UNIT - I

ANALYTICAL GEOMETRY OF 2D

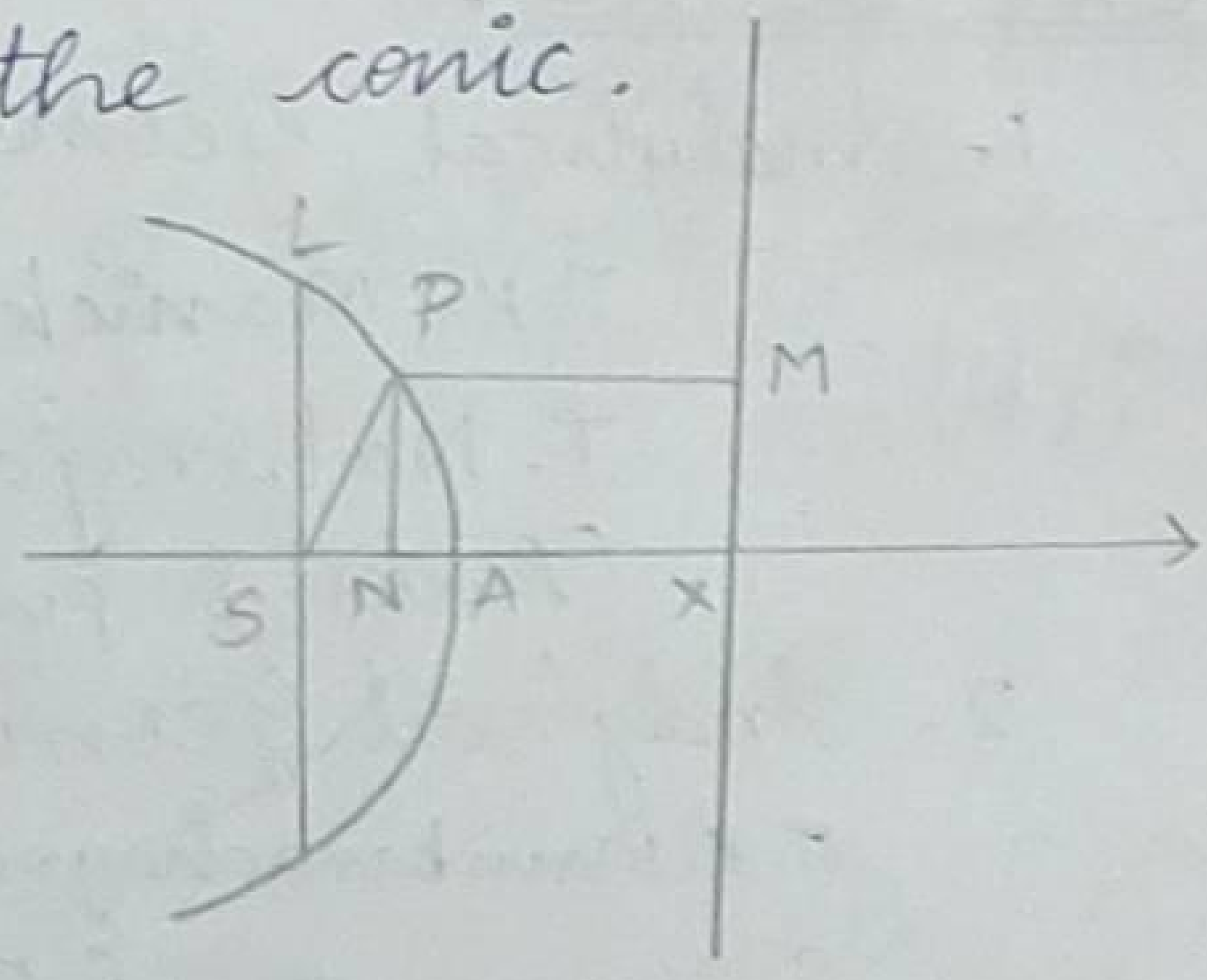
Section 9:

Polar equation of a conic

Let S be the focus and XM the directrix of the conic and let e be the eccentricity. Draw SX perpendicular to the directrix and take SX as the initial line and S as the pole.

Let P be any point on the conic and let its co-ordinates be (r, θ), so that $SP = r$ and the angle XSP be θ. Draw PM and PN perpendiculars respectively to the directrix and to the initial line.

Let LSL' be the latus rectum of the conic.



$SL = e.SX$

i.e $l = e.SX$

$\therefore SX = \frac{l}{e}$

$SP = e.PM$

$= e.NX$

$= e.(SX - SN)$

$= e.\left(\frac{l}{e} - SP\cos\theta\right)$

i.e., $r = e\left(\frac{l}{e} - r\cos\theta\right)$

$\therefore r = l - er\cos\theta$

i.e., $r(1 + e\cos\theta) = l$

$\therefore \frac{l}{r} = 1 + e\cos\theta$

COROLLARY

If the axis SX of the conic makes an angle α with the initial line SA, SP makes an angle $\theta - \alpha$ with the initial line and so the equation of the conic will be

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha)$$

DIRECTRIX CORRESPONDING TO THE POLE S.

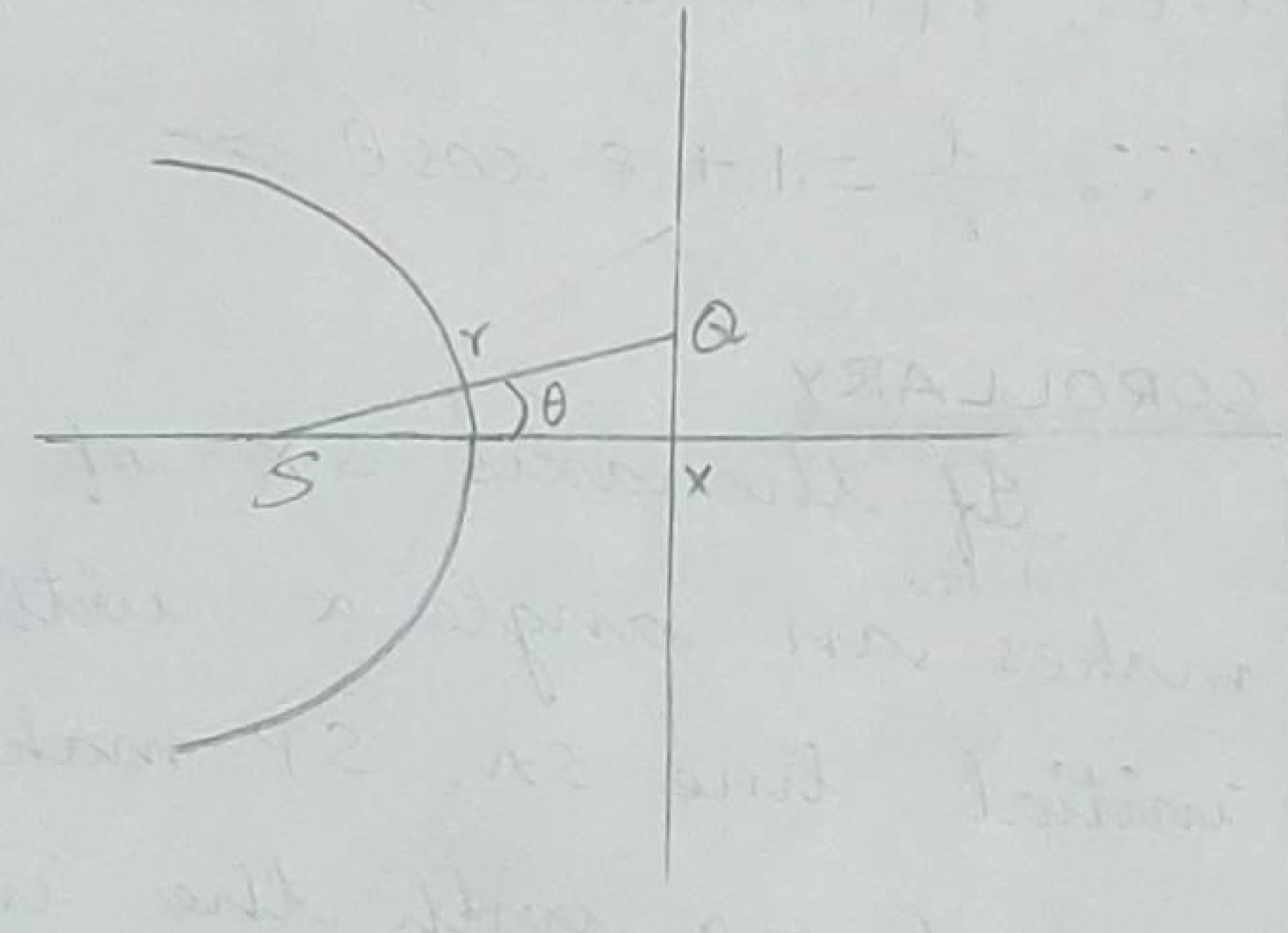
Let the coordinates of any point Q on the directrix be (r, θ).

$$SX = SQ \cos\theta$$

i.e., $\frac{l}{e} = r \cos\theta$

i.e., $\frac{l}{r} = e \cos\theta$

The ~~direction~~ of the conic $\frac{l}{r} = 1 + e\cos(\theta - \alpha)$ corresponding to the focus S is $\frac{l}{r} = e\cos(\theta - \alpha)$.



9.1 TRACING THE CONIC

$$\frac{l}{r} = 1 + e\cos\theta$$

Case 1. If $e = 0$, the equation

reduces to $r = l$.

The conic becomes a circle of radius l with its centre at the pole.

Case 2.

Let $0 < e < 1$.

$$r = \frac{l}{1+e} \text{ when } \theta = 0.$$

$\therefore$ The maximum value of $\frac{l}{r}$ is $1+e$.

