UNIT - IV

THE SPHERE

Chapter IV

Section 1: DEFINITION

A sphere is the locus of a point which moves in such a way that its distance from a fixed point is always constant.

The fixed point is called the centre of the sphere.

The constant distance is called the radius of the sphere.

Section 2:

The equation of a sphere when then centre and the radius given.

The equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \longrightarrow (1)$$

where (a, b, c) is the centre and r is the radius of the sphere.

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 + z^2 - 2cz + c^2 = r^2$

$x^2 + y^2 + c^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - r^2 = 0$

The reduced form of the sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

NOTE:

When the centre of the sphere is at the origin and its radius is ~~x~~ a, then equation of the sphere is $x^2 + y^2 + z^2 = a^2$.

Section 3:

The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ always represents a sphere and to find its centre and radius.

The equation ② can be written as

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + u^2 + v^2 + w^2 - u^2 + v^2 - w^2 = -d$$

$$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = (u^2 + v^2 + w^2 - d)$$

$$(x+u)^2+(y+v)^2+(z+w)^2=(u^2+v^2+w^2-d) \longrightarrow (3)$$

Comparing equation (1), (3) we get,

$a=-u$, $b=-v$, $c=-w$, $r^2=u^2+v^2+w^2-d$

$$r=\sqrt{u^2+v^2+w^2-d}$$

$\therefore$ The centre of the equation (2) is $(-u,-v,-w)$.

The radius of equation (2) is $\sqrt{u^2+v^2+w^2-d}$

<u>NOTE: 1</u>

i) When $r^2=u^2+v^2+w^2-d$ is positive, the locus is a real sphere.

ii) When $r^2=u^2+v^2+w^2-d=0$, then equation (3) reduces to

$$(x+u)^2+(y+v)^2+(z+w)^2=0$$

This is called a point sphere and the only real solution of the equation is $x=-u$, $y=-v$, $z=-w$. In this case, the sphere reduces to the point $(-u, -v, -w)$

iii) When $r^2 = u^2 + v^2 + w^2 - d$ is negative, the locus is an imaginary sphere.

NOTE 2:

The characteristics of the equation of a sphere.

i) It is of the second degree in x, y, z.

ii) The coefficients of x^2, y^2, z^2 are equal.

iii) The terms xy, yz, zx are absent.

NOTE 3:

The general equation of any sphere is

$$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0 \longrightarrow (4)$$

Divide by a.

$$x^2 + y^2 + z^2 + 2\frac{u}{a}x + \frac{2v}{a}y + \frac{2wz}{a} + \frac{d}{a} = 0$$

The centre of the sphere is $\left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right)$ and its radius is

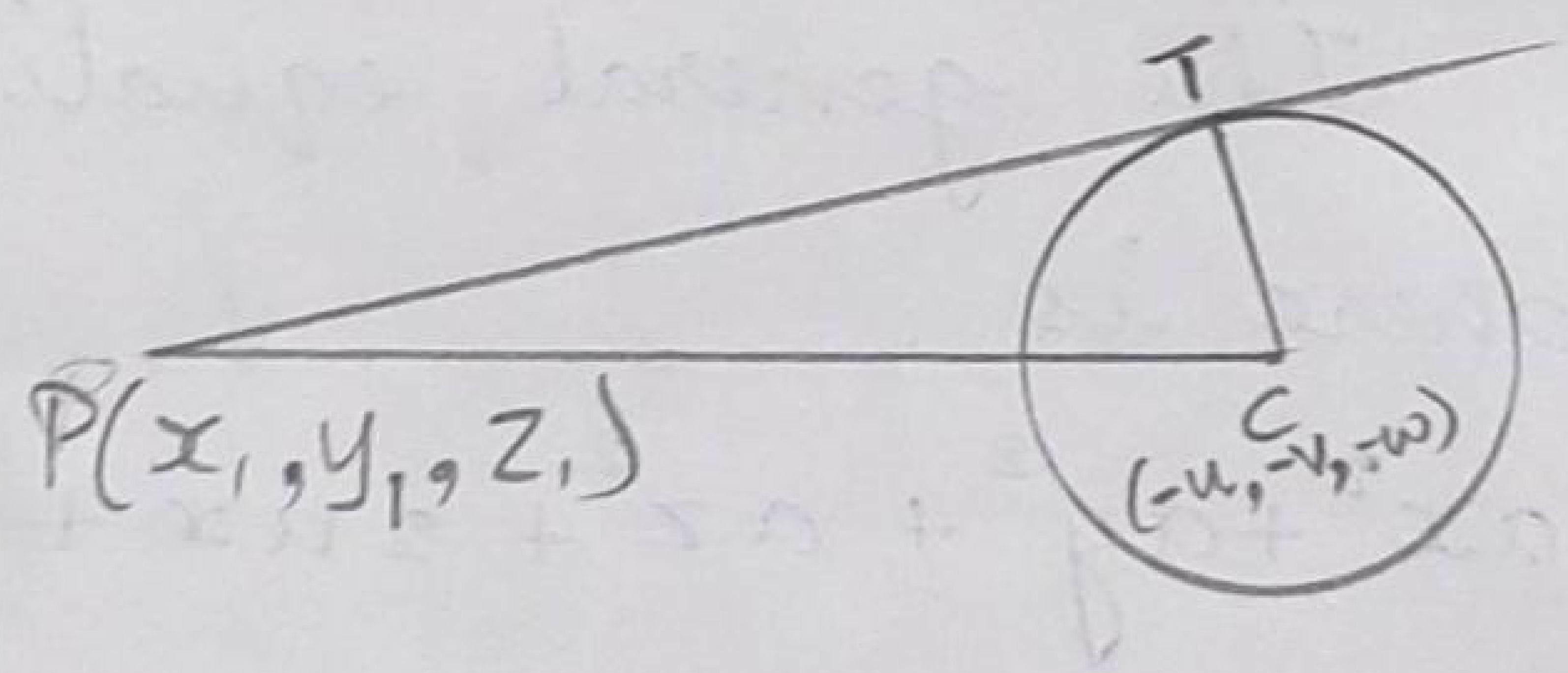
$$r = \sqrt{\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2} - \frac{d}{a}}$$

