UNIT - II

[CHAPTER - II
Sections 7 to 11]

ANALYTICAL GEOMETRY OF 3D.

Section 7. DIRECTION COSINES

If α, β, γ be the angles which any line makes with the positive direction of the axes, then

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

are called the direction cosines of the given line.

7.1 If P is the point (x, y, z), then the direction cosines of the line OP are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ where $OP = r$.

★ The direction cosines of the line O.P are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$

★ The direction cosines of the axes coordinates are 1, 0, 0 ; 0 1 0 ; 0, 0, 1.

7.2 Relation between the direction cosines of a straight line is

★ $l^2 + m^2 + n^2 = 1$

Sec: 8: Direction ratios

If we know three numbers which are proportional to the direction cosines of the straight line, we can calculate direction cosines of that line.

★ The direction cosines are

$$\pm \frac{p}{\sqrt{(p^2+q^2+r^2)}}, \pm \frac{q}{\sqrt{(p^2+q^2+r^2)}}, \pm \frac{r}{\sqrt{(p^2+q^2+r^2)}}$$

where the same sign positive or negative is to be chosen throughout.

Such set of numbers which are proportional to the direction cosines of a straight line are direction ratios.

Hence, to specify direction of straight line, direction ratios are enough.

Section 9:

★ The projection of line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on any other line with direction cosines l, m, n is $(x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$.

COR: ★ The projections of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the coordinate axes are x_2-x_1, y_2-y_1, z_2-z_1 since the direction cosines of the axes are $1, 0, 0$; $0, 1, 0$; $0, 0, 1$.

Section 10:

★ Direction cosines of line joining

the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2-x_1, y_2-y_1, z_2-z_1$.

Eg:

Find the direction cosines of the line joining the points $(3, -5, 4)$ and $(1, -8, -2)$.

Solution:

The direction cosines of the line are proportional to

$3-1, -5+8, 4+2$.

i.e proportional to $2, 3, 6$.

Let them be $2k, 3k, 6k$.

But $(2k)^2 + (3k)^2 + (6k)^2 = 1$

$$4k^2 + 9k^2 + 36k^2 = 1$$

$$49k^2 = 1$$

$$k^2 = \frac{1}{49}$$

$$k = \pm \frac{1}{7}$$

Taking the positive value for k, the direction cosines of the lines are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

Section 11:

Angle between the lines whose direction cosines are (l_1, m_1, n_1) and

(l_2, m_2, n_2)

$$\star\ \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

COR 1:

$$\star\ \sin\theta = \sqrt{(l_1 m_2 - m_1 l_2)^2 + (m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - l_1 n_2)^2}$$

COR 2:

$$\star\ \tan\theta = \frac{\sqrt{(l_1 m_2 - m_1 l_2)^2 + (m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - m_2 l_1)^2}}{l_1 l_2 + m_1 m_2 + n_1 n_2}$$

COR 3:

$\star$ If the direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 then

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

$$\sin\theta = \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

EXAMPLE:

Find the direction cosines of the line joining the points $(3, -5, 4)$ and $(1, -8, -2)$.

Sol:

The direction cosines of the lines are proportional to

$3-1, -5+8, 4+2$

i.e., $2, 3, 6$

$$\frac{l}{2} = \frac{m}{2} = \frac{n}{6} = k$$

$\therefore l = 2k, m = 3k, n = 6k$

but $l^2 + m^2 + n^2 = 1$

$(2k)^2 + (3k)^2 + (6k)^2 = 1$

$4k^2 + 9k^2 + 36k^2 = 1$

$49k^2 = 1$

$k^2 = 1/49$

$k = \pm 1/7$.

Taking the point values for k the direction cosines of the line are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

CHAPTER - II
THE PLANE

LOCUS

The plane where something is situated or occurs. (i.e location)

i.e. The set of all points that share a property.

i.e A set of points all meet at a designed location (i.e a particular position.

EQUATION OF THE PLANE

The equation which is satisfied by the coordinates of any point in the plane is called the equation of the plane.

Section: 1

The general equation of the first degree in x, y, z represents a plane.

The general equation of the degree in x, y, z is

$$Ax + By + Cz + D = 0 \longrightarrow (1)$$

where A, B, C and D are constants.

Every equation of the first degree in x, y, z represents a plane.

Section : 2

The equation of the plane making intercepts a, b, c on the axes OX, OY, OZ respectively.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

FORMULAE:

1. Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

2. The distance of the point $P(x_1, y_1, z_1)$ from the origin is

$$\sqrt{x_1^2 + y_1^2 + z_1^2}$$

3. The co-ordinates of the point dividing the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m:n$ then,

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

4. The middle point of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

5. The centroid of a triangle when the co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are given

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

DIRECTION RATIO

$$ax + by + cz + d = 0$$

A set of numbers which are proportional to the direction cosines l, m, n of a line are called the direction ratios of that line.