The maximum value of $\cos\theta$ is 1.

The maximum value of r is $\frac{l}{1+e}$.

The minimum value of $\cos\theta$ is -1 when $\theta = \pi$.

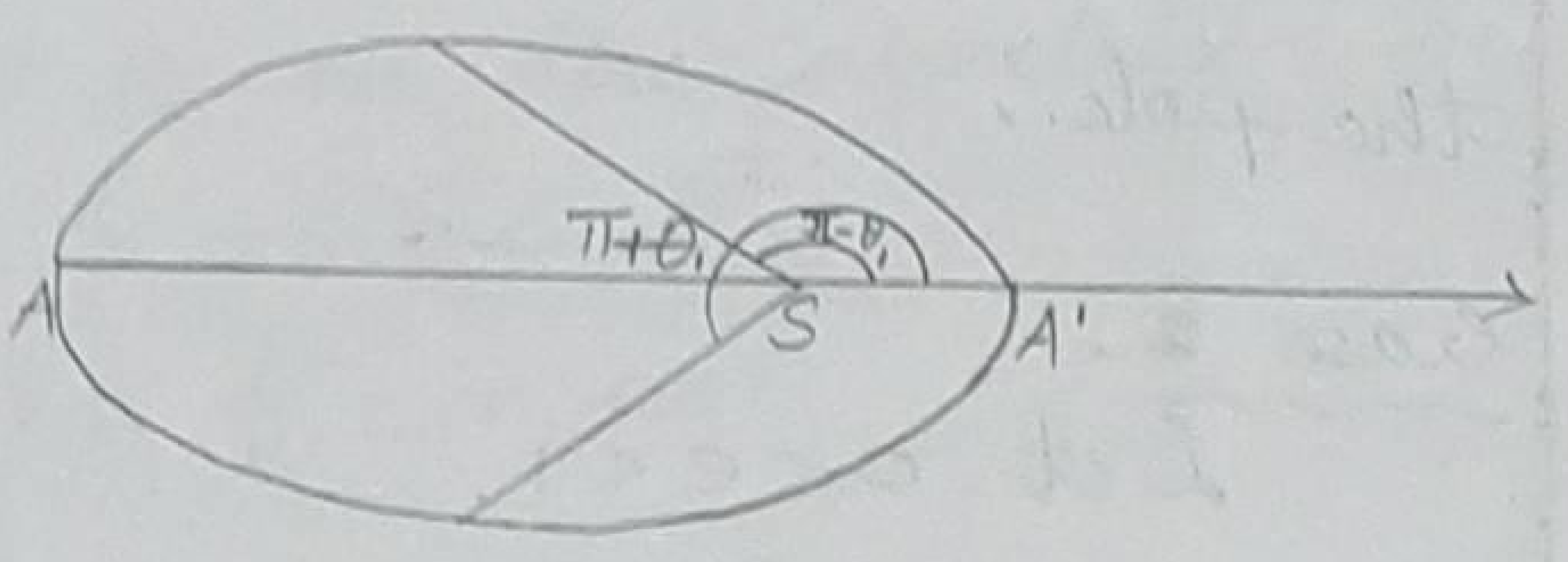
This minimum value of r is attained when $\theta = 0$.

$$r = \frac{l}{1-e} \text{ when } \theta = \pi$$

$\therefore$ The maximum value of r is $\frac{l}{1-e}$ and this maximum value is attained when $\theta = \pi$.

As θ increases from 0 to π, r increases from $\frac{l}{1+e}$ till its greatest value $\frac{l}{1-e}$ is reached when

$\theta = \pi$. When $\theta = \pi/2$ the value of r is l.



$$\cos(\pi+\theta_1) = \cos(\pi-\theta_1)$$

$\therefore$ Corresponding to the values $\pi+\theta_1$ and $\pi-\theta_1$ of θ we get the same value for r.

Hence the curve is symmetrical about the initial line.

Sine $e<1$ the curve is an ellipse.

Its shape is shown in the figure

Case 3. When $e=1$, the conic becomes a parabola and its equation is

$$\frac{l}{r} = 1+\cos\theta.$$

When $\theta=0$, $r=\frac{1}{2}l$. This is the minimum value of r. When θ ~~increasing~~ increases from 0, the value of r also increases. When θ approaches π, the value of r approaches infinity.

As in case 2 the curve is symmetrical about the initial line.

Case 4.

$e > 1$

Then the expression $1 + e\cos\theta$ becomes zero for a value of θ lying between $\pi/2$ and π.

Let us assume that value as α. Then $1 + e\cos\alpha = 0$. $\therefore \cos\alpha = -\frac{1}{e}$

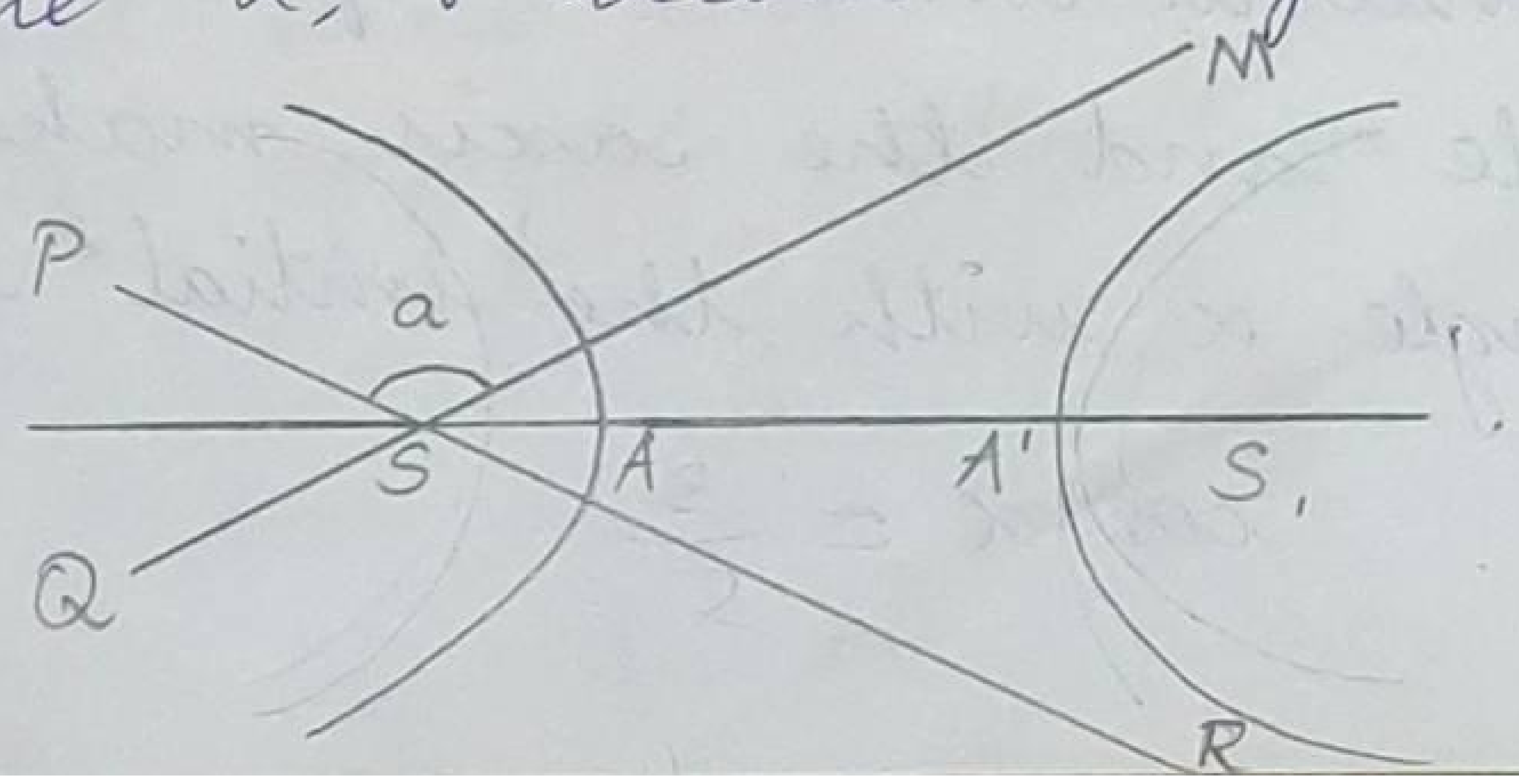
When $\theta = 0$, $r = \frac{l}{1+e}$ and when $\theta = \pi$, $r = \frac{l}{1-e}$.

As θ increases, r increases.

As θ approaches α, r approaches infinity.

The part of the curve given by these values of θ is AP.

When θ increases beyond the value α, r becomes negatives.



When $\theta = \pi$, then $r = -\frac{l}{e-1}$. These values of θ correspond to the portion RA'.

For values of θ between π and 2π, the curve can be drawn by symmetry, A'M corresponding to values of θ between π and $2\pi - \alpha$ and QA to values of θ between $2\pi - \alpha$ and 2π.

EXAMPLE 1.

Trace the curve $\frac{10}{r} = 3\cos\theta + 4\cos\theta + 5$.

Solution:

This equation can be written in the form

$$\frac{2}{r} = 1 + \frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta$$

If $\frac{3}{5} = \cos\alpha$, then $\sin\alpha = \frac{4}{5}$.

$\therefore \frac{2}{r} = 1 + \cos(\theta - \alpha)$ is the equation of the curve.

This equation represents a parabola with its focus at the pole and the axis makes an angle α with the initial line

$$\cos\alpha = \frac{3}{5}.$$

$\therefore\ \alpha = 53^\circ 8'$ (approximately).

The semi-latus rectum of the parabola $= 2$.