Section 4:

The length of the tangent from the point (x_1, y_1, z_1) to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

PROOF:

Let P be the point (x_1, y_1, z_1), C the centre of the sphere and PT a tangent from P to the sphere.



The coordinates of C are $(-u, -v, -w)$. (r is the radius of the sphere and is equal to $\sqrt{(u^2+v^2+w^2-d)}$.

CT is perpendicular to PT.

$$\therefore PC^2 = PT^2 + CT^2$$

$$(x_1+u)^2 + (y_1+v)^2 + (z_1+w)^2$$
$$= PT^2 + u^2 + v^2 + w^2 - d.$$

$$PT^2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d.$$

NOTE:

The value of PT^2 is called the power of P with respect to the circle.

Cor. 1.

The point (x_1, y_1, z_1) lies outside, on or inside the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

according as

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d >=< 0$$

Cor. 2.

If d is positive, the origin lies outside the sphere, if d is negative, the origin lies inside the sphere ; if $d=0$, the origin lies on the sphere.

EXAMPLE 1:

1. Find the equation of the sphere with centre $(-1,2,-3)$ and radius 3 units.

Solution:

We know that the equation of the sphere is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

$$\therefore (x+1)^2+(y-2)^2+(z+3)^2=3^2$$

$$x^2+2x+1+y^2-4y+4+z^2+6z+9=9$$

$$x^2+y^2+z^2+2x-4y+6z+1+4+9-9=0$$

$$x^2+y^2+z^2+2x-4y+6z+5=0.$$

2. Find the co-ordinates of the centre and radius of the sphere.

$$2x^2+2y^2+2z^2-2x+4y+2z-15=0$$

Solution:

$$2x^2+2y^2+2z^2-2x+4y+2z-15=0$$

$$x^2+y^2+z^2-x+2y+z-\frac{15}{2}=0 \longrightarrow (1)$$

We know that the general equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (2)$$

Comparing the co-efficients of x, y, z from (1) and (2), we get

$2ux = -x$ $2vy = 2y$ $z = 2wz$

$u = \frac{-1}{2}$ $v = 1$ $w = \frac{1}{2}$

$\therefore$ The centre $(-u, -v, -w)$ is

$\left(\frac{1}{2}, -1, \frac{-1}{2}\right)$.

The radius $\sqrt{u^2+v^2+w^2-d}$ is

$$\sqrt{\tfrac{1}{4}+1+\tfrac{1}{4}+\tfrac{15}{2}} = \sqrt{\frac{1+4+1+30}{4}} = \sqrt{\frac{36}{4}}$$

$$= \sqrt{\left(\frac{6}{2}\right)^2}$$

$$r = 3$$

$\therefore$ The radius, $r = 3$

3. Find the equation of the sphere which has its centre at the point $(6,-1,2)$ and touches the plane $2x-y+2z-2=0$.

<u>Solution</u>:

The radius of the sphere is perpendicular distances from $(6,-1,2)$ to the plane $2x-y+2z-2=0$