If a, b, c are direction ratios of a line then the direction cosines of that line are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

6. The equation of a straight line passing through a point (x_1, y_1, z_1) having direction ratios l, m, n is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

7. Equation of straight line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{y-y_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

8. The direction ratios of a line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$x_2-x_1, \; y_2-y_1, \; z_2-z_1$$

9. The condition for the lines (Planes).

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

a. The condition for the lines (Planes) to be $\perp^r$ is $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

b. The condition for the lines (Planes) to be $\parallel l$ is $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

10. If the point P divides AB externally where $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then $P\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$

11. Equation of any plane passing through the point (x_1, y_1, z_1) is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

12. Equation of the plane $ax+by+cz+d=0$ containing the line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$ is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0,$$

where $al + bm + cn = 0$

13. Consider equations of the plane $ax+by+cz+d=0$ and the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$, then the condition for the plane and the line to be parallel is

$$al+bm+cn=0$$

and to be perpendicular is

$$\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$$

14. The length of perpendicular from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ is

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

Corollary: 1

The length of the perpendicular from the origin to the plane is

$$\left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$$

Corollary: 2

The distance between the

parallel planes $ax+by+cz+d=0$ and $ax+by+cz+d_1=0$ is

$$\left|\frac{d-d_1}{\sqrt{a^2+b^2+c^2}}\right|$$

15. The relationship between the direction cosines of a straight line is $l^2+m^2+n^2=1$.

16. The projection of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on any other line with direction cosines l, m, n is

$$l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$$

17. The angle between the lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) is

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\sin\theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}$$

$$\tan\theta = \frac{\sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}}{l_1 l_2 + m_1 m_2 + n_1 n_2}$$

1. Find the distance between the points $(4,-2,3)$ and $(2,-3,1)$.

Solution:

Let (x_1, y_1, z_1) is $(4,-2,3)$ and (x_2, y_2, z_2) is $(2,-3,1)$

$$\therefore d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$d = \sqrt{(2-4)^2 + (-3+2)^2 + (1-3)^2}$$

$$d = \sqrt{4+1+4}$$

$$d = \sqrt{9}$$

$$\therefore d = 3$$

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2. Show that the points $(2,3,5)$, $(-1,5,-1)$ and $(4,-3,2)$ form an isoceles right angled triangle.

Solution:

Let $A(2,3,5)$, $B(-1,5,-1)$, $C(4,-3,2)$ be the points.

$$AB = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$AB = \sqrt{(-1-2)^2 + (5-3)^2 + (-1-5)^2}$$

$$AB = \sqrt{9+4+36}$$

$AB = \sqrt{49}$

$AB^2 = 49$

$BC = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$BC = \sqrt{(4+1)^2 + (-3-5)^2 + (2+1)^2}$

$BC = \sqrt{5^2 + (-8)^2 + (3)^2}$

$BC = \sqrt{25+64+9}$

$BC = \sqrt{98}$

$BC^2 = 98$

$CA = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$CA = \sqrt{(2-4)^2 + (3+3)^2 + (5-2)^2}$

$CA = \sqrt{(-2)^2 + 6^2 + (3)^2}$

$CA = \sqrt{4+36+9}$

$CA = \sqrt{49}$

$CA^2 = 49$

$\therefore AB^2 = CA^2$ and $AB^2 + CA^2 = BC^2$

$\angle A = 90^\circ$

$\therefore$ A, B, C forms isosceles right angled triangle.

3. Show that the points (5, 3, -2), (3, 2, 1) and (-1, 0, 7) are collinear.

Solution:

Let A(5, 3, -2), B(3, 2, 1) and C(-1, 0, 7) be the points, then

$$AB = \sqrt{(5-3)^2 + (3-2)^2 + (-2-1)^2}$$

$$AB = \sqrt{2^2 + 1^2 + 3^2}$$

$$AB = \sqrt{4 + 1 + 9}$$

$$AB = \sqrt{14}$$

$$BC = \sqrt{(3+1)^2 + (2-0)^2 + (1-7)^2}$$

$$BC = \sqrt{4^2 + 2^2 + 6^2}$$

$$BC = \sqrt{56}$$

$$BC = 2\sqrt{14}$$

$$CA = \sqrt{(5+1)^2 + (3-0)^2 + (-2-7)^2}$$

$$CA = \sqrt{6^2 + 3^2 + (-9)^2}$$

$$CA = \sqrt{126}$$

$$CA = 3\sqrt{14}$$

$\therefore CA = AB + BC$

$\therefore$ The points A, B, C are collinear.