$\therefore$ Locus rectum of the ~~rectum~~ parabola $= 4$.

If A is the vertex of the parabola, $AS = 1$.

If L and L' are the extremities of the latus rectum,

$SL = SL' = 2$.

When $\theta = 0$, $\frac{2}{r} = 1 + \cos\alpha$

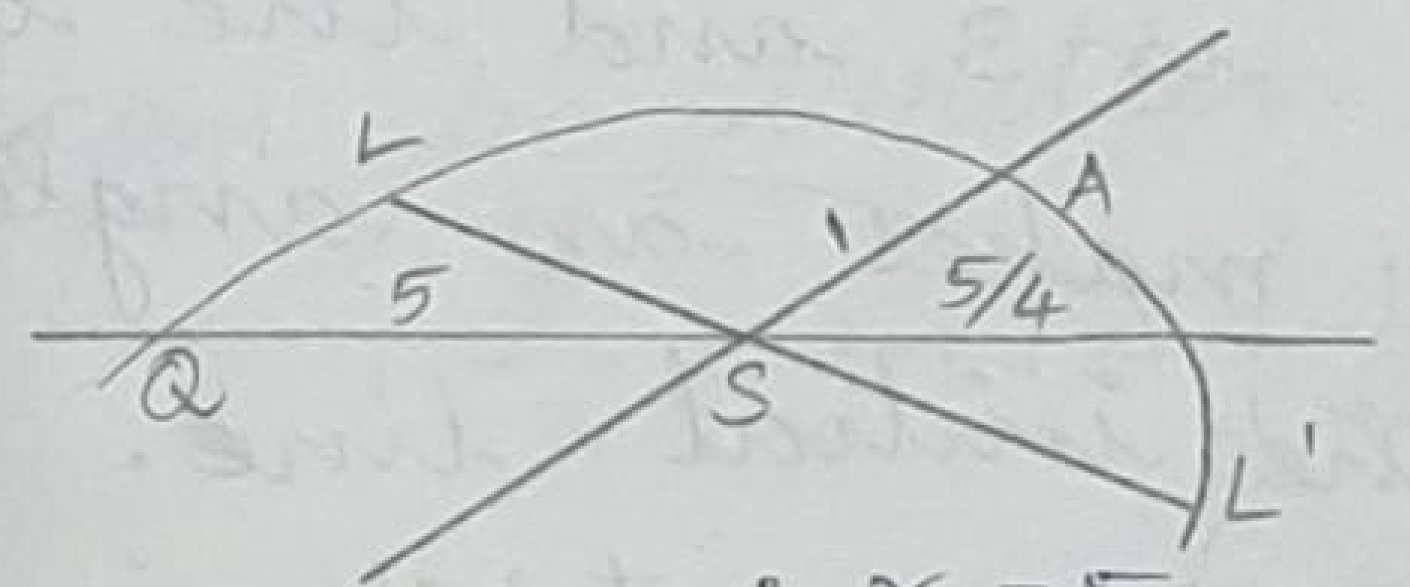
$= 1 + \frac{3}{5}$

$\therefore\ r = \frac{5}{4}$.

When $\theta = \pi$, $\frac{2}{r} = 1 + \cos(\pi - \alpha)$

$= 1 - \cos\alpha$

$= 1 - \frac{3}{5} = \frac{2}{5}$



$\therefore\ r = 5$.

If the parabola meets the initial line at P and its extension in the opposite direction in Q, then $SP = \frac{5}{4}$, $SQ = 5$.

From these we get the shape of the curve as below:

EXAMPLE 2.

Trace the curve $\frac{12}{r} = 4 + \sqrt{3}\cos\theta + \sin\theta$.

Solution:

Dividing the equation of the curve by 4, we get

$$\frac{3}{r} = 1 + \frac{\sqrt{3}}{4}\cos\theta + \frac{3}{4}\sin\theta.$$

This can be written as

$$\frac{3}{r} = 1 + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$

$$= 1 + \frac{\sqrt{3}}{2}\cos(\theta - \pi/3).$$

$\therefore$ One of the foci of the conic is at the pole; the eccentricity of the conic is $\frac{\sqrt{3}}{2}$, the semi-latus rectum is 3 and the axis of the conic makes an angle of 60° with the initial line.

Since the eccentricity is less than one, the equation represents an ellipse, $l = \frac{b^2}{a} = a(1-e^2)$

$$\therefore 3 = a\left(1 - \frac{3}{4}\right) = a(1/4)$$

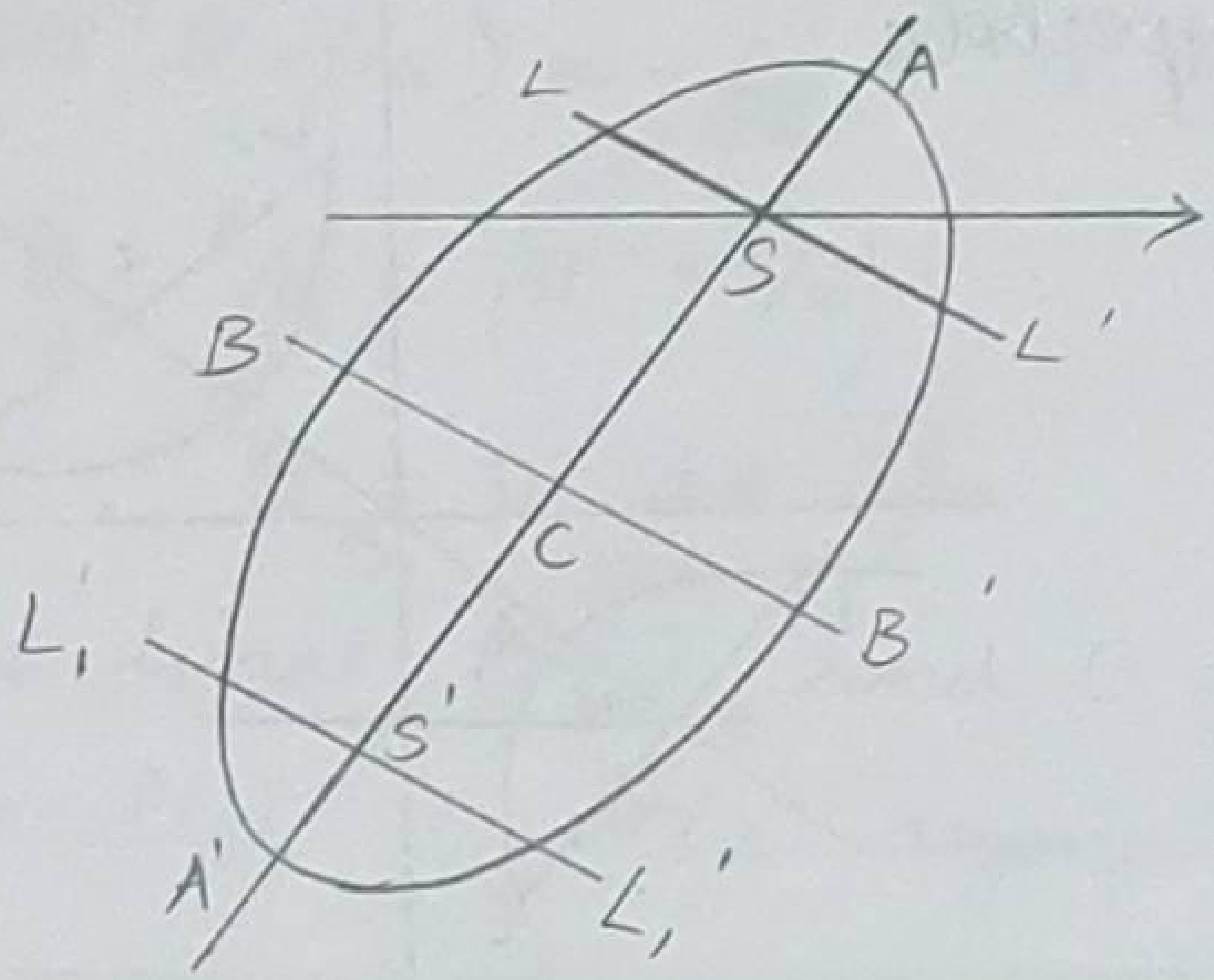
$$\therefore a = 12$$

$$b^2 = al = 36. \quad \therefore b = 6.$$

If the other focus is S',

$$SS' = 2ae = \frac{2(12)\sqrt{3}}{2} = 12\sqrt{3}.$$

If A is the vertex near to S, then $SA = 12 - 6\sqrt{3} = 6(2-\sqrt{3})$.



EXAMPLE 3.

