f'(0) \cdots f^{n-1}(0)$$

state and proof initial value theorem and final value theorem:

If $L\{f(t)\} = F(s)$ then

(a) $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$ (Initial value theorem)

(b) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ (Final value theorem)

Proof:

We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

(i) To prove

(a) $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

by Laplace Transforms of derivatives,
we get

$$L\{f'(t)\} = sF(s) - f(0) \rightarrow ①$$

Taking limit as $s \to \infty$ in ① on both sides

$$\lim_{s \to \infty} L\{f'(t)\} = \lim_{s \to \infty} [sF(s) - f(0)]$$

$$\lim_{s \to \infty} \int_0^\infty e^{-st} f(t)\, dt = \lim_{s \to \infty} [sF(s) - f(0)]$$

$$0 = \lim_{s \to \infty} sF(s) - f(0)$$

(B) To prove

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

by laplace transforms of derivatives,
we get

$$L\{f(t)\} = sF(s) - f(0) \rightarrow ①$$

Taking limit as $s \to 0$ in ① both sides. we get

$$\lim_{s \to 0} \int_0^\infty e^{-st} f'(t)\, dt = \lim_{s \to 0} [sF(s) - f(0)]$$

$$\int_0^\infty \frac{d\, f(t)}{dt}\, dt = \lim_{s \to 0} [sF(s) - f(0)]$$

$$[f(t)]_0^\infty = \lim_{s\to 0}[SF(s) - f(0)]$$

$$\lim_{t\to\infty} f(t) - \lim_{t\to 0} f(t) = \lim_{s\to 0} SF(s) - f(0)$$

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} SF(s) - f(0)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s)$$

Problems

1) Find $L\{t^2+2t+3\}$

Solution:

we know that

$$L\{1\} = \frac{1}{s}, \; s>0$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}, \; s>0$$

$$L\{t^2+2t+3\} = L\{t^2\} + 2L\{t\} + 3L\{1\}$$

$$= \frac{2!}{s^3} + 2\cdot\frac{1!}{s^2} + 3\frac{1}{s}$$

$$\boxed{L\{t^2+2t+3\} = \frac{2}{s^3} + \frac{2\cdot 1}{s^2} + \frac{3}{s}}$$

H.W.

1) $L\{t^3-3t^2+2\}$

2) $L\{at^2+bt+c\}$

2) Find $L\{\sin^2 t\}$

Solution:

we know that

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\sin^2 2t = \frac{1-\cos 4t}{2}$$

$$L\{\sin^2 2t\} = L\left\{\frac{1-\cos 4t}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$\boxed{= \frac{1}{2}\left(\frac{s^2+16-s^2}{s(s^2+16)}\right)}$$

3) Find $L\{\sin^3 2t\}$ (a)

Solution:

we know that

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$4\sin^3 A = 3\sin A - \sin 3A$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

$$L\{\sin^3 2t\} = \frac{1}{4}[3\sin^3 2t - \sin 6t]$$

$$L\{\sin^3 2t\} = \frac{1}{4}\{L\{3\sin^3 2t - \sin 6t\}\}$$

$$L\{\sin^3 2t\} = \frac{1}{4}[3L\{\sin^3 2t\} - L\{\sin 6t\}]$$

$$= \frac{1}{4}\left[3\cdot\frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

$$= \frac{3}{2}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{3}{2}\left[\frac{32}{(s^2+4)(s^2+36)}\right]$$

$$\boxed{L\{\sin^3 2t\} = \frac{48}{(s^2+4)(s^2+36)}}$$

W.H

(b) $L\{\cos^3 2t\}$

4. $L\{\cos t \cos 2t\}$

Soln:

we know that

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

$$\cos A\cos B = \frac{1}{2}(\cos(A+B)\cos(A-B))$$

$$\cos t\cos 2t = \frac{1}{2}[\cos 3t + \cos t]$$

$$L\{\cos t \cos t\} = \frac{1}{2}[L\{\cos 3t\} + L\{\cos t\}]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{s}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{s(s^2+1)+s(s^2+9)}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+s+s^3+9s}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3+10s}{(s^2+9)(s^2+1)}\right]$$

$$\boxed{L\{\cos t \cos 3t\} = \frac{s^3+5s}{(s^2+9)(s^2+1)}}$$

5. Find $L\{f(t)\}$ where $f(t)=0$, when $0<t\leq 2$
$= 3$ when $t>0$

Soln:
We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st}\, 3\,dt$$

$$= 3\int_2^\infty e^{-st}\,dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{-3}{s}\left[-e^{-2s}\right]$$

$$\boxed{L\{f(t)\} = \frac{3e^{-2s}}{s}}$$

H.W.

Find the laplace transform of the following.

(1) $f(t) = (t-1)^2$, when $t>1$
$= 0$ when $t<1$

(2) $f(t) = e^{-t}$, when $0<t<4$
$= 0$ when $t>4$

(3) $f(t) = \sin t$, when $0<t<\pi$
$= 0$ when $t>\pi$

(4) $f(t) = \cos t$, when $0 < t < \pi$

$= \sin t$ when $t > \pi$

(5) $f(t) = \sin t$, when $0 < t < \pi$

$= 0$ when $\pi < t < 2\pi$

6. Prove that $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$

Sol:

we know that

$$L\{t^n\} = \int_0^\infty e^{-st} f(t)\,dt$$

here $f(t) = t^n$

$$L\{t^n\} = \int_0^\infty e^{-st} t^n\,dt$$

$$st = x \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{1}{s}\,dx$$

$t = 0 \Rightarrow x = 0$

$t = \infty \Rightarrow x = \infty$

$$L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x^n}{s^n}\right)\frac{1}{s}\,dx$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\,dx$$

$$\boxed{L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}} \qquad \left[\because \int_0^\infty e^{-x} x^n\,dx = \Gamma(n+1)\right]$$

7. If $\left.\begin{array}{l} L\{\cos bt\} = s/(s^2+b^2) \\ L\{\sin bt\} = b/(s^2+b^2) \end{array}\right\}$ then

i) $L\{e^{-at}\cos bt\} = \dfrac{s+a}{(s+a)^2+b^2}$

ii) $L\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

iii) $L\{e^{-at}\sin bt\} = \dfrac{b}{(s+a)^2+b^2}$

iv) $L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

8. If $L\{t^n\} = \frac{n!}{s^{n+1}}$, $s>0$ n is a positive integer then $L\{e^{at}t^n\} = \frac{n!}{(s+a)^{n+1}}$, $L\{e^{st}t^n\} = \frac{n!}{(s-a)^{n+1}}$

9. If $L\{f(t)\} = F(s)$, then prove that $L\{tf(t)\} = \frac{-d}{ds} f(s)$

Proof:

we know that

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \rightarrow (1)$$

Taking differentiation on both side w.r.t s we get

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty (-t)\, e^{-st} f(t)\,dt$$

$$\frac{d}{ds}F(s) = -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$\boxed{L\{tf(t)\} = \frac{-d}{ds}(F(s))}$$

10. If $L\{f(t)\} = F(s)$ then prove that $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$

Proof:

we know that

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \rightarrow (1)$$

Also we have

$$L\{t(f(t))\} = \frac{-d}{ds}F(s)$$

$$= \frac{-d}{ds} L\{f(t)\}$$

$$L\{t^2 f(t)\} = L\{t.tf(t)\}$$

$$= \frac{-d}{ds} L\{t f(t)\}$$

$$= \frac{-d}{ds}\left[-\frac{d}{ds} L\{f(t)\}\right]$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{d^n s} F(s)$$