4. Show that (1,2,8), (0,3,4), (1,1,3) and (2,0,7) are the vertices of a parallelogram.

Solution:

Let A(1, 2, 8), B(0, 3, 4), C(1, 1, 3) and D(2, 0, 7).

$$AC = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$= \left(1, \frac{3}{2}, \frac{11}{2}\right)$$

$$= \left(\frac{0+2}{2}, \frac{3+0}{2}, \frac{4+7}{2}\right)$$

$$AC = \left(1, \frac{3}{2}, \frac{11}{2}\right)$$

$\therefore AC = BD$

$\therefore$ The A, B, C, D are the vertices of a parallelogram.

5. Find the direction cosines of the line joining points (3,-5,4) and (1,-8,-2).

Solution:

The direction cosines of the

line are proportional to

$(x_1-x_2, y_1-y_2, z_1-z_2)$.

i.e $(3-1, -5+8, 4+2) = (2, 3, 6)$

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$$

$l = ak,\ m = bk,\ n = ck$

Let the points be $(2k, 3k, 6k)$ which represent (l, m, n).

We know that $l^2 + m^2 + n^2 = 1$

$$(2k)^2 + (3k)^2 + (6k)^2 = 1$$

$$4k^2 + 9k^2 + 36k^2 = 1$$

$$49k^2 = 1$$

$$k^2 = \frac{1}{49}$$

$$k = \pm\frac{1}{7}$$

~~This is known as intercept form of the equation~~

The direction cosines of the lines are $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ is $\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This is known as the intercept form of the equation of a plane.

Section : 3

The equation of a plane in terms of p, the length of the perpendicular from the origin to it and l, m, n the direction cosines of that perpendicular.

$$\frac{lx}{p} + \frac{my}{p} + \frac{nz}{p} = 1$$

$$lx + my + nz = p$$

This is known as the normal form of the equation of a plane.

Section : 4

Several forms of the equations of a plane :

1. $Ax + By + Cz + D = 0$
2. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
3. $lx + my + nz = p$

DIRECTION COSINES :

The direction cosins of a vector are the cosines of the angles between the vector and the three coordinate axes.

Section : 5

The equation of the plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Section : 6

Direction cosines of the line which is perpendicular to a plane.

Let the equation of the plane be

$$Ax + By + Cz + D = 0.$$

Then the direction cosines of the line which is perpendicular to a plane.

$$\frac{A}{kl} = \frac{B}{km} = \frac{C}{kn}$$

Section : 7

Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

If θ be the angle between the two planes, then

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Corollary : 1

If the planes be at right angles $a_1a_2 + b_1b_2 + c_1c_2 = 0$

i.e $\theta = 90^\circ$; $\cos 90^\circ = 0$

Corollary : 2

If the planes are parallel then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Corollary : 3

If the equations of two planes differ only in the constant term, they are parallel. Thus the planes.

$ax+by+cz+d=0$ and

$ax+by+cz+d_1=0$ are parallel

SKEW LINES

Non intersecting straight lines are called skew lines.

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1. Find the equation to the plane through $(3, 4, 5)$ parallel to the plane $2x - 3y - z = 0$.

Solution:

The equation to any plane parallel to this plane is $2x + 3y - z + k = 0$ if it passes through the point $(3, 4, 5)$.

$$6 + 12 - 5 + k = 0$$

$$13 + k = 0$$

$$k = -13$$

Hence the equation of the required plane is $2x + 3y - z + 13 = 0$

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2. Find the equation of the plane through the point (x_1, y_1, z_1) parallel to the plane $ax + by + cz + d = 0$.

Solution:

The equation to any plane parallel to this plane is

$$ax + by + cz + d = 0 \longrightarrow (1).$$

If it passes through the point (x_1, y_1, z_1) then

$$ax_1 + by_1 + cz_1 + d = 0 \longrightarrow (2)$$

$$(1) - (2) \Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

3. Find the equation of the plane which passes through the point $(2,-4,5)$ and is parallel to the plane $4x+2y-7z+d=0$.

Solution:

The equation to any plane parallel to this plane is

$4x+2y-7z+d=0 \longrightarrow (1)$.

If it passes through the point $(\overset{x_1}{2}, \overset{y_1}{-4}, \overset{z_1}{5})$ then

$4x_1+2y_1-7z_1+d=0 \longrightarrow (2)$

$(1)-(2) \Rightarrow 4(x-2)+2(y+4)-7(z-5)=0$

$4x-8+2y+8-7z+35=0$

$4x+2y-7z+35=0$

4. Find the angle between the planes $2x-y+z=6$, $x+y+2z=3$.

Solution:

Given planes are

$2x-y+z=6$, $x+y+2z=3$

The direction cosines of the normals to the planes are proportional to $(2,-1,1)$ and $(1,1,2)$ respectively.