Trace the conic $\frac{2}{r} = 1 + \cos\theta + \sin\theta$

Solution:

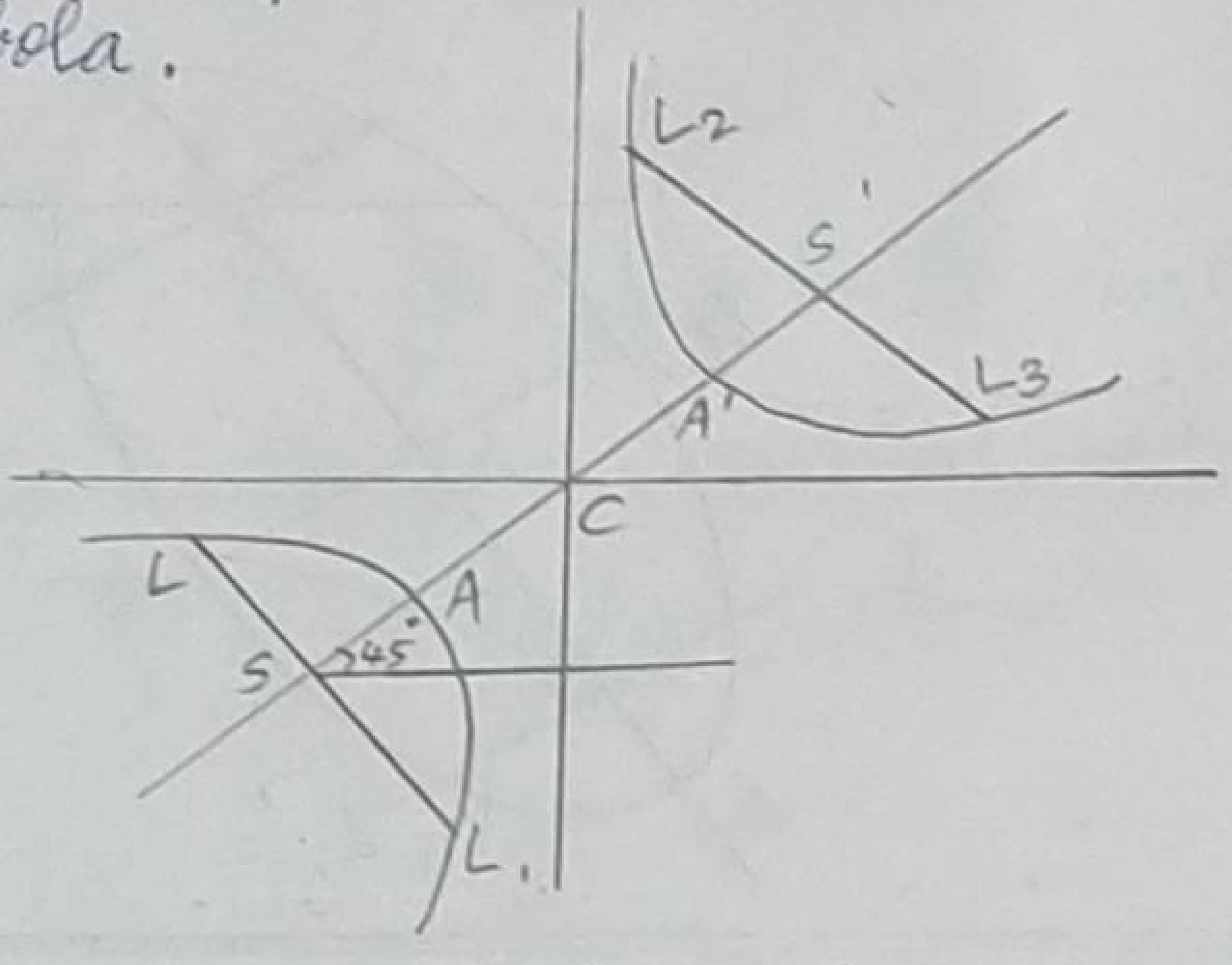
The equation of the conic can be written as

$$\frac{2}{r} = 1 + \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$= 1 + \sqrt{2}\cos(\theta - \pi/4)$$

From the form of the equation we get that one of the foci S of the conic is at the pole, the axis of the conic makes an angle of 45° with the initial line, the semi-latus rectum l is 2 and the eccentricity of the conic is $\sqrt{2}$.

Since the eccentricity is $\sqrt{2}$, the equation represents a rectangular hyperbola.



In the rectangular hyperbola,

$a = b$.

$$\therefore l = \frac{b^2}{a} = a$$

$$\therefore a = 2.$$

If the other focus is S', then $SS' = 2ae = 4\sqrt{2}$. If the vertex near to S is A, then $SA = CS - CA = ae - a$

$$= 2(\sqrt{2} - 1).$$

If the vertex near to S' is A',

then $SA' = 2(\sqrt{2}-1) + AA'$

$$= 2(\sqrt{2}-1) + 4$$

$$= 2(\sqrt{2}+1).$$

To find the asymptotes of the conic, find the values of θ when r approaches infinity.

$$\therefore 1+\sqrt{2}\cos(\theta - \pi/4) = 0$$

i.e., $\cos(\theta - \pi/4) = -\frac{1}{\sqrt{2}} = \cos(\pi \pm \pi/4)$

$$\therefore \theta - \pi/4 = (\pi \pm \pi/4)$$

$$\therefore \theta = \pi \text{ or } \frac{3\pi}{2}.$$

The lines $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ are parallel to the asymptotes.

~~EXERCISE 55~~

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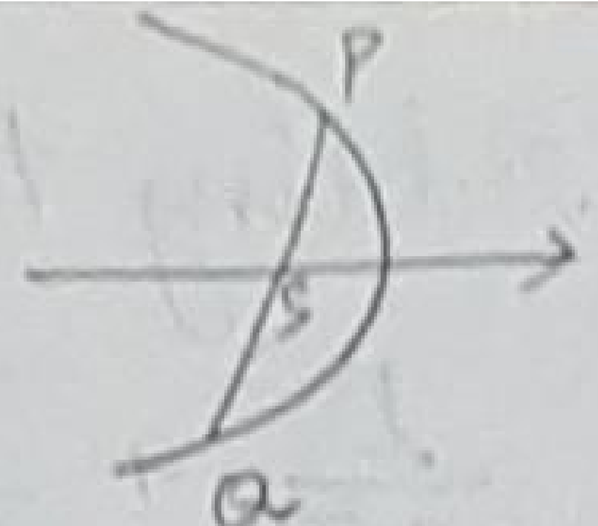

EXAMPLE 1:

Show that in a conic the semi-latus rectum is the harmonic mean between the segments of a fical chord.

Solution:

Taking the focus S as the pole and the axis as the initial line, we get the equation of the conic as

$$\frac{l}{r} = 1 + e\cos\theta$$

Let PQ be any focal chord and let the vectoral angle of P be α. Then the vectorial angle of Q is $\pi + \alpha$.

The co-ordinates of P and Q are (SP, α) and $(SQ, \pi+\alpha)$.

Since these points lie on the conic, we get

$$\frac{l}{SP} = 1 + e\cos\alpha$$

$$\frac{l}{SP} = 1 + e\cos(\pi+\alpha)$$

$$= 1 - e\cos\alpha.$$

Adding these two equations, we get

$$\frac{l}{SP} + \frac{l}{SQ} = 2 \quad \text{i.e.,} \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}$$

$\therefore$ SP, l, SQ are in H.P.

EXAMPLE 2:

A circle passing through the focus of a conic whose latus rectum is $2l$ meets the conic in four points whose distances from the focus are r_1, r_2, r_3 and r_4. Prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}$

Solution:

Taking one of the foci S as the pole and the axis as the initial line, the equation of the conic is

$$\frac{l}{r} = 1 + e\cos\theta$$

Let the diameter of the circle passing through S be d and the angle which the diameter makes with the initial line be α.

Then the equation of the circle is $r = d\cos(\theta - \alpha) \longrightarrow (2)$

Eliminating θ between (1) and (2), we will get an equation involving r whose roots are r_1, r_2, r_3, r_4.

From (1), we get $\cos\theta = \dfrac{l-r}{re}$

$$\therefore \sin\theta = \left\{1 - \frac{(l-r)^2}{r^2e^2}\right\}^{1/2}$$

Expanding (2) we get $r = d\cos\theta\cos\alpha + d\sin\theta\sin\alpha$.

i.e., $r = d\cos\alpha \cdot \dfrac{l-r}{re} + d\sin\alpha \cdot \left\{1 - \dfrac{(l-r)^2}{r^2e^2}\right\}^{1/2}$

i.e., $e^2r^4 + 2de\cos\alpha \cdot r^3 + (d^2 - 2del\cos\alpha - d^2e^2\sin^2\alpha)r^2 - 2ld^2r + d^2l^2 = 0$

$$\therefore r_1r_2r_3r_4 = \frac{d^2l^2}{e^2} \longrightarrow (3)$$

$$r_2r_3r_4 + r_3r_4r_1 + r_1r_2r_3 = \frac{2ld^2}{e^2} \longrightarrow (4)$$

Dividing (4) by (3), we get

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}$$

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EXAMPLE 3.

If two conics have a common focus, show that two of their common chords pass through the point of intersection of their directrices.

Solution:

Taking the common focus as the pole and the axis of one conic as the initial line, the equation of this conic is

$$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (1)$$

Let the axis of the other conic make an angle α with the initial line.

Then its equation is

$$\frac{L}{r} = 1 + E\cos(\theta - \alpha) \longrightarrow (2)$$

The equation of the directrices corresponding to the common focus

are $\frac{l}{r} = e\cos\theta$ and.

$$\frac{L}{r} = E\cos(\theta - \alpha)$$

Subtracting (2) ~~as~~ from (1), we get

$$\frac{l-L}{r} = e\cos\theta - E\cos(\theta - \alpha) \longrightarrow (3)$$

We can easily see that it represents a straight line and it passes through the common points of ① and ②.

Therefore ③ represents a common chord of the conics.

Since co-ordinates (r, θ), $(-r, \pi+\theta)$ represent the same point,

$\frac{l}{r} = 1 + e \cos\theta$ and

$-\frac{l}{r} = 1 + e \cos(\pi+\theta)$,

i.e., $\frac{l}{r} = -1 + e \cos\theta \longrightarrow ④$

represent the same conic.

Adding ④ and ②, we get

$$\frac{l+L}{r} = e \cos\theta + E \cos(\theta - \alpha)$$

This represents a straight line and it passes through the common points of ① and ②.