$$= (-1)^2 \frac{d^2}{d^2 s} L\{f(t)\}$$

$$\boxed{L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{d^2 s}(F(s))}$$

Problems

1. Find $L\{te^{-at}\}$

Solution:

we know that

$$L\{t^n f(t)\} = \frac{-d}{ds} F(s)$$

$$\therefore L\{te^{-at}\} = \frac{-d}{ds}\left(\frac{1}{s+a}\right)$$

$$= \frac{-d}{ds}\left[L\, e^{-st}\right]$$

$$= \frac{-d}{ds}(s+a)^{-1}$$

$$= (-1)(-1)(s+a)^{-2}$$

$$\boxed{L\{te^{-at}\} = \frac{1}{(s+a)^2}}$$

2. Find $L\{t^2 e^{-3t}\}$

Solution:

we know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{d^n s} F(s)$$

$$\therefore L\{t^2 e^{-3t}\} = (-1)^2 \frac{d^2}{d^2 s} L\, f(t)$$

$$= (-1)^2 \frac{d^2}{d^2 s}\left[L\, \{e^{-3t}\}\right]$$

$$= \frac{d^2}{d^2 s}\left[\frac{1}{s+3}\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s+3}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right]$$

$$= (-1) \frac{d}{ds}(s+3)^{-2}$$

$$= (-1)(-2)(s+3)^{-3}$$

$$\boxed{L\{t^2 e^{-3t}\} = \frac{2}{(s+3)^{-3}}}$$

3. $L\{t \sin at\}$

we know that

$$L\{t f(t)\} = \frac{-d}{ds} F(s)$$

$$L\{t \sin at\} = \frac{-d}{ds} L\{f(t)\}$$

$$= \frac{-d}{ds} \left[L\{\sin at\}\right]$$

$$= \frac{-d}{ds} \left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{(-1)(-2sa)}{(s^2+a^2)^2}$$

$$\boxed{L\{t \sin at\} = \frac{2sa}{(s^2+a^2)^2}}$$

4. $L\{t e^{-t} \sin t\}$

Solution:

we know that

$$L\{t f(t)\} = \frac{-d}{ds} F(s)$$

$$L\{e^{-at} f(t)\} = F(s+a)$$

$$L\{t e^{-t} \sin t\} = \frac{-d}{ds} L\{e^{-t} \sin t\}$$

$$= \frac{-d}{ds} \left(\frac{1}{(s+1)^2+1}\right)$$

$$= -\left[\frac{(s^2+2s+2) - (2s+2)(1)}{(s^2+2s+2)^2}\right]$$

$$= \frac{-(-1)(2s+2)}{(s^2+2s+2)^2}$$

$$\boxed{L\{t e^{-t} \sin t\} = \frac{2(s+1)}{(s^2+2s+2)^2}}$$

Find the laplace transforms of the following function

1) te^{-5t}
2) t^2e^{3t}
3) $t\cos 2t$
4) $t\cos^2 t$
5) $(1+te^{-t})^2$
6) $t^2\sin 2t$
7) $t\sin at$
8) $t^2\cosh at$
9) $\sin at - at\cos at$
10) $te^{-t}\cos t$

Home work sums

1) $L\{t^3-3t^2+2\}$

Solution:

We know that

$$L\{1\} = \frac{1}{s} \quad s>0$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \quad s>0$$

$$L\{t^3-3t^2+2\} = L\{t^3\} - 3L\{t^2\} + L\{2\}$$

$$= \frac{3!}{s^4} - \frac{3(2!)}{s^3} + \frac{2}{s}$$

$$\boxed{L\{t^3-3t^2+2\} = \frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}}$$

2) $L\{at^2+bt+c\}$

Solution:

we know that

$$L\{1\} = \frac{1}{s} \quad s>0$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \quad s>0$$

$$L\{at^2+bt+c\} = aL\{t^2\} + bL\{t\} + cL\{1\}$$

$$= a\frac{2!}{s^3} + b\left(\frac{1}{s^2}\right) + \frac{c}{s}$$

$$\boxed{L\{at^2+bt+c\} = \frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s}}$$

