UNIT - III

THE STRAIGHT LINE

[CHAPTER - III
Sections 1 to 8]

Section : 1

A straight line may be determined as the intersection of two planes.

Let the equations of two planes be

$Ax + By + Cz + D = 0,$

$A_1x + B_1y + C_1z + D_1 = 0.$

Any set of values of x, y, z which satisfy the two equations simultaneously, will give the coordinates of a point, which lies in the line of intersection of the two planes. Hence the equations of the planes taken together.

$$Ax + By + Cz + D = 0 = A_1x + B_1y + C_1z + D_1$$

gives the equation of the line of intersection of the two planes.

COROLLARY:

The intersection of XZ and XY planes is the x-axis. Hence the are $\left(\dfrac{bd_1-b_1d}{ab_1-a_1b}, \dfrac{da_1-d_1a}{ab_1-a_1b}, 0\right)$

$\therefore$ The equations of the line are

$$\frac{x-\dfrac{bd_1-b_1d}{ab_1-a_1b}}{bc_1-b_1c} = \frac{y-\dfrac{da_1-d_1a}{ab_1-a_1b}}{ca_1-c_1a} = \frac{z}{ab_1-a_1b}$$

EXAMPLE.

Find the symmetrical form of the equations of line of intersection of the planes.

$$x+5y-z-7=0, \quad 2x-5y+3z+1=0.$$

Solution:

The normals to the two planes are in the direction given by

$1:5:-1$ and $2:-5:3$

Each of these two directions is perpendicular to that of the line of intersection.

If $l:m:n$ are the direction ratios of the line of intersection, we get

$$l+5m-n=0$$

$$2l-5m+3n=0$$

$\therefore \frac{l}{10} = \frac{m}{-5} = \frac{n}{-15}$ i.e., $\frac{l}{2} = \frac{m}{-1} = \frac{n}{-3}$

We have to find coordinates of any fixed point on it and there is an unlimited number of points from which to choose.

We shall take the point in which the line meets the plane $z=0$.

The x and y coordinates of this point are given by

$$x+5y-7=0$$

$$2x-5y+1=0$$

$$\therefore x=2,\ y=1.$$

Thus one point on this line is $(2,1,0)$.

Hence the equations of the line are

$$\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z}{-3}$$

Section: 4

Equation of a straight line passing through two given points.

If the given points are (x_1, y_1, z_1) and (x_2, y_2, z_2), the direction ratios of the line passing through them are x_2-x_1, y_2-y_1, and z_2-z_1.

$\therefore$ The equations of the line are $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

EXAMPLE 1.

Find the point where the line $\dfrac{x-2}{2} = \dfrac{y-4}{-3} = \dfrac{z+6}{4}$ meets the plane $2x+4y-z-2=0$.

Solution:

Let $\dfrac{x-2}{2} = \dfrac{y-4}{3} = \dfrac{z+6}{4} = r$

$\therefore$ The coordinates of any point on the line are

$(2+2r,\ 4-3r,\ -6+4r)$

If this point lies on the plane $2x+4y-z-2=0$, we get

$2(2+2r)+4(4-3r)-(-6+4r)-2=0$

i.e $r=2$

Hence the coordinates of the required points are $(6,-2,2)$.

EXAMPLE 2.

Find the perpendicular distance from $P(3, 9, -1)$ to the line

$$\frac{x+8}{-8} = \frac{y-31}{1} = \frac{z-13}{5}.$$

Solution:

Let the foot of the perpendicular from P to the line be A.

Since A is on the line, its coordinates are $(-8r-8,\ r+31,\ 5r+13$

The direction ratios of the line AP are proportional to

$8r-8-3,\ r+31-9,\ 5r+13+1$

i.e., $-8r-11,\ r+22,\ 5r+14$

AP is perpendicular to the given line.

$\therefore -8(-8r-11)+1(r+22)+5(5r+14) = 0$

Simplifying, we get $r=-2$.

$\therefore$ A is the point $(8, 29, 3)$.