If θ be the angle between the planes, then

$$\cos\theta = \frac{\pm a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

here $a_1 = 2$, $b_1 = -1$, $c_1 = 1$

$a_2 = 1$, $b_2 = 1$ $c_2 = 2$

$$\cos\theta = \pm\frac{2(1) + (-1)(1) + 1(2)}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+1^2+2^2}}$$

$$\cos\theta = \pm\frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

$$\theta = \pi/3$$

H.W

Find the angle btwn the planes:

5. $2x+4y-6z=11$ and $3x+6y+5z+4=0$

6. $x-y+2z-9=0$ and $2x+y+z=7$

7. $6x-3y-2z=7$ and $x+2y+2z+9=0$

8. Find the distance of the origin from the plane $6x-3y+2z-14=0$

Solution:

Let the equation of the plane in the normal form be $lx+my+nz=p$ $\longrightarrow (1)$

Given plane is

$$6x - 3y + 2z - 14 = 0 \longrightarrow (2)$$

$$lx + my + nz - p = 0$$

If equations (1), (2) represents the same plane, then

$$\frac{A}{l} = \frac{B}{m} = \frac{C}{n} = \frac{-D}{p}$$

$\therefore A = 6,\ B = -3,\ C = 2,\ D = -14,$

Each ratio is equal to

$$\frac{6}{l} = \frac{-3}{m} = \frac{2}{n} = \frac{14}{p}.$$

$$\therefore \frac{14}{p} = \frac{\sqrt{A^2+B^2+C^2}}{\sqrt{l^2+m^2+n^2}} = \sqrt{A^2+B^2+C^2}$$

$(\because l^2+m^2+n^2=1)$

$$\frac{14}{p} = \sqrt{36+9+4}$$

$$\frac{14}{p} = 7$$

$$\frac{14}{7} = p$$

$$\therefore \boxed{p = 2}$$

$$\frac{l}{6} = \frac{m}{-3} = \frac{n}{2} = \frac{p}{14}$$

$$(\text{ie}) \frac{p}{14} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{6^2+(-3)^2+(2)^2}}$$

$\because l^2+m^2+n^2=1$

$$= \frac{1}{\sqrt{36+9+4}} = \frac{1}{\sqrt{49}}$$

$$p/14 = 1/7 \quad \therefore p = 2$$

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9. Find the equation of the plane passing through the points (3, 1, 2), (3, 4, 4) and perpendicular to the plane $5x + y + 4z = 0$

Solution:

Let the equation of the required plane be $Ax+By+Cz+D=0 \rightarrow ①$

This plane passes through the points $(3,1,2)$ and $(3,4,4)$ we get

$$3A+B+2C+D=0 \longrightarrow ②$$

$$3A+4B+4C+D=0 \longrightarrow ③$$

The plane is $\perp r$ to the plane $5x+y+4=0$

Put $x=A, y=B, z=C$.

$$\therefore 5A+B+4C=0 \longrightarrow ④$$

$② - ③ \Rightarrow \quad -3B-2C=0$

$$-3B=2C \longrightarrow ⑤$$

$④ \Rightarrow \quad 5A=-B-4C.$

$$A=\frac{-1}{5}(B+4C)$$

Substitute A in ②

$$\Rightarrow 3\left(-\tfrac{1}{5}(B+4C)\right)+2C+B+D=0$$

$$\frac{-3}{5}B-\frac{12}{5}C+2C+B+D=0$$

$$-3B-12C+10C+5B+5D=0$$

$$2B-2C+5D=0 \longrightarrow ⑥$$

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Substitute 5 in 6.

$$6 \Rightarrow 2B + 3B + 5D = 0$$

$$5B = -5D$$

$$\boxed{B = -D}$$

Put $B = -D$ in 5.

$$2C = 3D$$

$$C = \frac{3D}{2}$$

Put B, C in 2

$$2 \Rightarrow 3A - D + 2\left(\frac{3D}{2}\right) + D = 0$$

$$3A - D + 3D + D = 0$$

$$3A = -3D$$

$$\boxed{A = -D}$$

Substitute A, B, C in 1

$$-Dx - Dy + \frac{3Dz}{2} + D = 0$$

$$-2Dx - 2Dy + 3Dz + 2D = 0$$

$$D(-2x - 2y + 3z + 2) = 0$$

$$-2x - 2y + 3z + 2 = 0$$

$$2x + 2y - 3z - 2 = 0$$

H.W.

10. Find the equation of the plane

through the points (9, 3, 6) and (2, 2, 1) and perpendicular to the plane $2x+6y+6z-9=0$.

Ans: $3x+4y-5z-9=0$

11. Find the equation of the plane through the points (2, 1, 1) and (3, 2, 2) and perpendicular to the plane $x+2y-5z=3$.