$\therefore$ Two of the common chords of the conics are

$$\frac{l-L}{r} = e \cos\theta - E \cos(\theta - \alpha)$$

$$\frac{l+L}{r} = e \cos\theta + E \cos(\theta - \alpha)$$

i.e., $\left(\frac{l}{r} - e\cos\theta\right) - \left\{\frac{L}{r} - E\cos(\theta-\alpha)\right\} = 0$

and $\left(\frac{l}{r} - e\cos\theta\right) + \left\{\frac{L}{r} - E\cos(\theta-\alpha)\right\} = 0$

$\therefore$ These chords pass through the intersection of the lines

$$\frac{l}{r} - e\cos\theta = 0 \text{ and}$$

$$\frac{L}{r} - E\cos(\theta-\alpha) = 0 \text{; which are}$$

the directrices of the conics corresponding to the common focus.

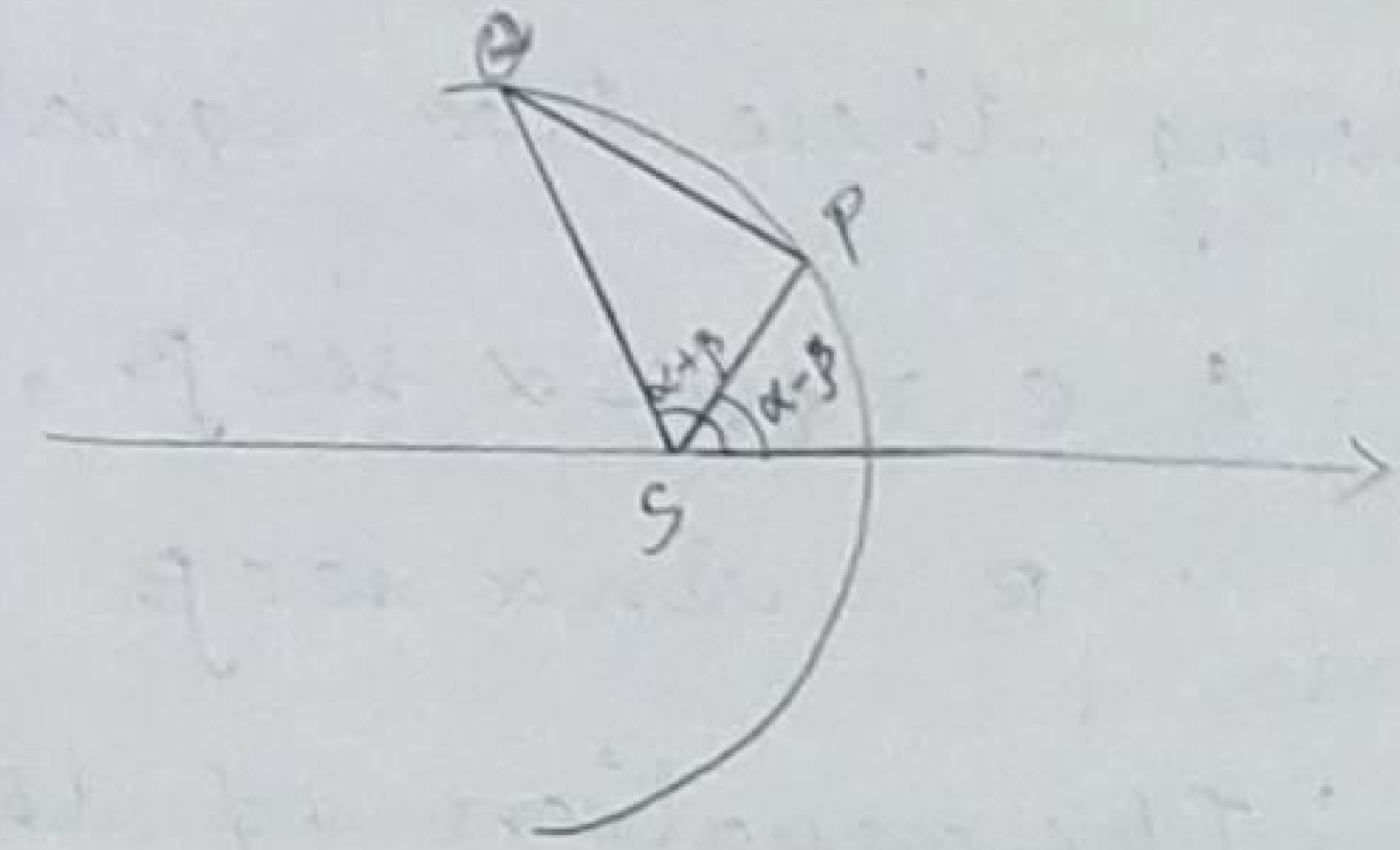
SECTION: 10

The equation of the chord of the conic, $\frac{l}{r} = 1 + e\cos\theta$ joining the points whose vectorial angles are $\alpha - \beta$ and $\alpha + \beta$.

Let the vectorial angles of the points P, Q be $\alpha - \beta$ and $\alpha + \beta$.

The equation of any line not passing through the pole is of the form

$$\frac{l}{r} = A\cos\theta + B\sin\theta$$



Since the points P and Q are on the conic, we get

$$\frac{l}{SP} = 1 + e\cos(\alpha - \beta) \longrightarrow (2)$$

$$\frac{l}{SQ} = 1 + e\cos(\alpha + \beta) \longrightarrow (3)$$

P and Q lie on the line $\frac{l}{r} = A\cos\theta + B\sin\theta$

$$\therefore \frac{l}{SP} = A\cos(\alpha - \beta) + B\sin(\alpha - \beta) \rightarrow (4)$$

$$\frac{l}{SQ} = A\cos(\alpha + \beta) + B\sin(\alpha + \beta) \rightarrow (5)$$

From (2), (3), (4) and (5), we get

$$1 + e\cos(\alpha - \beta) = A\cos(\alpha - \beta) + B\sin(\alpha - \beta)$$

$$1 + e\cos(\alpha + \beta) = A\cos(\alpha + \beta) + B\sin(\alpha + \beta)$$

i.e., $(A - e)\cos(\alpha - \beta) + B\sin(\alpha - \beta) = 1$

$$(A - e)\cos(\alpha + \beta) + B\sin(\alpha + \beta) = 1$$

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Solving these two equations, we get $A - e = \cos\alpha \sec\beta$,

$$B = \sin\alpha \sec\beta$$

$\therefore$ The equation of the chord PQ is

$$\frac{l}{r} = (e + \cos\alpha \sec\beta)\cos\theta + \sin\alpha \sec\beta \sin\theta$$

$$= e\cos\theta + (\cos\alpha\cos\theta + \sin\alpha\sin\theta)\sec\beta$$

$$= e\cos\theta + \sec\beta\cos(\theta - \alpha)$$

When $\beta = 0$, the points P and Q coincide and the chord PQ becomes the tangent at P and in that case the vectorial angle of P becomes α and the equation of the tangent at P becomes

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$$

COROLLARY:

To find the equation of the tangent at 'α' to the conic $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ we have to substitute $\theta - \gamma$ and $\alpha - \gamma$ for θ and

α in (7).

$\therefore$ The tangent at 'α' is

$$\frac{l}{r} = e\cos(\theta-\gamma) + \cos(\theta - \gamma - \overline{\alpha-\gamma})$$

i.e., $\dfrac{l}{r} = e\cos(\theta-\gamma) + \cos(\theta-\alpha)$

EXAMPLE 1.

Find the condition in order that the line $\dfrac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic $\dfrac{l}{r} = 1 + e\cos\theta$.

Solution:

Let the vectorial angle of the point of contact be α. Then the tangent at α is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha).$$

Identifying the tangent with the given line, we get

$A = e + \cos\alpha$ and $B = \sin\alpha$.

Eliminating α, we get $(A-e)^2 + B^2 = 1$

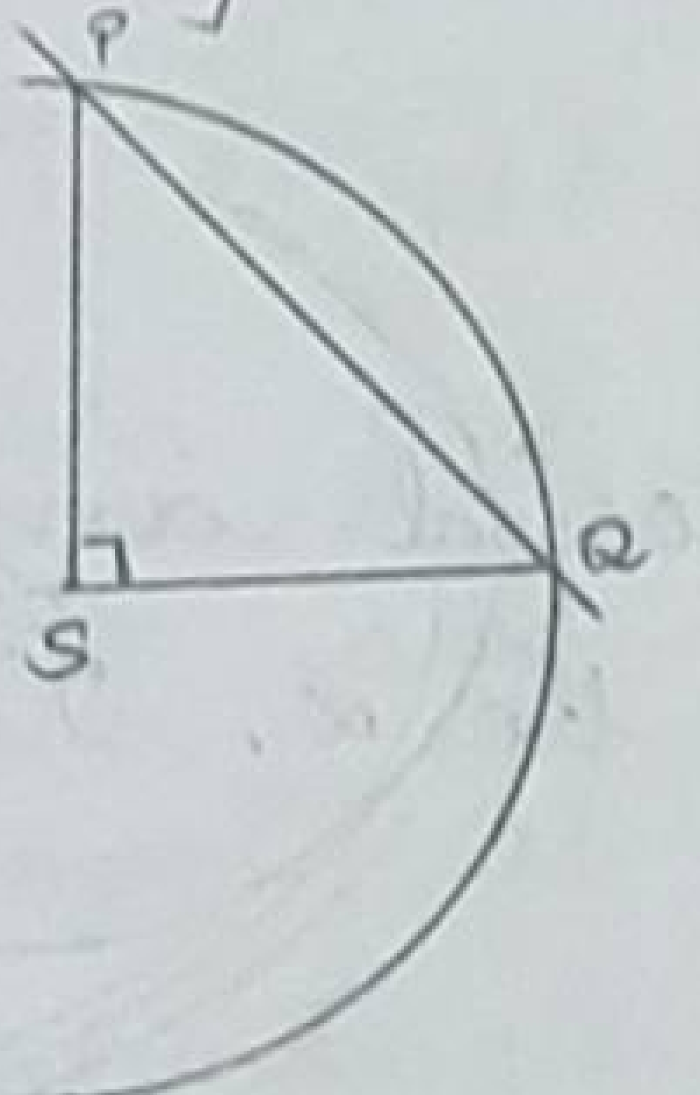
EXAMPLE 2.