3) $L\{\sin^2 t\}$

Solution:

we know that

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\left(\frac{s^2+4-s^2}{s(s^2+4)}\right)$$

$$= \frac{1}{2}\left(\frac{4}{s(s^2+4)}\right)$$

$$\boxed{L\{\sin^2 t\} = \frac{2}{s(s^2+4)}}$$

4) $L\{\cos^2 3t\}$

Soln.:

we know that

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$\cos^2 3t = \frac{1+\cos 6t}{2}$$

$$L\{\cos^2 3t\} = L\left\{\frac{1+\cos 6t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right] = \frac{1}{2}\left[\frac{s^2+36+s^2}{s(s^2+36)}\right]$$

$$\boxed{L\{\cos^2 3t\} = \frac{1}{2}\left(\frac{2s^2+36}{s(s^2+36)}\right) = \frac{s^2+18}{s(s^2+36)}}$$

5. $L\{\sinh 3t\}$

Soln

we know that

$$L\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$L\{\sinh 3t\} = \frac{3}{s^2-9}$$

6. $L\{\cos^3 2t\}$

Soln:

we know that

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$4\cos^3 A = \cos 3A - 3\cos A$$

$$\cos^3 A = \frac{1}{4}(\cos 3A - 3\cos A)$$

$$L\{\cos^3 2t\} = \frac{1}{4} L\{\cos 6t + 3\cos 2t\}$$

$$= \frac{1}{4}\left[L\{\cos 6t\} + 3L\{\cos 2t\}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$$

$$= \frac{1}{4}\left(\frac{s^3+4s+3s^3+108s}{(s^2+36)(s^2+4)}\right)$$

$$= \frac{1}{4}\left[\frac{4s^3+112s}{(s^2+36)(s^2+4)}\right]$$

$$\boxed{L\{\cos^3 2t\} = \frac{s^3+28s}{(s^2+36)(s^2+4)}}$$

7. Find the laplace transform of the following

$f(t) = (t-1)^2$ when $t > 1$

$= 0$ $\quad t < 1$

Soln:

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \int_0^1 e^{-st}(0)\,dt + \int_1^\infty e^{-st}(t-1)\,dt$$

$$= \int_1^\infty e^{-st}(t^2-2t+1)\,dt$$

$$= \int_1^\infty e^{-st}(t^2 dt) - 2\int_1^\infty e^{-st}\,t\,dt + \int_1^\infty e^{-st}\,dt$$

$$\int_1^\infty e^{-st}t^2\,dt \qquad u = t^2 \qquad dv = e^{-st}\,dt$$

$$u' = 2t \qquad v = \frac{e^{-st}}{-s}$$

$$u'' = 2 \qquad v_1 = \frac{e^{-st}}{s^2}, \quad v_2 = \frac{e^{-st}}{-s^3}$$

$$= \left[t^2\left(\frac{e^{-st}}{-s}\right)\right]_1^\infty - \left[2t\left(\frac{e^{-st}}{s^2}\right)\right]_1^\infty + \left[2\left(\frac{e^{-st}}{-s^3}\right)\right]_1^\infty$$

$$= \left(\frac{-e^{-s}}{-s}\right) + \left(\frac{-2e^{-s}}{s^2}\right) + \left(\frac{-2e^{-s}}{-s^3}\right)$$

$$= \frac{e^{-s}}{s}\left(1 - \frac{2}{s} + \frac{2}{s^2}\right)$$

$$-2\int_1^\infty e^{-3t}\,dt \qquad u = t \qquad dv = e^{-st}\,dt$$

$$du = dt \qquad v = \frac{e^{-s}}{-s}$$

$$= 2\left\{\left[t\left(\frac{e^{-st}}{-s}\right)\right]_1^\infty - \int_1^\infty e^{-st}\,dt\right\}$$