Ans: $7x-6y-z=7$.

12. Find the equation of the plane which passes through the point (−1, 3, 2) and perpendicular to the two planes $x+2y+2z=5$, $3x+3y+2z=8$

1 pt, 2 eq.

Sol

Let the equation of the required plane be

$$Ax+By+Cz+D=0 \longrightarrow ①$$

This plane passes through the points (−1, 3, 2).

Substitute (−1, 3, 2) in ①,

$$-A+3B+2C+D=0 \longrightarrow ②$$

The plane is perpendicular to the planes, $x+2y+2z=5$, $3x+3y+2z=8$.

$$A+2B+2C=0 \longrightarrow (3)$$

$$3A+3B+2C=0 \longrightarrow (4)$$

From equations (3), (4) we get

$$\frac{A}{4-6} = \frac{-B}{2-6} = \frac{C}{3-6}$$

$$\frac{A}{-2} = \frac{-B}{-4} = \frac{C}{-3}$$

$$\frac{A}{-2} = \frac{B}{4} = \frac{C}{-3} = k$$

then $A=-2k$, $B=4k$, $C=-3k$

Substituting the values of A, B, C in equation (2),

$$2k+12k-6k+D=0$$

$$8k+D=0$$

$$D=-8k$$

Hence the equation of the plane is

$$(1) \Rightarrow -2kx+4ky-3kz-8k=0$$

$$k(2x-4y+3z+8)=0$$

$$2x-4y+3z+8=0$$ which is the required equation of the plane

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H.W

13. Find the equation of the plane through $(1,1,2)$ ⊥r to the planes $2x-2y-4z=3$, $3x+y+6z=4$.

14. Find the equation of the plane which passes through the point $(1,-2,1)$ and is perpendicular to each of the planes $3x+y+z-2=0$, $x-2y+z+4=0$.

15. Find the equation to the plane which passes through the point $(4,1,1)$ and perpendicular to each of the planes $x-3y+5z+2=0$ and $3x-y+7z-3=0$.

16. Find the equation of the plane passing through the points $(2,5,-3)$ $(-2,-3,5)$ and $(5,3,-3)$.

3 pts.

Solution:

Let the equation of the plane be $Ax+By+Cz+D=0 \longrightarrow (1)$

Equation (1) passes through the points $(2,5,-3)$, $(-2,-3,5)$ and $(5,3,-3)$, we get

$$2A+5B-3C+D=0 \longrightarrow (2)$$

$$-2A-3B+5C+D=0 \longrightarrow (3)$$

$$5A+3B-3C+D=0 \longrightarrow (4)$$

(2) − (3) ⇒

$$2A + 5B - 3C + D = 0$$

$$-2A - 3B + 5C + D = 0$$

$$4A + 8B - 8C = 0$$

$$A + 2B - 2C = 0 \longrightarrow (5)$$

(3) - (4) $\Rightarrow$

$$-2A - 3B + 5C + D = 0$$

$$5A + 3B - 3C + D = 0$$

$$-7A - 6B + 8C = 0$$

$$7A + 6B - 8C = 0 \longrightarrow (6)$$

From (5), (6) we get

$$A + 2B - 2C = 0$$

$$7A + 6B - 8C = 0$$

$$\frac{A}{-16+12} = \frac{-B}{-8+14} = \frac{C}{6-14}$$

$$\frac{A}{-4} = \frac{B}{-6} = \frac{C}{-8}$$

$$\frac{A}{2} = \frac{B}{3} = \frac{C}{4} = k$$

$$A = 2k, \quad B = 3k, \quad C = 4k$$

substituting A, B, C in (2)

$$4k + 15k - 12k + D = 0$$

$$D = -7k$$

(1) $\Rightarrow 2kx + 3ky + 4kz - 7k = 0$

$2x+3y+4z-7=0$ which is the required equation of the plane.

NOTE:

The ratio in which the plane $ax+by+cz+d=0$ divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\lambda = -\left(\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$$

NOTE:

(i) If λ is positive, then (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on opposite sides of the plane $ax+by+cz+d=0$

(ii) If λ is negative, then (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side of the plane.

$\therefore$ Thus we get that two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same or different sides of the plane $ax+by+cz+d=0$

according as the expressions $ax_1 + by_1 + cz_1 + d$, $ax_2 + by_2 + cz_2 + d_2$ are of the same or different signs.

Equations of the plane through the line of intersection of two given planes.

Let $a_1x + b_1y + c_1z + d_1 = 0 \rightarrow ①$

$a_2x + b_2y + c_2z + d_2 = 0 \rightarrow ②$

be the equations of the two given planes.