$\therefore AP^2 = (8-3)^2+(29-9)^2+(3+1)^2 = 441$

$\therefore AP = 21.$

$\therefore$ The perpendicular distance from A to the line is 21 units.

EXAMPLE 3.

Find the image of the point $(1,-2,3)$ in the plane $2x-3y+2z+3=0$.

Solution:

Let P be the point $(1,-2,3)$ and let its image w.r.to the plane $2x-3y+2z+3=0$ be Q.

The PQ is perpendicular to the plane.

The direction cosines of PQ are proportional to $2, -3, +2$.

$\therefore$ The equations of the line PQ are

$$\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-3}{2}$$

$\therefore$ The coordinates of Q are of the form $(2r+1, -3r-2, 2r+13)$.

The mid-point of PQ is

$$\left(\frac{2r+1+1}{2}, \frac{-3r-2-2}{2}, \frac{2r+3+3}{2}\right) \text{ ie}$$

$$\left(r+1, \frac{-3r-4}{2}, r+3\right)$$

This point lies on the plane $2x-3y+2z+3=0$.

$\therefore 2(r+1) - \left(\frac{-3r-4}{2}\right) + 2(r+3) + 3 = 0$

$\therefore r = -2.$

Hence Q is the point $(-3, 4, -1)$

EXAMPLE 4.

Find the equations of the image of the line

$\frac{x-1}{2} = \frac{y+2}{-5} = \frac{z-3}{2}$ in the plane

$2x - 3y + 2z + 3 = 0.$

Solution:

The image of this line in the plane is the straight line joining the images in the plane of two points on the given line.

$(1, -2, -3)$ is a point on the line and its image in the plane $2x - 3y + 2z + 3 = 0$ is $(-3, 4, -1)$.

The coordinates of the point R in which the line meets the plane are given by $(1+2r, -2-5r, 3+2r)$ where

$$2(1+2r)-3(-2-5r)+2(3+2r)+3=0$$

i.e., $23r+17=0$.

i.e $\quad r=\frac{-17}{23}$

$\therefore$ R is the point $\left(\frac{-11}{23}, \frac{39}{23}, \frac{35}{23}\right)$

Hence its equations are

$$\frac{x+3}{\frac{-58}{23}}=\frac{y-4}{\frac{53}{23}}=\frac{z+1}{\frac{-58}{23}}$$

i.e., $\frac{x+3}{-58}=\frac{y-4}{53}=\frac{z+1}{-58}$

EXAMPLE 5.

The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Find the coordinates of the orthocentre of the triangle ABC.