$$= 2\left\{\left(\frac{e^{-s}}{-s}\right) - \left(\frac{e^{-st}}{s}\right)^\infty\right\}$$

$$= 2\left(\frac{e^{-s}}{-s} + \frac{e^{-s}}{s}\right)$$

$$\int_1^\infty e^{-st}\,dt = \left(\frac{e^{-st}}{-s}\right)_1^\infty$$

$$= \left(\frac{e^{-s}}{s}\right)$$

$$L\{f(t)\} = \left(\frac{e^{-s}}{s}\right)\left(1 - \frac{2}{s} + \frac{2}{s^2}\right) - 0 + \frac{e^{-s}}{s}$$

$$= \frac{e^{-s}}{s}\left[1 + \frac{2}{s} + \frac{2}{s^2} + 1\right]$$

$$= \frac{e^{-s}}{s}\left[2 - \frac{2}{s} + \frac{2}{s^2}\right]$$

$$\boxed{L\{f(t)\} = \frac{2e^{-s}}{s}\left[1 - \frac{1}{s} + \frac{1}{s^2}\right]}$$

8. Find $f(t) = e^{-t}$ when $0 < t < 4$

$= 0$ when $t > 4$

Soln

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{f(t)\} = \int_0^4 e^{-st}\, dt + \int_4^\infty e^{-st}(0)\, dt$$

$$= \int_0^4 e^{-st} e^{-t}\, dt$$

$$= \int_0^4 e^{-t(s+1)}\, dt$$

$$= \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^4$$

$$= \left[\frac{e^{-(s+1)4}}{-(s+1)} - \frac{e^{-(s+1)0}}{-(s+1)}\right]$$

$$\boxed{L\{f(t)\} = \frac{1}{s+1} - \frac{e^{-(s+1)4}}{s+1}}$$

9. Find the L.T of the following functions

1) $t e^{-5t}$

we know that

$$L\{t(f(t))\} = \frac{-d}{ds} F(s)$$

$$\therefore L\{t\{e^{-5t}\}\} = \frac{-d}{ds}\left(\frac{1}{s+5}\right)$$

$$= \frac{-d}{ds}(s+5)^{-1}$$

$$= (-1)(s+5)^{-2}$$

$$= -(s+5)^{-2}$$

$$\boxed{L\{t(e^{-5t})\} = \frac{-1}{(s+5)^2}}$$

10. $t^2 e^{3t}$

we know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{d^n s} F(s)$$

$$L\{t^2 e^{3t}\} = (-1)^2 \frac{d^2}{ds^2} L\{e^{3t}\}$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s-3}\right)\right)$$

$$= \frac{d}{ds}(-1)(s-3)^{-2}$$

$$= (-1)(-2)(s-3)^{-3}$$

$$\boxed{L\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}}$$

11. $t \cos 2t$

we know, that

$$L\{t f(t)\} = \frac{-d}{ds} F(s)$$

$$L\{t(\cos 2t)\} = \frac{-d}{ds} L\{f(t)\}$$

$$= \frac{-d}{ds} L\{\cos 2t\}$$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= (-1)\left[\frac{(s^2+4)(1) - (s)(2s)}{(s^2+4)^2}\right]$$

$$= -1\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$\boxed{L\{t(\cos 2t\} = (-1)\left[\frac{-s^2+4}{(s^2+4)^2}\right] = \frac{s^2-4}{(s^2+4)^2}}$$