$\therefore$ Radius of the sphere

$$r = \frac{lx+my+nz+p}{\sqrt{l^2+m^2+n^2}}$$

$$r = \frac{2(6)-(-1)+2(2)-2}{\sqrt{2^2+1^2+2^2}}$$

$$r = \frac{12+1+4-2}{\sqrt{4+1+4}}$$

$$r = \frac{15}{3}$$

$$r = 5$$

$\therefore$ The equation of the sphere is

$$(x-6)^2+(y+1)^2+(z-2)^2 = 5^2.$$

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$$x^2-12x+36+y^2+2y+1+z^2-4z+4=25$$

$$x^2+y^2+z^2-12x+2y-4z+36+1+4-25=0$$

$$x^2+y^2+z^2-12x+2y-4z+16=0$$

4. Find the equation to the sphere through the focus points $(2,3,1)$, $(5,-1,2)$, $(4,3,-1)$ and $(2,5,3)$.

Solution:

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (A)$$

Since the sphere passes through these four points the co-ordinates of these points must satisfy the equation of the sphere.

Hence, at $(2,3,1)$

$$2^2+3^2+1^2+4u+6v+2w+d=0$$

$$4+9+1+4u+6v+2w+d=0$$

$$4u+6v+2w+d=-14 \longrightarrow (1)$$

At $(5,-1,2)$

$$5^2+1^2+2^2+10u-2v+4w+d=0$$

$$25+1+4+10u-2v+4w+d=0$$

$$10u-2v+4w+d=-30 \quad \longrightarrow (2)$$

At $(4,3,-1)$

$$4^2+3^2+1^2+8u+6v-2w+d=0$$

$$16+9+1+8u+6v-2w+d=0$$

$$8u+6v-2w+d=-26 \quad \longrightarrow (3)$$

At $(2,5,3)$

$$2^2+5^2+3^2+4u+10v+6w+d=0$$

$$4+25+9+4u+10v+6w+d=0$$

$$4u+10v+6w+d=-38 \quad \longrightarrow (4)$$

From (1) and (2),

$$(1)-(2) \Rightarrow \begin{array}{r} 4u+6v+2w+d=-14 \\ 10u-2v+4w+d=-30 \\ \hline -6u+8v-2w \quad =16 \end{array} \quad \longrightarrow (5)$$

(2) - (3) $\Rightarrow$

$$10u - 2v + 4w + d = -30$$
$$8u + 6v - 2w + d = -26$$
$$2u - 8v + 6w = -4 \longrightarrow (6)$$

(3) - (4) $\Rightarrow$

$$8u + 6v - 2w + d = -26$$
$$4u + 10v + 6w + d = -38$$
$$4u - 4v - 8w = 12 \longrightarrow (7)$$

(5) + 3 × (6)

$$-6u + 8v - 2w = 16$$
$$6u - 24v + 18w = -12$$
$$-16v + 16w = 4$$
$$-4v + 4w = 1 \longrightarrow (8)$$

(6) × 2 - (7)

$$4u - 16v + 12w = -8$$
$$4u - 4v - 8w = 12$$
$$-12v + 20w = -20$$
$$4(-3v + 5w) = -20$$
$$-3v + 5w = -5 \longrightarrow (9)$$

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$⑧ \times 3 \Rightarrow \quad -12v + 12w = 3$

$⑨ \times 4 \Rightarrow \quad -12v + 20w = -20$

$-8w = 23$

$w = \frac{-23}{8}$

Put $w = \frac{-23}{8}$ in ⑧, we get

$-4v + 4\left(\frac{-23}{8}\right) = 1$

$-4v = 1 + \frac{23}{2}$

$-4v = \frac{2+23}{2}$

$v = \frac{-25}{8}$

Put v, w in ⑥, we get

$2u - 8\left(\frac{-25}{8}\right) + 6\left(\frac{-23}{8}\right) = -4$

$2u + 25 - \frac{69}{4} = -4$

$2u + \frac{100-69}{4} = -4$

$2u + \frac{31}{4} = -4$

$2u = -4 - \frac{31}{4}$

$2u = \frac{-16-31}{4}$

$$u = \frac{-47}{8}$$

Put u, v, w in ①, we get

$$4\left(\frac{-47}{8}\right) + 6\left(\frac{-25}{8}\right) + 2\left(\frac{-23}{8}\right) + d = -14$$

$$\frac{-188 - 150 - 46}{8} + d = -14$$

$$\frac{-384}{8} + d = -14$$

$$-48 + d = -14$$

$$d = -14 + 48$$

$$\therefore d = 34.$$

Put u, v, w and d in Ⓐ, we get

$$x^2 + y^2 + z^2 + 2\left(\frac{-47}{8}\right)x + 2\left(\frac{-25}{8}\right)y + 2\left(\frac{-23}{8}\right)z + 34 = 0$$

$$x^2 + y^2 + z^2 - \frac{47}{4}x - \frac{25}{4}y - \frac{23}{4}z + 34 = 0$$

$$4x^2 + 4y^2 + 4z^2 - 47x - 25y - 23z + 136 = 0$$

5. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2) = 4k^2$.