Consider the equation,

$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0 \longrightarrow ③$

where λ is any constant.

Equation ③ is an equation of the first degree and therefore represents a plane.

---

1. Find the equation of the plane through the point $(1, -2, 3)$ and the intersection of the planes

1 point, 2 planes

$2x - y + 4z = 7$ and $x + 2y - 3z + 8 = 0$

Solution:

Consider the equation of plane through the line of intersection of two given planes:

1 point, 2 planes

$$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

∴ The plane

$$(2x - y + 4z - 7) + \lambda(x + 2y - 3z + 8) = 0 \quad \rightarrow (1)$$

Passes through the intersection of the given planes for all values of λ.

Equation (1) passes through the points $(1, -2, 3)$, we get

$$(2 + 2 + 12 - 7) + \lambda(1 - 4 - 9 + 8) = 0$$

$$9 + \lambda(-4) = 0$$

$$-4\lambda + 9 = 0$$

$$\lambda = +9/4$$

Put $\lambda = 9/4$ in (1), we get

$$(2x - y + 4z - 7) + (9/4)(x + 2y - 3z + 8) = 0$$

$$17x + 14y - 11z + 44 = 0$$

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H.W.

(2). Find the equation of the plane through (1, 1, 1) and the line of intersection of the plane

$x+2y-z+1=0, \quad 3x-y+4z+3=0$

(3) Find the equation of the plane through the origin $(0,0,0)$ and the line of intersection of the planes

$3x-y+2z=4$ and $x+y+z=1$.

$(0,0,0)$

$(-4)+\lambda(-1)=0 \quad -4-\lambda=0$

$\lambda=4$

4. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$, $2x+3y+4z-7=0$ and perpendicular to the plane $x-5y+3z=5$.

Solution:

Consider the equation of the plane through the line of intersection of two given planes:

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

The equation of the required

plane is of the form

$$(x+y+z-1)+\lambda(2x+3y+4z-7)=0 \longrightarrow (1)$$

$$(x+2\lambda x)+(y+3\lambda y)+(z+4\lambda z)-(1+7\lambda)=0$$

$$(1+2\lambda)x+(1+3\lambda)y+(1+4\lambda)z-(1+7\lambda)=0 \longrightarrow (2)$$

This plane is perpendicular to $x-5y+3z=5$.

$$1(1+2\lambda)-5(1+3\lambda)+3(1+4\lambda)=0$$

$$1+2\lambda-5-15\lambda+3+12\lambda=0$$

$$-\lambda-1=0$$

$$-\lambda=1$$

$$\boxed{\lambda=-1}$$

(1) $\Rightarrow$

$$(x+y+z-1)-1(2x+3y+4z-7)=0$$

$$(x-2x)+(y-3y)+(z-4z)+(-1+7)=0$$

$$-x-2y-3z+6=0$$

$x+2y+3z-6=0$ which is the required equation of the plane.

plane is of the form

$$(x+y+z-1)+\lambda(2x+3y+4z-7)=0 \longrightarrow (1)$$

$$(x+2\lambda x)+(y+3\lambda y)+(z+4\lambda z)-(1+7\lambda)=0$$

$$(1+2\lambda)x+(1+3\lambda)y+(1+4\lambda)z-(1+7\lambda)=0 \longrightarrow (2)$$

This plane is perpendicular to $x-5y+3z=5$.

$$1(1+2\lambda)-5(1+3\lambda)+3(1+4\lambda)=0$$

$$1+2\lambda-5-15\lambda+3+12\lambda=0$$

$$-\lambda-1=0$$

$$-\lambda=1$$

$$\boxed{\lambda=-1}$$

(1) $\Rightarrow$

$$(x+y+z-1)-1(2x+3y+4z-7)=0$$

$$(x-2x)+(y-3y)+(z-4z)+(-1+7)=0$$

$$-x-2y-3z+6=0$$

$x+2y+3z-6=0$ which is the required equation of the plane.

5. Find the equation of the plane passing through the line of intersection of the planes $2x-y+5z-3=0$ and $4x+2y-z+7=0$ and parallel to the z-axis.

Ans:

6. Find the equation of the plane passing through the line of intersection of the planes $2x+y+3z-4=0$ and $4x-y+5z-7=0$ and which is ⊥r to the plane $x+3y-4z+6=0$. Ans: $x-4y+6z=106$

Section : 10

Length of perpendicular.

The length of the ⊥r from the origin $(0,0,0)$ to the plane $ax+by+cz+d=0$ is $\pm \dfrac{d}{\sqrt{a^2+b^2+c^2}}$

the positive sign is to be taken when d is positive and the negative sign, when d is negative.