12. $t\cos^2 t$

we know that

$$L\{t f(t)\} = \frac{-d}{ds} F(s) \qquad \cos^2 t = \frac{1+\cos 2t}{2}$$

$$L\{t(\cos^2 t)\} = \frac{-d}{ds}\{L f(t)\}$$

$$= \frac{-d}{ds} L\left(\frac{1+\cos 2t}{2}\right)$$

$$= \frac{1}{2}\frac{-d}{ds}\left[L\{1\} + L\{\cos 2t\}\right]$$

$$\frac{1}{s} + \quad 1/s^2+4$$

$$= \frac{1}{2}\frac{-d}{ds}\left(\frac{s^2+4+s^2}{s(s^2+4)}\right)$$

$$= (-1)\left[\frac{(s^3+4s)(2s) - (s^2+2)(3s^2+4)}{(s^3+4s)^2}\right]$$

$$= (-1)\left[\frac{2s^4+8s^2 - (3s^4+4s^2+6s^2+8)}{(s^3+4s)^2}\right]$$

$$= (-1)\left(\frac{-s^4-2s^2-8}{(s^3+4s)^2}\right)$$

$$\boxed{L\{t\cos^2 t\} = \frac{s^4+2s^2+8}{(s^3+4s)^2}}$$

13. $(1+te^{-t})^2$

we know that

$$L\{t f(t)\} = \frac{-d}{ds} F(s) \qquad L\{t^n f(t)\} = (-1)^n \frac{d^n}{d^n s} F(s)$$

$$L\{1\} = \frac{1}{s}$$

$$(1+te^{-t})^2 = 1 + 2te^{-t} + t^2 t^{-2t}$$

$$L\{(1+te^{-t})^2\} = L\{1\} + 2L\{te^{-t}\} + L\{t^2 e^{-2t}\}$$

$$= \frac{1}{s} + 2\frac{-d}{ds}(e^{-t}) + (-1)^2 \frac{d^2}{d^2 s}(e^{-2t})$$

$$= \frac{1}{s} + 2\left(-\frac{d}{ds}\left(\frac{1}{(s+1)}\right)\right) + \frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{s} - 2\left[\frac{(s+1)(0) - (1)(1)}{(s+1)^2}\right] + \frac{d}{ds}\left[\frac{(s+2)0 - 1(1)}{(s+2)^2}\right]$$

$$= \frac{1}{s} + \frac{2}{(s+1)^2} + \frac{d}{ds}\left(\frac{-1}{(s+2)^2}\right)$$

$$= \frac{1}{s} + \frac{2}{(s+1)^2} - \frac{(-2)}{(s+2)^3}$$

$$\boxed{L\{(1+te^{-t})^2\} = \frac{1}{s} + \frac{2}{(s+1)^2} + \frac{2}{(s+2)^3}}$$

14. $t^2 \sin 2t$

we know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}(\sin 2t)$$

$$= \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right)$$

$$= \frac{d}{ds}\left[2(-1)(s^2+4)^{-2}\right]$$

$$= 2(-2)(-1)(s^2+4)^{-3}$$

$$\boxed{L\{t^2 \sin 2t\} = \frac{4}{(s^2+4)^3}}$$

15^{th} last page.

15. ~~[illegible]~~ $t^2 \cosh at$

Soln:

we know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$L\{t^2 \cosh at\} = (-1)^2 \frac{d^2}{ds^2}[\cosh at]$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2-a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2-a^2)(1)-s(2s)}{(s^2-a^2)^2}\right]$$

$$= \frac{d}{ds}\left(\frac{-(s^2+a^2)}{(s^2-a^2)^2}\right)$$

$$= (-1)\left[\frac{(s^2-a^2)^2(2s)-(s^2+a^2)(4s)}{(s^2-a^2)^4}\right]$$

---

1. If $L\{f(t)\} = F(s)$ and if $\frac{f(t)}{t}$ has a limit as $t \to 0$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Prove:

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Taking integration with respect to s from s to ∞ on both sides, we get