Prove that the chords of a rectangular hyperbola which subtend at ~~any~~ right angle at a focus touch a fixed parabola.

Solution:

Let the equation of the rectangular hyperbola be

$\frac{l}{r} = 1 + \sqrt{2} \cos\theta$ and let the chord which subtends a right angle at S the focus be PQ and let the vectorial angles of Q and P be $\alpha - \beta$ and $\alpha + \beta$ respectively.



$\angle PSQ = (\alpha + \beta) - (\alpha - \beta)$

$= 2\beta$

$\therefore 2\beta = 90°$

$\therefore \beta = 45°.$

The equation of PQ is

$$\frac{l}{r} = \sqrt{2}\cos\theta + \sec\beta \cos(\theta - \alpha).$$

Substituting the value of β in this equation, we get

$$= \sqrt{2}\cos\theta + \sec 45° \cos(\theta - \alpha)$$

$$= \sqrt{2}\cos\theta + \sqrt{2}\cos(\theta - \alpha)$$

i.e., $$\frac{l}{r\sqrt{2}} = \cos\theta + \cos(\theta - \alpha)$$

The line touches the conic

$$\frac{l}{r\sqrt{2}} = 1 + \cos\theta \text{ at } '\alpha'.$$

This conic is a parabola having the pole as its focus.

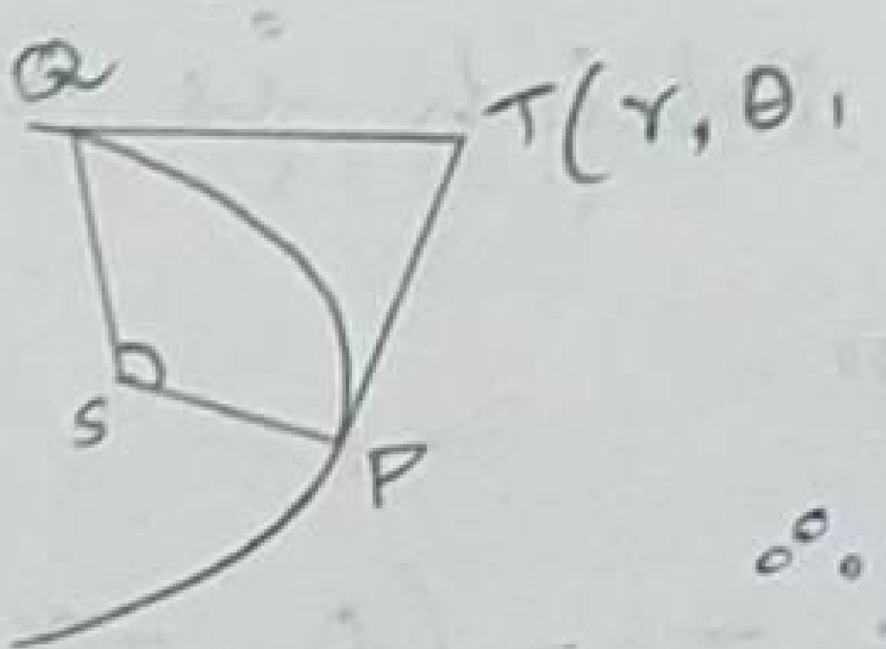
EXAMPLE 3.

A chord PQ of a conic subtends an angle of 2β of constant magnitude at the pole. Find the locus of the intersection of the tangents at P and Q.

Solution:

Let the vectorial angles of P and Q be $\alpha-\beta$ and $\alpha+\beta$ respectively.

Then $\angle QSP = (\alpha+\beta) - (\alpha-\beta)$



$= 2\beta$

$=$ constant $2k$.

$\therefore \beta = k.$

Let the ~~angles~~ tangents at P and Q intersect at T whose co-ordinates are (r_1, θ_1).

The tangents at P is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha + \beta)$$

The tangents at Q is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \beta)$$

Since (r_1, θ_1) is a point on these tangents, we get

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha + \beta)$$

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha - \beta)$$

Subtracting ② from ①, we get

$$\cos(\theta_1 - \alpha + \beta) = \cos(\theta_1 - \alpha - \beta)$$

$$\therefore \theta_1 - \alpha + \beta = \pm(\theta_1 - \alpha - \beta)$$

If we take the positive sign, we get $\beta = \alpha$ and so the points P and Q coincide which is contrary to our assumption.

$$\therefore \theta_1 - \alpha + \beta = -(\theta_1 - \alpha - \beta)$$

$$\text{i.e., } 2\theta_1 = 2\alpha$$

$$\theta_1 = \alpha$$

Substituting this value in ①, we get

$$\frac{l}{r_1} = e \cos\theta_1 + \cos\beta_1$$

i.e., $\frac{l}{r_1} = e \cos\theta_1 + \cos k$.

$\therefore$ Locus of (r_1, θ_1) is

$$\frac{l}{r} = e \cos\theta + \cos k$$

i.e., $\frac{l \sec k}{r} = 1 + e \sec k \cos\theta$

Putting $l \sec k$ as L and $e \sec k$ as E, we get the locus as

$$\frac{L}{r} = 1 + E \cos\theta$$

This represents a conic with the pole as one of its foci and the initial line as its axis.

The directrix of the conic is $\frac{L}{r} = E \cos\theta$

i.e., $\frac{l \sec k}{r} = e \sec k \cos\theta$

i.e., $\frac{l}{r} = e \cos\theta$.

This line is also the directrix of the given conic.

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SECTION : 11

The asymptotes of the conic

$$\frac{l}{r} = 1 + e\cos\theta.$$

The tangent at 'α' to the conic is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \rightarrow (1)$$

This tangents becomes an asymptote if the point of contact lies at infinity, i.e.; the point whose vectorial angle α lies on the conic at infinite distance from the pole.

$$\therefore \theta = 1 + e\cos\alpha \longrightarrow (2)$$

If we eliminate α between (1) and (2), we will get the equation of the asymptotes.

Expanding (1), we get

$$\frac{l}{r} = e\cos\theta + \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$= (e + \cos\alpha)\cos\theta + \sin\theta \cdot \sin\alpha \rightarrow (3)$$

From (2), $\cos\alpha = -\frac{1}{e}$

$\therefore \sin\alpha = \pm\left(1 - \frac{1}{e^2}\right)^{1/2}$

Substituting these values of $\cos\alpha$ and $\sin\alpha$ in ③, we get.

$$\frac{l}{r} = \cos\theta\left(e - \frac{1}{e}\right) \pm \sin\theta\left(1 - \frac{1}{e^2}\right)^{1/2}$$

$$= \frac{e^2-1}{e}\cos\theta \pm \frac{\sqrt{e^2-1}}{e}\sin\theta$$

$$\frac{l}{r} = \frac{e^2-1}{e}\left(\cos\theta \pm \frac{\sin\theta}{\sqrt{e^2-1}}\right)$$

SECTION : 12.

Equation of the normal at the point P whose vectorial angle is α.

The equation of the tangent to the conic 'α' is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$$

The equation of any line perpendicular to this tangent is of the form

$$\frac{k}{r} = e\cos\left(\frac{\pi}{2} + \theta\right) + \cos\left(\frac{\pi}{2} + \theta - \alpha\right)$$

i.e, $\frac{k}{r} = -e\sin\theta - \sin(\theta - \alpha)$

If it is a normal at 'α', it passes through P whose polar co-ordinates are (SP, α).

$$\therefore \frac{k}{SP} = -e\sin\alpha - \sin(\alpha - \alpha)$$

$$= -e\sin\alpha$$

P is a point on the conic.

$$\therefore \frac{l}{SP} = 1 + e\cos\alpha$$

$$\therefore SP = \frac{l}{1 + e\cos\alpha}.$$

$$\therefore k = -\frac{le\sin\alpha}{1 + e\cos\alpha}$$

The equation of the normal becomes

$$-\frac{e\sin\alpha}{1 + e\cos\alpha} \cdot \frac{l}{r} = -e\sin\theta - \sin(\theta - \alpha)$$

i.e., $\frac{e\sin\alpha}{1 + e\cos\alpha} \cdot \frac{l}{r} = e\sin\theta + \sin(\theta - \theta)$

EXAMPLE 1.

If the normal at L, one of the extremities of the latus rectum of the conic $\frac{l}{r} = 1 + e\cos\theta$, meets the curve again in Q, show that $SQ = l \cdot \dfrac{1+3e^2+e^4}{1+e^2-e^4}$.

Solution :

The coordinates of L are $(l, \pi/2)$.

The normal to the conic at 'α' is

$$\frac{e \sin\alpha}{1+e\cos\alpha} \cdot \frac{l}{r} = e\sin\theta + \sin(\theta-\alpha)$$

At L, $\alpha = \pi/2$.

$\therefore$ Normal at L is $\dfrac{el}{r} = e\sin\theta - \cos\theta \longrightarrow (1)$

The equation of the conic is

$$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (2)$$

Solving (1) and (2), the co-ordinates of Q can be got.

$\therefore$ $e(1+e\cos\theta) = e\sin\theta - \cos\theta$

i.e., $(1+e^2)\cos\theta - e\,\cancel{\cos}\,\sin\theta + e = 0$

i.e., $\{e + (1+e^2)\cos\theta\}^2 = e^2\sin^2\theta$

i.e., $e^2 + 2e(1+e^2)\cos\theta + (1+e^2)^2\cos^2\theta$
$= e^2\sin^2\theta$

i.e., $e^2\cos^2\theta + 2e(1+e^2)\cos\theta + (1+e^2)^2 \cos^2\theta = 0$

$\cos\theta \neq 0$.