Solution:

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (1)$$

It passes through the origin $(0,0,0)$.

$$\therefore 0+0+0+2u(0)+2v(0)+2w(0)+d=0$$

$$\therefore d=0$$

The radius is $k = \sqrt{u^2+v^2+w^2-d}$

$$k^2 = u^2+v^2+w^2-d^2$$

The sphere meets the x-axis at points given by $x^2+2ux=0$

$$x(x+2u)=0$$

$$x=0,\ x=-2u$$

$\therefore$ The coordinates of A are $(-2u,0,0)$.

The co-ordinates of B are

$$y^2+2vy=0.$$

$y(y+2v)=0$

$y=0,\ y=-2v$

$\therefore$ The coordinates of B are

$(0,-2v,0)$.

The co-ordinates of C are

$(0,0,-2w)$.

Let the centroid of the triangle be (x_1,y_1,z_1).

$$\frac{-2u+0+0}{3}=x_1;\quad x_1=\frac{-2u}{3}\Rightarrow u=\frac{-3x_1}{2}$$

$$\frac{0-2v+0}{3}=y_1;\quad y_1=\frac{-2v}{3}\Rightarrow v=\frac{-3y_1}{2}$$

$$\frac{0+0-2w}{3}=z_1;\quad z_1=\frac{-2w}{3}\Rightarrow w=\frac{-3z_1}{2}$$

Put u,v,w in ② we get

$$\left(\frac{-3x_1}{2}\right)^2+\left(\frac{-3y_1}{2}\right)^2+\left(\frac{-3z_1}{2}\right)^2=k^2$$

$$9x_1^2+9y_1^2+9z_1^2=4k^2$$

$$9(x_1^2+y_1^2+z_1^2)=4k^2$$

(i.e) Locus of (x_1,y_1,z_1) is

$$9(x_1^2+y_1^2+z_1^2)=4k^2$$

6. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Solution:

Let the equation of the plane be $lx + my + nz = p$.

Since it passes through the point (a, b, c), $la + mb + nc = p$ $\longrightarrow$ (1)

The co-ordinates of the points where the plane $lx + my + nz = p$ meets the x-axis are obtained by putting $y = 0$, $z = 0$ in that equation.

$$la + 0 + 0 = p$$
$$a = \frac{p}{l}$$

$\therefore$ The co-ordinates of A are $\left(\frac{p}{l}, 0, 0\right)$ similarly the points B and C are respectively $\left(0, \frac{p}{m}, 0\right), \left(0, 0, \frac{p}{n}\right)$.

$\therefore$ The sphere which passes through the points $(0,0,0)$, $(\frac{p}{l},0,0)$, $(0,\frac{p}{m},0)$ and $(0,0,\frac{p}{n})$.

Hence the equation of the sphere be

$$x^2+y^2+z^2-\frac{p}{l}x-\frac{p}{m}y-\frac{p}{n}z=0$$

Let the centre of this sphere be (x_1, y_1, z_1).

We know that the centre of the sphere be $(-u,-v,-w)$

(i.e) $-\frac{p}{l}x = 2ux$

$$u = \frac{-p}{2l} \Rightarrow -u = \frac{p}{2l}$$

$$-v = \frac{p}{2m}, \quad -w = \frac{p}{2n}$$

$$\therefore (x_1, y_1, z_1) = \left(\frac{p}{2l}, \frac{p}{2m}, \frac{p}{2n}\right).$$

(i.e) $x_1 = \frac{p}{2l} \Rightarrow l = \frac{p}{2x_1}$

$$m = \frac{p}{2y_1}; \quad n = \frac{p}{2z_1}$$

Substituting the values of l, m, n in ① we get

$$\left(\frac{p}{2x_1}\right)a + \left(\frac{p}{2y_1}\right)b + \left(\frac{p}{2z_1}\right)c = p$$

$$\frac{p}{2}\left(\frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}\right)=p$$

$$\frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=2$$

H.W

1. Find the radius and the co-ordinates of the centre of each of the following spheres.

(1) $x^2+y^2+z^2-2x+6y+4z-35=0$

Solution:

Given: Equation of the sphere is

$$x^2+y^2+z^2-2x+6y+4z-35=0 \longrightarrow (1)$$

We know that the general equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (2)$$

Comparing the coefficients of x, y, z from (1) and (2), we get

$2ux=-2x$ $\quad$ $2v=6$ $\quad$ $2w=4$ $\quad$ $d=-35$

$u=-1$ $\quad$ $v=3$ $\quad$ $w=2$

$\therefore$ The centre of the sphere $(-u,-v,-w)$ is $(+1,-3,-2)$.