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_s^\infty e^{-st} f(t)\,dt.ds$$

on integrating the order of integration we get,

$$\int_s^\infty F(s)\,ds = \int_0^\infty \int_s^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_0^\infty f(t)\, dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$$

$$\boxed{\int_s^\infty F(s)\, ds = L\left\{\frac{f(t)}{t}\right\}}$$

---

(1) Evaluate : Find $L\left\{\frac{1-e^{t}}{t}\right\}$

we know that

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

$$= \int_s^\infty L\{f(t)\}\, ds$$

$$L\left\{\frac{1-e^{t}}{t}\right\} = \int_s^\infty L\{1-e^{t}\}\, ds$$

$$= \int_s^\infty L\{1\} - L\{e^{t}\}\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-1}\right] ds$$

$$= -\left[\log s - \log(s-1)\right] = \log(s-1) - \log s$$

$$\boxed{L\left\{\frac{1-e^{t}}{t}\right\} = \log \frac{s-1}{s}}$$

---

(2) Find $L\left\{\frac{\sin at}{t}\right\}$

Soln:

we know that

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\, dt = \int_s^\infty F(s)\, ds.$$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty L(f(\sin at))\, ds$$

$$= \int_s^\infty \left[\frac{a}{s^2+a^2}\right] ds \quad \left[\int \frac{a}{x^2+a^2}\, dx = \tan^{-1}\frac{x}{a}\right]$$

$$= \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

$$\boxed{L\left\{\frac{\sin at}{t}\right\} = \cot^{-1}\frac{s}{a}}$$

---

3. Evaluate $\int_0^\infty e^{-2t} \sin 3t\, dt$

Soln:

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \quad -①$$

$$\int_0^\infty e^{-st} \sin 3t\, dt = L(\sin 3t) - \frac{3}{s^2+9} \rightarrow ②$$

put $s = 2$ in ①

$$\boxed{\int_0^\infty e^{-2t} \sin 3t\, dt = \frac{3}{4+9} = \frac{3}{13}}$$

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4. Evaluate $\int_0^\infty e^{-3t}\, t \cos t\, dt$

Soln:

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{t\cos t\} = \int_0^\infty e^{-st}\, t\cos t\, dt$$

$$L(t\cos t) = -\frac{d}{ds} L(\cos t)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]$$

$$= -\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)$$

$$L(t\cos t) = \frac{s^2-1}{(s^2+1)^2} \rightarrow (1)$$

Put $s=3$ in (1)

$$\int_0^\infty e^{-3t}(t\cos t)\,dt = \frac{9-1}{(9+1)^2} = \frac{8}{100}$$

$$\boxed{\int_0^\infty e^{-3t}(t\cos t)\,dt = \frac{2}{25}}$$

5. Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt$

Soln: we know that

$$\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$$

$$\int_s^\infty F(s)\,ds = L\left\{\frac{f(t)}{t}\right\}$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left(\frac{e^{-t}-e^{-2t}}{t}\right)$$

$$= \int_s^\infty L(e^{-t}-e^{-2t})\,ds$$

$$= \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds$$

$$= \left[\log(s+1) - \log(s+2)\right]_s^\infty$$

$$= -\left[\log(s+1) - \log(s+2)\right].$$

$$= \log(s+2) - \log(s+1)$$

$$= \log(s+2) - \log(s+1)$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = \log\left(\frac{s+2}{s+1}\right) \rightarrow (1)$$

put $s=0$ in (1)

$$\boxed{\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt = \log\frac{2}{1} = \log 2.}$$

15. $t \sinh at$

Soln:

we know that

$$L\{t f(t)\} = \frac{-d}{ds} F(s)$$

$$L\{t \sinh at\} = \frac{-d}{ds}\left[L\{\sinh at\}\right]$$

$$= \frac{-d}{ds}\left(\frac{a}{s^2-a^2}\right)$$

$$= (-1)\left[\frac{(s^2-a^2)(0) - a(2s)}{(s^2-a^2)^2}\right]$$

$$= (-1)\frac{-2as}{(s^2-a^2)^2}$$

$$\boxed{L\{t \sinh at\} = \frac{2as}{(s^2-a^2)^2}}$$