Section 10.1

The length of the $\perp^r$ from the point (x_1, y_1, z_1) on the plane $lx + my + nz = p$ (is $p - p_1$)

where $p_1 = lx_1 + my_1 + nz_1$

$\therefore$ The length of the $\perp^r$ from the point (x_1, y_1, z_1) on the given plane is $p - p_1$

$$\Rightarrow p - lx_1 - my_1 - n = 1.$$

Section 10.2

The length of the perpendicular from the point (x_1, y_1, z_1) on the plane $ax + by + cz + d = 0$.

Let the normal form of the equation of the plane $ax + by + cz + d = 0$ be $lx + my + nz = p$.

$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{-d}$$

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{-p}{d} = \pm \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$p = \pm \frac{d}{\sqrt{a^2+b^2+c^2}}$$

We know that the length of the perpendicular from the point (x_1, y_1, z_1) on the plane $lx + my + nz$ is

$$p - lx_1 - my_1 - nz_1$$

$$\Rightarrow \pm \frac{d}{\sqrt{a^2+b^2+c^2}} - \left\{ \pm \frac{a}{\sqrt{a^2+b^2+c^2}} x_1 \pm \frac{b}{\sqrt{a^2+b^2+c^2}} y_1 \pm \frac{c}{\sqrt{a^2+b^2+c^2}} z_1 \right\}$$

$$= \pm \frac{d}{\sqrt{a^2+b^2+c^2}} \pm \frac{ax_1}{\sqrt{a^2+b^2+c^2}} \pm \frac{by_1}{\sqrt{a^2+b^2+c^2}} \pm \frac{cz_1}{\sqrt{a^2+b^2+c^2}}$$

$$= \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}$$

Section: 11

The equation of the planes bisecting the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

NOTE: $\theta < 45^\circ \Rightarrow$ the bisection of the acute angle.

$\theta > 45^\circ \Rightarrow$ the bisection of the obtuse angle.

---

1. Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$.

Solution:

Claim: Find a point on the plane $2x - 2y + z + 3 = 0$

Put $x = 0, y = 0 \Rightarrow z = -3$

$\therefore$ The first plane meets the z-axis at the point $(0, 0, -3)$.

Claim: To find the distance between the two parallel planes.

i.e., to find the perpendicular distance from that point to the plane $4x-4y+2z+5=0$

$\therefore$ The length of the perpendicular from $(0,0,-3)$ to the plane.

$4x-4y+2z+5=0$ is

$$\pm\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

$a=4,\ b=-4,\ c=2,\ d=5$

$x_1=0,\ y_1=0,\ z_1=-3$

$$\Rightarrow \pm\frac{4(0)-4(0)+2(-3)+5}{\sqrt{4^2+(-4)^2+2^2}}$$

$$=\pm\frac{-6+5}{\sqrt{16+16+4}}$$

$$=\pm\frac{(-1)}{\sqrt{36}}$$

$$=\pm\frac{1}{6}$$

$\therefore$ The distance between the parallel planes is $\frac{1}{6}$.

---

2. Show that the origin lies in the acute angle between the planes $x+2y+2z=9$, $4x-3y+12z+13=0$.

Find the planes bisecting the angles between them and point out which bisects the obtuse angle.

Solution:

The equation of the planes which bisect the angles between the given planes are given by

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\frac{x+2y+2z-9}{\sqrt{1^2+2^2+2^2}} = \pm\frac{4x-3y+12z+13}{\sqrt{4^2+3^2+12^2}}$$

$$\frac{x+2y+2z-9}{\sqrt{1^2+2^2+3^2}} = \frac{4x-3y+12z+13}{13}.$$

$$-13x-26y-26z+117$$
$$=12x-9y+36z+39$$

$$12x+13x-9y+26y+36z+26z+39-117=0$$

$25x+17y+62z-78=0$ is the bisector of the acute angle.

We have shown that the bisector of the angle containing the origin is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\left(\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

$$\therefore \frac{-x-2y-2z+9}{3} = \frac{4x-3y+12z+13}{13}$$

$$-13x-26y-26z+117 = 12x-9y+36z+39$$

$$12x+13x-9y+26y+36z+26z+39-117=0$$

$$25x+17y+62z-78=0$$

Hence the plane bisecting the other angle is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\frac{x+2y-2z-9}{3} = \frac{4x-3y+12z+13}{13}$$

$$13x+26y+26z-117 = 12x-9y+36z+39$$

$$13x-12x+26y+9y+26z-36z-117-39=0$$

$$x+35y-10z-156=0$$

Claim: Find the angle between

$25x+17y+62z-78=0$ and

$x+2y+2z-9=0$,

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{25+34+124}{\sqrt{25^2+17^2+62^2}\cdot\sqrt{1^2+2^2+2^2}}$$

$$= \frac{183}{3\sqrt{4758}}$$

$$= \frac{61}{\sqrt{4758}}$$

$$\cos\theta = \frac{61}{\sqrt{4758}}$$

$$\sec\theta = \frac{\sqrt{4758}}{61}$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$= \frac{4758}{61^2} - 1$$

$$= \frac{4758-3721}{3721}$$

$$\tan^2\theta = \frac{1037}{(61)^2}$$

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$$\tan\theta = \frac{\sqrt{1031}}{61}$$

$$\tan\theta = \frac{32.2025}{61}$$

$$\tan\theta = 0.5279 < 1$$

$$\therefore \theta < 45^\circ$$

$\therefore 25x + 17y + 62z - 78 = 0$ is the bisector of the acute angle.