Radius, $r=\sqrt{u^2+v^2+w^2-d}$

$$r=\sqrt{(-1)^2+(3)^2+(2)^2+35}$$

$$r=\sqrt{1+9+4+35}=\sqrt{49}$$

$r = 7$ units.

(2) $x^2+y^2+z^2-6x-2y-4z-11=0$

Solution:

Given: Equation of the sphere

$$x^2+y^2+z^2-6x-2y-4z-11=0 \rightarrow (1)$$

General equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (2)$$

Comparing (1) and (2),

$2u=-6$, $2v=-2$, $2w=-4$ $d=-11$

$u=-3$, $v=-1$, $w=-2$

$\therefore$ The centre of the sphere $(-u,-v,-w)$ is $(3,1,2)$.

Radius, $r = \sqrt{u^2+v^2+w^2-d}$

$= \sqrt{9+1+4+11}$

$= \sqrt{25}$

$r = 5$ units

3. $16x^2+16y^2+16z^2-16x-8y-16z-55=0$.

Solution:

Given: Equation of sphere is

$$16x^2+16y^2+16z^2-16x-8y-16z-55=0$$

$$x^2+y^2+z^2-x-\frac{y}{2}-z-\frac{55}{16}=0 \rightarrow (1)$$

Comparing ① with the general equation of the sphere,

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$2u=-1$, $2v=-\frac{1}{2}$, $2w=-1$, $d=\frac{-55}{16}$

we get

$u=-\frac{1}{2}$, $v=-\frac{1}{4}$, $w=-\frac{1}{2}$.

$\therefore$ The center of the sphere $(-u,-v,-w)$ is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$.

Radius, $r=\sqrt{u^2+v^2+w^2-d}$

$$r=\sqrt{\frac{1}{4}+\frac{1}{16}+\frac{1}{4}+\frac{55}{16}}$$

$$r=\sqrt{\frac{1}{2}+\frac{1}{16}+\frac{55}{16}}$$

$$r=\sqrt{\frac{64}{16}}$$

$$r=\frac{8}{4}$$

$r=2$ units

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4. $2x^2+2y^2+2z^2+8x-8y-6z-1=0$

Solution: $x^2+y^2+z^2+4x-4y-3z-\frac{1}{2}=0 \rightarrow ①$

Comparing the given equation with the general equation of the sphere,

$\cancel{2x^2+12}$

we get, $2u=4$, $2v=-4$, $2w=-3$

$u=2$, $v=-2$, $w=\frac{-3}{2}$

The centre of the sphere $(-u,-v,-w)$ is $(-2, 2, \frac{3}{2})$.

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Radius, $r = \sqrt{u^2+v^2+w^2-d}$

$r = \sqrt{4+4+9/4+1/2}$

$r = \sqrt{8+9/4+1/2}$

$r = \sqrt{\frac{32+9+2}{4}}$

$r = \sqrt{43/4}$

$r = \frac{\sqrt{43}}{2}$ units

2. Find the equation of the sphere

(1) centre at (1,2,3) ; radius 4.

Solution:

We know that the equation of the sphere is

$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

$(x-1)^2+(y-2)^2+(z-3)^2 = 16$

$x^2+1-2x+y^2+4-4y+z^2+9-6z = 16$

$x^2+y^2+z^2-2x-4y-6z+14 = 16$

$x^2+y^2+z^2-2x-4y-6z-2 = 0$

(2) centre at (-6,-2,3); radius 5

Solution:

We know that the equation of the sphere is

H. W:

4. Find the equation of the sphere through the four points and determine its radius.

(1) $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

(2) $(0,1,3)$, $(1,2,4)$, $(2,3,1)$, $(3,0,2)$.

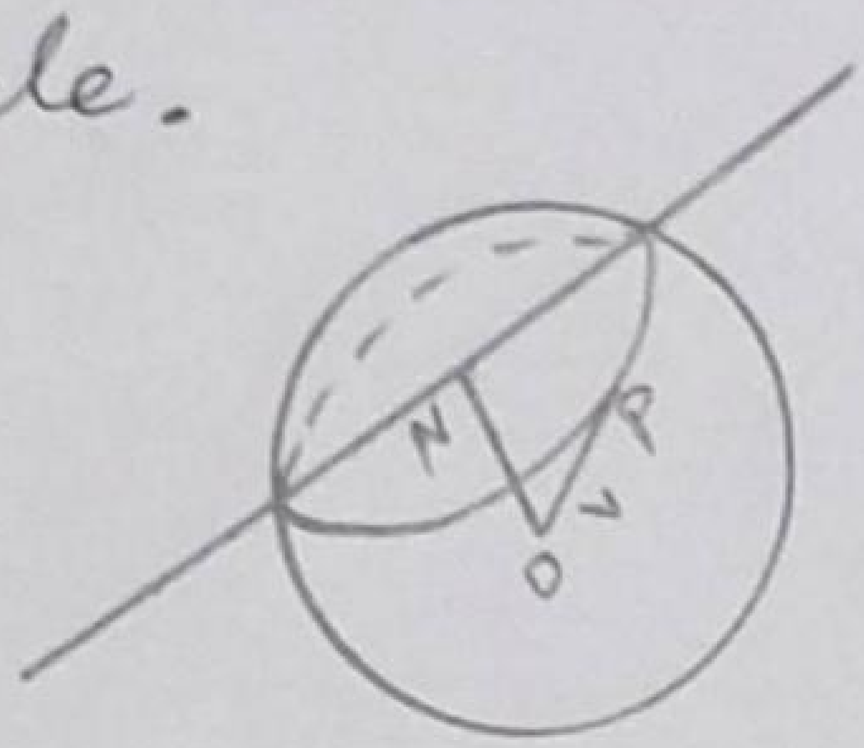
(3) $(0,0,0)$, $(-a,b,c)$, $(a,-b,c)$, $(a,b,-c)$