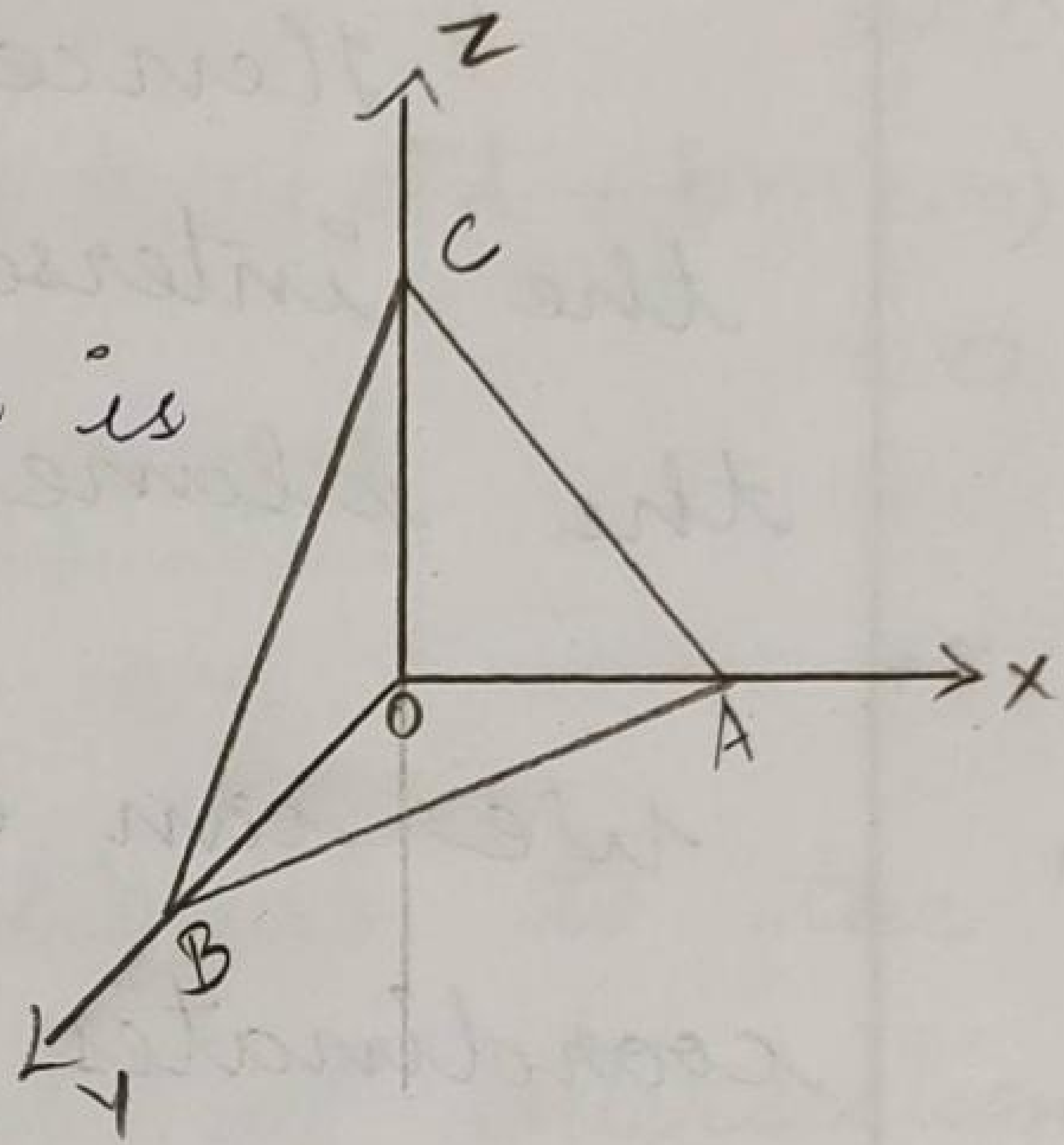
Solution:

The points A, B, C are respectively $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

The direction cosines of the line BC are proportional to $0, b, -c$.

Hence the equation of the line BC is $\frac{x}{0} = \frac{y-b}{b} = \frac{z}{-c}$

Let the line through A, perpendicular to B.C have direction cosines proportional to l, m, n.



Then its equation is

$$\frac{x-a}{l} = \frac{y}{m} = \frac{z}{n}$$

$\therefore l(0) + m(b) + n(-c) = 0$

i.e., $mb = nc$.

Hence the equations of the line become $\frac{x-a}{l} = \frac{by}{mb} = \frac{cz}{nc}$

Hence the equation of the plane passing through OX perpendicular to BC is

$$by = cz$$

Similarly the equation of the plane through OY perpendicular to CA is $cz = ax$.

These two planes will intersect on the line

$$ax = by = cz.$$

i.e., $\dfrac{x}{\frac{1}{a}} = \dfrac{y}{\frac{1}{b}} = \dfrac{z}{\frac{1}{c}}$

Hence the orthocentre is the intersection of this line with the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

We can easily show that the coordinates of the orthocentre are

$$\frac{\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}, \frac{\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}, \frac{\frac{1}{c}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

THE PLANE AND THE STRAIGHT LINE

Section : 5.

The condition for the line

$\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ to be parallel to the plane $ax+by+cz+d=0$.