We have shown that the bisection of the angle containing the origin is

$$25x + 17y + 62z - 78 = 0$$

$\therefore$ The origin lies in the acute angle.

Hence the bisector of the obtuse angle is

$$x + 35y - 10z - 156 = 0.$$

3. Prove that the reflection of the plane $ax+by+cz+d=0$ in the plane $a_1x+b_1y+c_1z+d_1=0$ is the plane

$$2(aa_1+bb_1+cc_1)(a_1x+b_1y+cz+d_1)$$
$$=(a_1^2+b_1^2+c_1^2)(ax+by+cz+d)$$

Solution:

The reflection of the plane

$$ax+by+cz+d=0 \longrightarrow (1)$$

in the plane $a_1x+b_1y+c_1z+d_1=0$

$$\longrightarrow (2)$$

is the plane passing through the intersection of (1) and (2).

We know that the equation of a plane through the line of intersection of two given planes is

Let (x_1, y_1, z_1) be any points ~~in~~ (2) implies,

$a_1 x_1 + b_1 y_1 + c_1 z_1 + d_1 = 0$

$① \Rightarrow a x_1 + b y_1 + c z_1 + d = 0$

$② \Rightarrow (a x_1 + b y_1 + c z_1 + d) +$

$\lambda (a_1 x_1 + b_1 y_1 + c_1 z_1 + d) = 0$

$(a_1 x + b_1 y + c_1 z + d_1) +$

$\lambda (a_2 x + b_2 y + c_2 z + d_2) = 0$

From ①, ②,

$(ax + by + cz + d) + \lambda (a_1 x + b_1 y + c_1 z + d_1)$

$= 0$

Also the perpendicular from any point on ② to ① and ③ are equal.

We know that from any point (x, y, z) on the required plane, the perpendicular on the two given planes will be equal. Hence the required equation is

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\therefore \frac{a x_1 + b y_1 + c z_1 + d}{\sqrt{a^2 + b^2 + c^2}} = (a x_1 + b y_1 + c z_1 + d$$

$$+\frac{\lambda(a_1x_1+b_1y_1+c_1z_1+d_1)}{\sqrt{(a+\lambda a_1)^2+(b+\lambda b_1)^2+(c+\lambda c_1)^2}}$$

Squaring on both sides,

$$\frac{(ax_1+by_1+cz_1+d)^2}{a^2+b^2+c^2}-\left[(ax_1+by_1+cz_1+d)+\frac{\lambda(a_1x_1+b_1y_1+c_1z_1+d_1)}{(a+\lambda a_1)^2+(b+\lambda b_1)^2+(c+\lambda c_1)^2}\right]$$

$$a^2+b^2+c^2=(a+\lambda a_1)^2+(b+\lambda b_1)^2+(c+\lambda c_1)^2$$

$$=a^2+\lambda^2a_1^2+2aa_1\lambda+b^2+\lambda^2b^2+2bb_1\lambda+c^2+\lambda^2c_1^2+2cc_1\lambda$$

$$\lambda^2(a_1^2+b_1^2+c_1^2)+2\lambda(aa_1+bb_1+cc_1)=0$$

$$\lambda^2(a_1^2+b_1^2+c_1^2)=-2\lambda(aa_1+bb_1+cc_1)$$

$$\lambda=\frac{-2(aa_1+bb_1+cc_1)}{a_1^2+b_1^2+c_1^2}$$

Substitute the value of λ in (3),

$$(ax+by+cz+d)-\left[\frac{2(aa_1+bb_1+cc_1)}{(a_1^2+b_1^2+c_1^2)}\right](a_1x+b_1y+c_1z+d_1)=0$$

$$(a_1^2+b_1^2+c_1^2)(ax+by+cz+d)-2(aa_1-bb_1+cc_1)(a_1x+b_1y+c_1z+d_1)=0$$

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$$\therefore 2(aa_1+bb_1+cc_1)(a_1x+b_1y+c_1z+d_1)$$
$$=(a_1^2+b_1^2+c_1^2)(ax+by+cz+d)$$