$\therefore e^2\cos\theta + 2e(1+e^2) + (1+e^2)^2\cos\theta = 0$

$$\therefore \cos\theta = -\frac{2e(1+e^2)}{e^2+(1+e^2)^2}$$

$$= -\frac{2e(1+e^2)}{e^4+3e^2+1}$$

Since Q lies on the conic $\frac{l}{r}$

$$\frac{l}{r} = 1+e\cos\theta,$$

$$\frac{l}{SQ} = 1+e\cos\theta.$$

and at Q, $\cos\theta = -\frac{2e(1+e^2)}{e^4+3e^2+1}$

$$\therefore \frac{l}{SQ} = 1 - \frac{2e^2(1+e^2)}{1+3e^2+e^4} = \frac{1+e^2-e^4}{1+3e^2+e^4}$$

$$\therefore SQ = l \cdot \frac{1+3e^2+e^4}{1+e^2-e^4}$$

EXAMPLE 2.

If the normal at α, β, γ on $\frac{l}{r} = 1+\cos\theta$ meet in the point (e, ϕ), show that

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(1) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = 0$.

Solution: (2) $\alpha + \beta + \gamma = 2n\pi + 2\phi$.

The normal at 'θ_1' to the conic $\frac{l}{r} = 1 + \cos\theta$ is

$$\frac{\sin\theta_1}{1+\cos\theta_1} \cdot \frac{l}{r} = \sin\theta + \sin(\theta - \theta_1)$$

If this normal passes through (ρ, ϕ),

$$\frac{\sin\theta_1}{1+\cos\theta_1} \cdot \frac{l}{\rho} = \sin\phi + \sin(\phi - \theta_1).$$

If we put $\tan\frac{\theta_1}{2} = t$ in the equation, we get

$$\frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{l}{\rho} = \sin\phi + \sin\phi \cdot \frac{1-t^2}{1+t^2} - \cos\phi\, \frac{2t}{1+t^2}$$

i.e., $\frac{2lt^{2}}{2\rho} = \sin\phi + \sin\phi \cdot \frac{1-t^2}{1+t^2} - \cos\phi \frac{2t}{1+t^2}$

i.e., $lt^3 + (l + 2\rho\cos\phi)t - 2\rho\sin\phi = 0$

This is a cubic equation and so it has three roots and let the

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roots be t_1, t_2 and t_3.

Corresponding to these values of t, let the values of θ, be α, β and γ respectively.

$\therefore t_1 = \tan\frac{\alpha}{2}$, $t_2 = \tan\frac{\beta}{2}$ and $t_3 = \tan\gamma/2$.

From (1), we get $t_1 + t_2 + t_3 = 0$ → (2)

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = \frac{l + 2p\cos\phi}{l} \rightarrow (3)$$

$$t_1 t_2 t_3 = \frac{2p\sin\phi}{l} \rightarrow (4)$$

$\therefore$ From (2), we get $\tan\frac{\alpha}{2} + \tan\beta/2 + \tan\frac{\gamma}{2} = 0$

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2}\right) = \frac{\Sigma\tan\frac{\alpha}{2} - \tan\frac{\alpha}{2}.\tan\beta/2.\tan\gamma/2}{1 - \Sigma\tan\frac{\alpha}{2}.\tan\beta/2}$$

$$= \frac{t_1 + t_2 + t_3 - t_1 t_2 t_3}{1 - t_1 t_2 - t_2 t_3 - t_3 t_1}$$

$$= \frac{0 - \frac{2p\sin\phi}{l}}{1 - \frac{l + 2p\cos\phi}{l}}$$

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$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)=\tan\phi$

$\therefore \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=n\pi+\phi$

i.e., $\alpha+\beta+\gamma=2n\pi+2\phi$

SECTION: 13

Some properties of the general conic.

(1) If the tangent at P to a conic meets the directrix at K, then $\angle KSP = 90^\circ$.

Let the equation of the conic be $\frac{l}{r}=1+e\cos\theta$ and the vectorial angle of P be α.

i.e., $\angle ZSP=\alpha$

The tangent at P to the conic is $\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha) \longrightarrow (1)$

The equation of the directrix is

$$\frac{l}{r}=e\cos\theta \longrightarrow (2)$$

The lines (1) and (2) intersect at K whose vectorial angle is given by $e\cos\theta+\cos(\theta-\alpha)=e\cos\theta$

$\cos(\theta - \alpha) = 0$

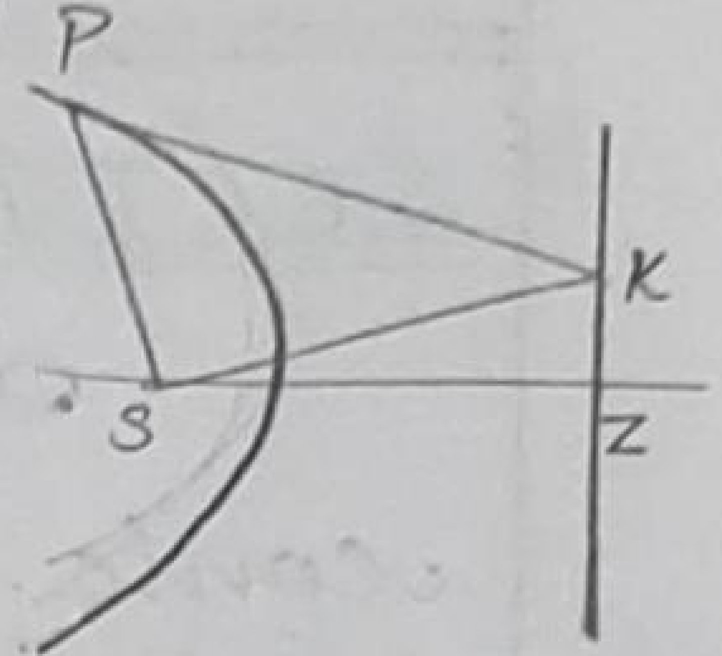
$$\theta - \alpha = \pm \pi/2$$

$\therefore$ At K, $\theta = \alpha \pm \pi/2$

$\angle KSP = \angle ZSP - \angle ZSK$

$= \alpha - (\alpha \pm \pi/2)$

$= \pm \pi/2$



(2) The tangents at the extremities of any focal chord of a conic intersect on the corresponding directrix.

Let α be the vectorial angle of the extremity of P of the focal chord PSP' of the conic.

$$\frac{l}{r} = 1 + e\cos\theta.$$

Then the vectorial angle of P' is $\alpha + \pi$.

The tangents at P and P' are

$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$ and

$\frac{l}{r} = e\cos\theta + (\theta - \alpha - \pi)$.

If (r_1, θ_1) be the point of intersection of the two tangents, we get

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha) \longrightarrow (1)$$

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha - \pi)$$

i.e., $\frac{l}{r_1} = e\cos\theta_1 - \cos(\theta_1 - \alpha) \longrightarrow (2)$

Adding (1) and (2), we get

$$\frac{2l}{r_1} = 2e\cos\theta_1$$

i.e., $\frac{l}{r_1} = e\cos\theta_1$.

$\therefore$ Locus of (r_1, θ_1) is

$\frac{l}{r} = e\cos\theta$ which is the directrix of the conic.

---

(3) If the tangents at P and Q on a conic intersect at T, then

a) ST bisects $\angle PSQ$

b) $\angle TSK = 90^\circ$ if PQ intersects the directrix at K.

c) $ST^2 = SP \cdot SQ$ if the conic is a parabola.

Let the vectorial angles of P, Q be α and β and the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta$$

The tangents at P and Q are respectively

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha) \text{ and}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\beta).$$

At the point where the tangents intersect.

$$\cos(\theta-\alpha) = \cos(\theta-\beta)$$

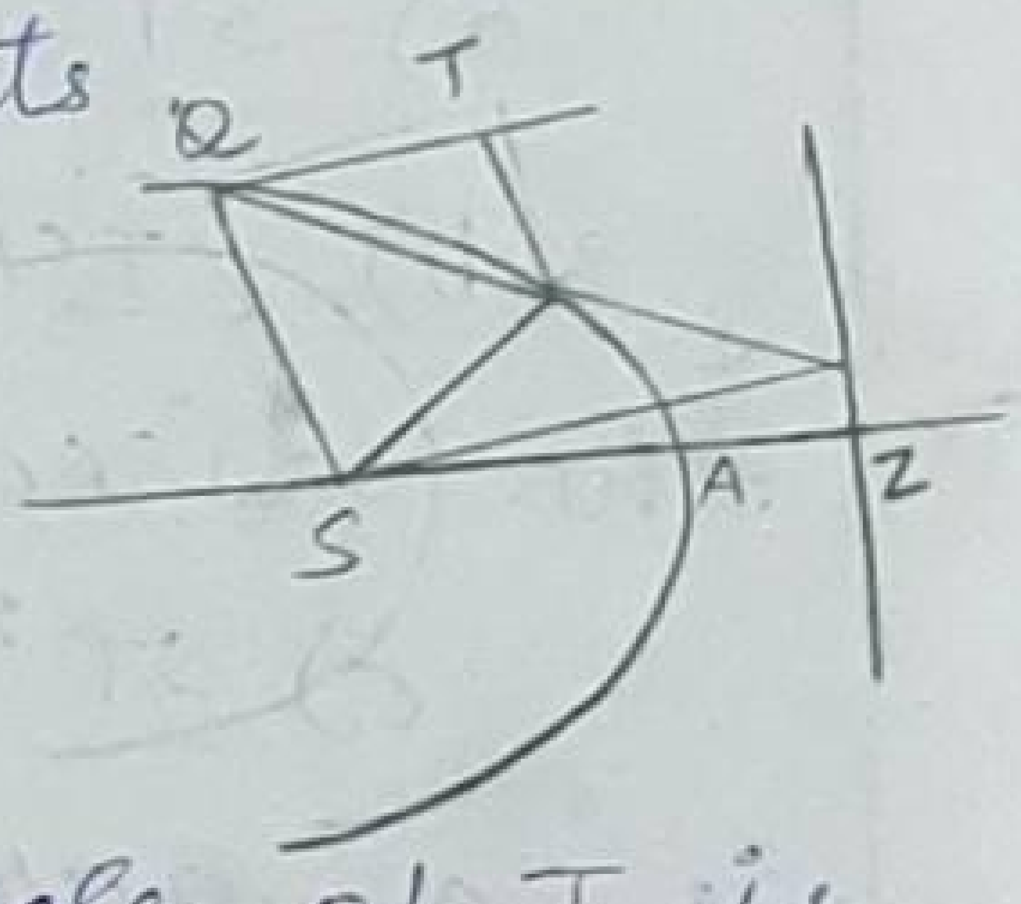
$$\therefore \theta-\alpha = \pm(\theta-\beta).$$