Any point on this line can

be put in the form

$(x_1+lr,\ y_1+mr,\ z_1+nr)$

If this point lies on the plane, we get

$$a(x_1+lr)+b(y_1+mr)+c(z_1+nr)+d=0$$

i.e., $ax_1+by_1+cz_1+d+r(al+bm+cn)=0$

i.e., $$r=\frac{-ax_1+by_1+cz_1+d}{al+bm+cn}$$

Here r is proportional to the distance of the point of intersection from (x_1, y_1, z_1)

Hence the line is parallel to the plane if

$al+bm+cn=0$ and

$ax_1+by_1+cz_1 d\neq 0$

COROLLARY:

If the line lies in the plane, then $al+bm+cn=0$ and

$ax_1+by_1+cz_1+d=0$

These conditions lead to the

geometrical facts that a line will lie in a given plane if

(1) the normal to the plane is perpendicular to the line and

(2) any one point on the line lies in the plane.

COROLLARY:

The equation of any plane containing the line.

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$ is a

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$ subject to the condition $al + bm + cn = 0$

EXAMPLE 1.

Find the equations of the orthogonal projection of the line $\frac{x-2}{4} = \frac{y-1}{2} = \frac{z-4}{3}$ on to the plane $8x + 2y + 9z - 1 = 0$.

Solution:

The required orthogonal projection lies in the plane drawn through

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the given line perpendicular to the given plane.

The equation of any plane containing the given line is

$$A(x-2) + B(y-1) + C(z-4) = 0$$

subject to the condition

$$4A + 2B + 3C = 0$$

Plane (1) is perpendicular to the plane $8x + 2y + 9z - 1 = 0$

$$\therefore 8A + 2B + 9C = 0$$

From 2 and 3, we get

$$\frac{A}{12} = \frac{B}{-12} = \frac{C}{-8}$$

i.e., $\frac{A}{3} = \frac{B}{-3} = \frac{C}{-2}$.

Substituting the value of A, B, C in (1), we get the equation of the plane (1) as

$$3(x-2) - 3(y-1) - 2(z-4) = 0$$

i.e $3x - 3y - 2z + 5 = 0$

EXAMPLE 2.

If 'l' is the line $\frac{x}{-1} = \frac{y-1}{2} = \frac{z+2}{1}$, find the equation of the plane through 'l' which is parallel to the line of intersection of the planes $5x+2y+3z=4$ and $x-y+5z+6=0$

Solution:

The equation of any plane passing through 'l' is

$$Ax+B(y-1)+C(z-2)=0$$

where $-A+2B+C=0$

Let l, m, n be the direction of the line of intersection of the planes $5x+2y+3z=4$ and $x-y+5z+6=0$

Then $5l+2m+3n=0$

and $l-m+5n=0$

From (3) and (4) we get,

$$\frac{l}{13}=\frac{m}{22}=\frac{n}{-7}$$

Hence the plane (1) is parallel to the line whose direction ~~cosines~~ ratios are proportional to $13, 22, -7$.

$\therefore 13A + 22B - 7C = 0$

From (2) and (5), we get

$$\frac{A}{-36} = \frac{B}{6} = \frac{C}{-48} \quad \text{i.e.,} \quad \frac{A}{6} = \frac{B}{-1} = \frac{C}{8}$$

Substituting the values of A, B, C in (1), we get the required plane as $6x - y + 8z + 17 = 0$

Section : 6

Angle between the plane $ax + by + cz + d = 0$ and the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.

Let the required angle be θ.

The $90° - \theta$ is the angle between the line and the normal to the plane.

The direction ratios of the normal to the plane are a, b, c.

$$\therefore \cos(90° - \theta) = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)} \cdot \sqrt{l^2 + m^2 + n^2}}$$

$$\text{i.e } \sin\theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)} \cdot \sqrt{l^2 + m^2 + n^2}}$$

COROLLARY:

The line is parallel to the plane if $\theta = 0$.

i.e., $al + bm + cn = 0$

COPLANAR LINES

Section: 7

The condition that two given straight lines should be coplanar.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

NOTE:

If the two lines are coplanar, they must intersect.

EXAMPLE 1.

Find the condition for the lines $ax + by + cz + d = 0 = a_1x + b_1y + c_1z + d_1$

$a_2x + b_2y + c_2z + d_2 = 0 = a_3x + b_3y + c_3z + d_3$

to be coplanar.