~~(4)~~

15. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is the sphere $x^2+y^2+z^2=r^2$.

SECTION : 5

The plane section of a sphere is a circle.



Let O be the centre of the sphere of radius r and P any point common to the sphere and the plane.

Then OP = radius of the sphere.

Draw ON perpendicular to the plane.

Then $NP^2 = OP^2 - ON^2$

$NP^2 = r^2 - ON^2$

O and N are fixed points.

$\therefore$ ON is constant.

$\therefore$ NP = constant.

Hence the locus of P is a circle whose centre is N, the foot of the perpendicular from the centre of the sphere to the plane. Such a circle is called a small circle on the sphere.

Definition:

GREAT CIRCLE, SMALL CIRCLE

A circle on a sphere whose plane passes through the centre of the sphere is called a great circle. Otherwise it is a small circle.

SECTION: 6

Equation of a circle on a sphere.

The section of a sphere is a circle. Therefore the circle can be represented by two equations, one being of a sphere and the other of a plane. Thus the equations $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, $lx+my+nz=p$ taken together represent a circle.

SECTION 6.1:

Equation of a sphere passing through a given circle.

The equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d+k(lx+my+nz-p)=0$$

in which k is any constant represent a sphere. in which k is any constant represent a

SECTION: 7

Intersection of two sphere is a circle.

Let the equations of the two spheres be

$$S_1=x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$$S_2=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

The co-ordinates of points common to any two spheres satisfy both these equations and therefore they also satisfy the equation

$S_1 - S_2 = 2x(u-u_1) + 2y(v-v_1) + 2z(w-w_1) + (d-d_1) = 0$

which being of the first degree, represent a plane.

Thus the points of intersection of the two sphere are the same as those of any one of them on this plane and therefore lie on a circle.

EXAMPLE : 1

Find the equation of the sphere having the circle $x^2+y^2+z^2-2x+4y-6z+7=0$, $2x-y+2z=5$ for a great circle.

Solution:

Any sphere through the circle is

$x^2+y^2+z^2-2x+4y-6z+7+k(2x-y+2z-5)=0$

Since the centre of a great circle coincides with the centre of the sphere,

$\left(\frac{2-2k}{2}, \frac{-4+k}{2}, \frac{6-2k}{2}\right)$

lies on the plane $2x-y+2z-5=0$.

$\therefore 2\left(\frac{2-2k}{2}\right) - \left(\frac{-4+k}{2}\right) + 2\left(\frac{6-2k}{2}\right) - 5 = 0$

Simplifying, $k = \frac{5}{9}$.

Required equation is

$x^2+y^2+z^2-2x+4y-6z+7+\frac{5}{9}(2x-y-2z-5)=0$

(ie) $9(x^2+y^2+z^2) - 8x + 31y - 44z - 18 = 0$.

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$$\frac{-10(k+3)}{10} = \left(\frac{7k^2+6k+10}{2}\right)^{\frac{1}{2}}$$