Taking the positive sign, we get

$$\theta-\alpha = \theta-\beta$$

i.e., $\alpha = \beta$

which is not true since P and Q are two different points



$$\therefore \theta-\alpha = -(\theta-\beta)$$

i.e., $\theta = \dfrac{\alpha+\beta}{2}$

$\therefore$ The vectorial angle of T is

$\frac{\alpha+\beta}{2}$.

(a) $\angle ZSP = \alpha$, $\angle ZSQ = \beta$ and

$\angle ZST = \frac{\alpha+\beta}{2}$

$\angle PST = \frac{\alpha+\beta}{2} - \alpha = \frac{\beta-\alpha}{2}$.

$\angle TSQ = \beta - \frac{\alpha+\beta}{2} = \frac{\beta-\alpha}{2}$

$\therefore \angle PST = \angle TSQ$.

$\therefore$ TS bisects $\angle PSQ$.

---

b) The equation of the line PQ is

$$\frac{l}{r} = e\cos\theta + \sec\frac{\alpha-\beta}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right)$$

The equation of the directrix is

$$\frac{l}{r} = e\cos\theta$$

$\therefore$ At K, $e\cos\theta = e\cos\theta + \sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right)$

i.e., $\sec\frac{\alpha-\beta}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) = 0$

i.e., $\cos\left(\theta - \frac{\alpha+\beta}{2}\right) = 0$

i.e., $\theta - \frac{\alpha+\beta}{2} = \pm\pi/2$

$\therefore \theta = \frac{\alpha+\beta}{2} \pm \pi/2$

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i.e., $\angle ZSK = \frac{\alpha+\beta}{2} \pm \pi/2$ and

$\angle ZST = \frac{\alpha+\beta}{2}$.

$\therefore \angle KST = \angle ZST - \angle ZSK$.

$$= \frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2} \pm \frac{\pi}{2}\right)$$

$$= \pm \frac{\pi}{2}$$

---

c) Here $e = 1$.

The equation of the tangent PT is $\frac{l}{r} = \cos\theta + \cos(\theta - \alpha)$.

The co-ordinates of T are $\left(ST, \frac{\alpha+\beta}{2}\right)$.

Since this point PT lies on the line PT, we get

$$\frac{l}{ST} = \cos\frac{\alpha+\beta}{2} + \cos\left(\frac{\alpha+\beta}{2} - \alpha\right)$$

$$= \cos\frac{\alpha+\beta}{2} + \cos\frac{\beta-\alpha}{2}$$

$$= 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$$

$$\therefore ST = \frac{l}{2\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}}$$

Since the points P and Q are on the conic, we get

$$\frac{l}{SP} = 1 + \cos\alpha,$$

$$\frac{l}{SQ} = 1 + \cos\beta$$

$$\therefore \frac{l^2}{SP.SQ} = (1+\cos\alpha)(1+\cos\beta)$$

$$= 4\cos^2\frac{\alpha}{2}\cos^2\beta/2$$

$$\therefore SP.SQ = \frac{l^2}{4\cos^2\frac{\alpha}{2}\cdot\cos^2\frac{\beta}{2}}$$

$$\therefore ST^2 = SP.SQ$$

---

(4) The feet of the perpendicular drawn from a focus to the tangents at a conic lie on a circle.

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$ and the vectorial angle of P any point on it be α.

Tangent at P is $\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$ $\longrightarrow$ (1)

The equation of the line perpendicular to this through the pole

$$0 = e\sin\theta + \sin(\theta - \alpha) \longrightarrow (2)$$

The point of intersection of (1) and (2) is the foot of the perpendicular from the focus on the tangent at P and let the coordinates of the foot of the perpendicular be (r_1, θ_1).

Then $\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha)$ $\longrightarrow (3)$

$$0 = e\sin\theta_1 + \sin(\theta_1 - \alpha) \longrightarrow (4)$$

Eliminating from (3) and (4), we get

$$\left(\frac{l}{r_1} - e\cos\theta_1\right)^2 + e^2\sin^2\theta_1 = 1$$

i.e., $\frac{l^2}{r_1^2} - \frac{2le}{r_1}\cos\theta_1 + e^2 = 1$

i.e., $r_1^2(1-e^2) + 2ler_1\cos\theta_1 - l^2 = 0$

$\therefore$ Locus of (r_1, θ_1) is $r^2(1-e^2) + 2ler\cos\theta - l^2 = 0$ which is a circle.

In the case of an ellipse

$$l = a(1-e^2)$$

$\therefore$ The equation of the circle can be written as

$$r^2(1-e^2) + 2er\cos\theta \cdot a(1-e^2) - a^2(1-e^2)^2 = 0$$

i.e., $r^2 + 2aer\cos\theta - a^2(1-e^2) = 0$

i.e., $r^2 + 2aer\cos\theta + a^2e^2 = a^2$

$\therefore$ The centre of this circle is (ae, π) which is also the center of the ellipse.

In the parabola $e = 1$.

In that case the equation of the circle becomes $2lr\cos\theta = l^2$

i.e., $\frac{l}{r} = 2\cos\theta$ which is the tangent to the parabola at $\theta = 0$, i.e., at the vertex.

So in the case of the parabola the circle reduces itself to the tangent at the vertex.

(5) The locus of the intersection of the perpendicular tangents to a conic is a circle.

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos\theta$ and let the co-ordinates of intersection of a pair of perpendicular tangents be (r_1, θ_1) and let the vectorial angles of points of contact of these tangents be α and β.

∴ The equations of the perpendicular tangents are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \longrightarrow (1)$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta) \longrightarrow (2)$$

(r_1, θ_1) is a point on these tangents.

$$\therefore \frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha) \longrightarrow (3)$$

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \beta) \longrightarrow (4)$$

Solving (3) and (4), we get

$$\theta_1 = \frac{\alpha + \beta}{2} \longrightarrow (5)$$

and substituting this value in ③, we get

$$\frac{l}{r_1} = e\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2} \longrightarrow ⑥$$

Expanding ① and ②, we get

$$\frac{l}{r} = \cos\theta\,(e+\cos\alpha) + \sin\theta\,\sin\alpha$$

$$\frac{l}{r} = \cos\theta\,(e+\cos\beta) + \sin\theta\,\sin\beta$$

These two lines are at right angles to each other.

$$\therefore (e+\cos\alpha)(e+\cos\beta) + \sin\alpha\,\sin\beta = 0$$

i.e., $e^2 + e(\cos\alpha + \cos\beta) + \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta = 0$

i.e., $e^2 + 2e\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} + \cos(\alpha-\beta) = 0$

i.e., $e^2 + 2e\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\alpha-\beta}{2} - 1 = 0$

Substituting the values of $\frac{\alpha+\beta}{2}$ and $\cos\frac{\beta-\alpha}{2}$ from (5) and (6) in this equation, we get

$$e^2 + 2e\cos\theta_1\left(\frac{l}{r_1} - e\cos\theta_1\right) + 2\left(\frac{l}{r_1} - e\cos\theta_1\right)^2 - 1 = 0$$

i.e., $(e^2-1) - \frac{2el\cos\theta_1}{r_1} + \frac{2l^2}{r_1^2} = 0$

i.e., $(1-e^2)r_1^2 + 2el\,r_1\cos\theta_1 - 2l^2 = 0$

$\therefore$ Locus of (r_1, θ_1) is $(1-e^2)r^2 + 2elr\cos\theta - 2l^2 = 0$.

We can easily see that it is a circle with (ae, π) as its centre.

This is the director circle of the ellipse.

In the case of a parabola $e=1$.

$\therefore$ The circle becomes $2lr\cos\theta = 2l^2$.

i.e., $\frac{l}{r} = \cos\theta$ which is the directrix of the parabola.

So in the case of a parabola the director circle reduces itself to the directrix.

(b) If the normal at P on a conic meets the axis in G, then $SG = e.SP$.

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$ and let the vectorial angle of P on it be α.

The normal at P is $\frac{e\sin\alpha}{1+e\cos\alpha}.\frac{l}{r} = e\sin\theta + \sin(\theta-\alpha)$

This line meets the axis at G, i.e., the point (SG, π) lies on the normal.

$$\therefore \frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{SG} = \sin\alpha$$

$$\therefore SG = \frac{le}{1+e\cos\alpha}$$

The vectorial angle of P is α.

$$\therefore \frac{l}{SP} = 1+e\cos\alpha$$

$$\therefore SP = \frac{l}{1+e\cos\alpha}$$

$$\therefore SG = e.S.P.$$