Solution:

Let the lines intersect at the point (x_1, y_1, z_1).

Then (x_1, y_1, z_1) lies on the planes.

$$ax + by + cz + d = 0$$
$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

$$\therefore ax_1 + by_1 + cz_1 + d = 0$$
$$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$$
$$a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$$
$$a_3x_1 + b_3y_1 + c_3z_1 + d_3 = 0$$

Eliminating x_1, y_1, z_1 from the above four equations, we get the condition

$$\begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0$$

EXAMPLE 2.

Prove that the lines

$$\frac{x+1}{-3} = \frac{y+10}{8} = \frac{z-1}{2}; \quad \frac{x+3}{-4} = \frac{y+1}{7} = \frac{z-4}{1}$$

are coplanar. Find also their point of intersection and the plane through them.

Solution:

The coordinates of the points on the two lines are respectively of the form

$(-3r-1, 8r-10, 2r+1)$ and

$(-4r_1-3, 7r_1-1, r_1+4)$

The lines are coplanar if the lines intersect, i.e., if the three equations

$$-3r-1 = 4r_1-3$$

$$8r-10 = 7r_1-1$$

$$2r+1 = r_1+4$$ are

simultaneously true.

$$\therefore 3r-4r_1 = 2$$

$$8r-7r_1 = 9$$

$$2r-r_1 = 3$$

Solving the first two equations, we get $r=2$ and $r_1=1$.

These values satisfy the third equation also.

$\therefore$ The lines are coplanar.

Substituting the value of r in (1) or the value of r_1 in (2), we get the co-ordinates of the intersecting point.

The intersecting point is $(-7, 6, 5)$.

The equation of the plane containing the lines is

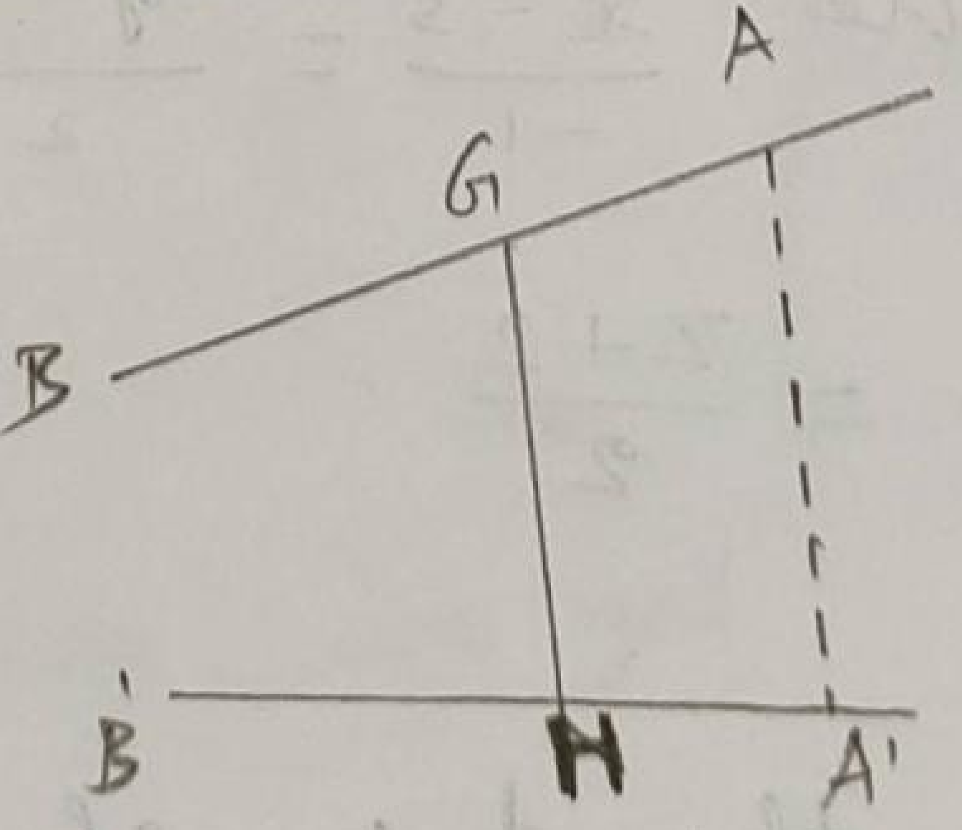
$$\begin{vmatrix} x+1 & y+10 & z-1 \\ -3 & 8 & 2 \\ -4 & 7 & 1 \end{vmatrix} = 0$$