$$2(k+3)^2 = 7k^2+6k+10$$

$$2(k^2+6k+9) = 7k^2+6k+10$$

$$2(k^2+6k+9) - 7k^2+6k+10$$

$$2k^2-7k^2+12k-6k+18-10=0$$

$$-5k^2+6k+8=0$$

$$5k^2-6k-8=0$$

$$5k^2-10k+4k-8=0$$

$$5k(k-2)+4(k-2)=0$$

$$(k-2)(5k+4)=0$$

$$k=2,\ k=\frac{-4}{5}.$$

Substituting the values of k in (1), we get for $k=2$,

$$(x^2+y^2+z^2-2x-4y)+2(x+2y+3z-8)=0$$

$$x^2+y^2+z^2-2x-4y+2x+4y+6z-16=0$$

$$x^2+y^2+z^2+6z-16=0.$$

For $k = \frac{-4}{5}$,

$$(x^2+y^2+z^2-2x-4y) - \frac{4}{5}(x+2y+3z-8) = 0$$

$$5x^2+5y^2+5z^2-10x-20y-4x-8y-12z+32 = 0$$

$$5x^2+5y^2+5z^2-14x-28y-12z+32=0$$

$$5(x^2+y^2+z^2)-14x-28y-12z+32=0$$

---

3. The plane ABC, whose equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. meets the axes in A, B, C. Find the equation to the circumcircle of the triangle ABC and obtain the co-ordinates of its center and radius.

Solution:

The points A, B, C are respectively $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.

Let O be the origin

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (1)$$

$(a,0,0) \Rightarrow (1)$. $-u = \frac{a}{2}$.

$x^2+y^2+z^2-ax-by-cz=0$ is the equation to the sphere OABC.

The equation of the circumcircle of the triangle ABC is

$$x^2+y^2+z^2-ax-by-cz=0,$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

$\therefore$ The centre of the sphere is $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

The centre of the circle is the foot of the perpendicular from $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ to the plane.

The equation of the perpendicular to the plane through $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ is

$$\frac{x-\frac{a}{2}}{1/a}=\frac{y-\frac{b}{2}}{1/b}=\frac{z-\frac{c}{2}}{1/c}=\lambda$$

$\therefore$ The coordinates of any point on this line are of the form (x, y, z).

(i.e) $\left(\frac{a}{2}+\frac{\lambda}{2}, \frac{b}{2}+\frac{\lambda}{b}, \frac{c}{2}+\frac{\lambda}{c}\right)$

If this point lies on the plane we get

$$\frac{1}{a}\left(\frac{a}{2}+\frac{\lambda}{a}\right)+\frac{1}{b}\left(\frac{b}{2}+\frac{\lambda}{b}\right)+\frac{1}{c}\left(\frac{c}{2}+\frac{\lambda}{c}\right)=1$$

(i.e) $\lambda\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=-\frac{1}{2}$

$$\lambda=\frac{-1}{2(a^{-2}+b^{-2}+c^{-2})}$$

$$\therefore x=\frac{a}{2}+\frac{\lambda}{a}$$

$$=\frac{a}{2}-\frac{1}{2a(a^{-2}+b^{-2}+c^{-2})}$$

$$=\frac{aa(a^{-2}+b^{-2}+c^{-2})}{2a(a^{-2}+b^{-2}+c^{-2})}$$

$$=\frac{a^2.a^{-2}+a^2b^{-2}+a^2c^{-2}-1}{2a(a^{-2}+b^{-2}+c^{-2})}$$

$$=\frac{1+a^2(b^{-2}+c^{-2})-1}{2a(a^{-2}+b^{-2}+c^{-2})}$$

$$=\frac{a^2(b^{-2}+c^{-2})}{2a(a^{-2}+b^{-2}+c^{-2})}$$

$$x=\frac{\frac{a}{2}(b^{-2}+c^{-2})}{a^{-2}+b^{-2}+c^{-2}}$$

We can find,

$$y = \frac{\frac{b}{2}(c^{-2}+a^{-2})}{a^{-2}+b^{-2}+c^{-2}}$$

$$= \frac{\frac{c}{2}(a^{-2}+b^{-2})}{a^{-2}+b^{-2}+c^{-2}}$$

∴ The coordinates of the circumcentre of the triangle ABC are (x, y, z).

$$\left[\frac{\frac{a}{2}(b^{-2}+c^{-2})}{a^{-2}+b^{-2}+c^{-2}}, \frac{\frac{b}{2}(c^{-2}+a^{-2})}{a^{-2}+b^{-2}+c^{-2}}, \frac{\frac{c}{2}(a^{-2}+b^{-2})}{a^{-2}+b^{-2}+c^{-2}}\right]$$

Let R be the radius of the circle, d be the perpendicular distance from

$\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ to the plane

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and r the radius of the sphere.