i.e., $6x + 5y - 11z + 67 = 0$

Section : 8

The shortest distance between two given lines.

The shortest distance is the line of intersection of the planes containing the lines AB and GH; and A'B' and GH.



Hence GH is the line

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

COROLLARY:

The two lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$ and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$ are coplanar

if the shortest distance between them is zero.

i.e., $$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

EXAMPLE 1.

Find the shortest distance between the $\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$;

$$\frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}.$$

Solution:

Let the d.c. of the line perpendicular to both the lines be

l, m, n.

Then $-l + 2m + n = 0$

$$l + 3m + 2n = 0$$

$$\therefore \frac{l}{1} = \frac{m}{3} = \frac{n}{-5}$$

$$\therefore l = \frac{1}{\sqrt{35}},\ m = \frac{3}{\sqrt{35}},\ n = \frac{-5}{\sqrt{35}}$$

The magnitude of the shortest distance is the projection of the line joining the points $(3, 4, -2)$ and $(1, -7, -2)$ on the line of shortest distance.

$$\therefore S.D = (3-1)\frac{1}{\sqrt{35}} + (4+7)\frac{3}{\sqrt{35}} + (-2+2)\frac{1}{\sqrt{35}}$$

$$= \sqrt{35}.$$

$\cancel{\frac{4}{\sqrt{35}}} \quad \cancel{=}$

The equation of the shortest distance between them is

$$\begin{vmatrix} x-3 & y-4 & z+2 \\ -1 & 2 & 1 \\ 1 & 3 & -5 \end{vmatrix} = 0 = \begin{vmatrix} x-1 & y+7 & z+2 \\ 1 & 3 & 2 \\ 1 & 3 & -5 \end{vmatrix}$$

Simplifying, we get

$$13x + 4y + 5z - 45 = 0 = 3x - y - 10$$

Section 8.1

If $u_1 = 0 = v_1$ and $u_2 = 0 = v_2$

be two straight lines, then the general equations of a straight line intersecting them both are $u_1 + \lambda_1 v_1 = 0 = u_1 + \lambda_1 v_2$ where λ_1, λ_2 are constants.

The line $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$ lies in the plane $u_1 + \lambda_1 v_1 = 0$ which again ~~too~~ contains the line $u_1 = 0 = v_1$.

The two lines $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$ and $u_1 = 0 = v_1$ are ∴ coplanar and hence they intersect.

Similarly, the same line intersects the line $u_2 = 0 = v_2$.

Section 8.2

The equations of two skew lines in a simplified form.

$$\frac{x}{\cos\alpha} = \frac{y}{-\sin\alpha} = \frac{z-c}{0}, \quad \text{i.e., } y = -x\tan\alpha, \; z = c$$

and

$$\frac{x}{\cos\alpha} = \frac{y}{\sin\alpha} = \frac{z+c}{0},$$

i.e., $y = x\tan\alpha$, $z = -c$.

NOTE 1.

$(r, -r\tan\alpha, c)$ and $(p, p\tan\alpha, -c)$ are the general coordinates of points on the two lines, r and p being any two constants.

NOTE 2.

Solutions to problems relating to two non-intersecting given straight lines are often simplified by taking the equations of the lines in the simplified form.

NOTE 3.

Putting $\tan\alpha = m$, we can also express the equation of the lines in the forms $y = -mx$, $z = c$ & $y = mx$, $z = -c$.

Hence the general co-ordinates of points on the two lines are respectively $(r, -mr, c)$ and $(p, mp, -c)$.