$$x^2+y^2+z^2-ax-by-cz=0$$

We have $R^2 = r^2 - d^2$

$$r^2 = \frac{-(a^2+b^2+c^2)}{4}$$

and $d = \dfrac{\frac{1}{2}}{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{1/2}}$

$$\therefore R^2 = \frac{a^2+b^2+c^2}{4} - \frac{1}{4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

$$= \frac{a^2+b^2+c^2}{4} - \frac{1}{4\left(\frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2}\right)}$$

$$= \frac{(a^2+b^2+c^2)(b^2c^2+a^2c^2+a^2b^2) - a^2b^2c^2}{4(b^2c^2+a^2c^2+a^2b^2)}$$

$$= \frac{a^2b^2c^2 + a^2a^2c^2 + a^2a^2b^2 + b^2b^2c^2 + a^2b^2c^2 + a^2b^2b^2 + b^2c^2c^2 + a^2c^2c^2 + a^2b^2c^2 - a^2b^2c^2}{4(b^2c^2+a^2c^2+a^2b^2)}$$

$$= \frac{a^2a^2c^2 + a^2a^2b^2 + b^2b^2c^2 + b^2b^2a^2 + c^2c^2b^2 + c^2c^2a^2 + 2a^2b^2c^2}{4(b^2c^2+a^2c^2+a^2b^2)}$$

$$= \frac{b^2c^2(b^2+c^2) + c^2a^2(a^2+c^2) + a^2b^2(a^2+b^2) + 2a^2b^2c^2}{4(b^2c^2+a^2c^2+a^2b^2)}$$

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$$R^2 = \frac{(b^2+c^2)(a^2+c^2)(a^2+b^2)}{4(b^2c^2+a^2c^2+a^2b^2)}$$

$$R = \frac{1}{2}\left\{\frac{(b^2+c^2)(a^2+c^2)(a^2+b^2)}{b^2c^2+a^2c^2+a^2b^2}\right\}^{1/2}$$

SECTION: 8

The equation of the tangent plane to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

at point (x_1, y_1, z_1).

Let P be the point (x_1, y_1, z_1).

P lies on the sphere.

$$\therefore x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \text{(1)}$$

The tangent plane passes through the point (x_1, y_1, z_1) and is perpendicular to OP where O is the centre of the sphere.

$$\therefore x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$$

The co-ordinates of the point O are $(-u, -v, -w)$.

$\therefore$ The direction cosines of OP are proportional to

$$u + x_1 \,,\ v + y_1 \,,\ w + z_1 .$$

Hence the equation of the plane is

$$(x - x_1)(u + x_1) + (y - y_1)(v + y_1) + (z - z_1)(w + z_1) = 0$$

(i.e) $xx_1 + yy_1 + zz_1 + ux + vy + wz$

$$= x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1$$

But from ①,

$$x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + z_1 w$$

$$= -ux_1 - vy_1 - wz_1 - d.$$

$\therefore$ The equation of the tangent plane is

$$xx_1 + yy_1 + zz_1 + ux + vy + wz$$

$$= -ux_1 - vy_1 - wz_1 - d$$

(i.e)., $xx_1 + yy_1 + zz_1 + u(x + x_1) +$

$$v(y + y_1) + w(z + z_1) + d = 0$$

COR: The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$

at the point (x_1, y_1, z_1) is

$xx_1 + yy_1 + zz_1 = r^2$.

EXAMPLE 1:

Show that the plane $2x - y - 2z = 16$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y + 2z - 3 = 0$ and find the point of contact.

Solution:

Equation of the tangent plane is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

The sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

$$x^2 + y^2 + z^2 - 4x + 2y + 2z - 3 = 0$$

$\therefore$ The centre of the sphere is $(-u, -v, -w)$

(i.e) $(2, -1, -1)$

$\because$ $2u = -4$
$u = -2$
$2v = 2$
$v = 1$
$2w = 2$
$w = 1$

Radius, $r = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{4 + 1 + 1 + 3}$

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$r = \sqrt{9}$

$\therefore r = 3$ units

$\therefore$ The length of the perpendicular from $(2,-1,-1)$ to the plane

$$L = \frac{lx+my+nz}{\sqrt{l^2+m^2+n^2}}$$

$$L = \frac{2(2)-(-1)-2(-1)-16}{\sqrt{2^2+1^2+2^2}}$$

$$L = \frac{4+1+2-16}{\sqrt{9}}$$

$$L = \frac{-9}{\sqrt{9}}$$

$$L = \frac{-9}{3}$$

$\therefore L = -3$

$\therefore L = 3$

Hence the plane touches the sphere.

Let (x_1, y_1, z_1) be the point of contact.

$\therefore$ The equation of the tangent plane at (x_1, y_1, z_1) is

$-2x_1 + y_1 + z_1 - 2(-2+x_1) + 1(1+y_1) + (1+z_1) - 3 = 0$

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$xx_1 + yy_1 + zz_1 - 2(x+x_1) + (y+y_1) +$

$(z+z_1) - 3 = 0$

$(x_1-2)x + (y_1+1)y + (z_1+1)z - 2x_1 + y_1 + z_1 - 3 = 0$

$\therefore \frac{x_1-2}{2} = \frac{y_1+1}{-1} = \frac{z_1+1}{-2} = \frac{2x_1-y_1-z_1+3}{16}$

$\frac{x_1-2}{2} = \frac{y_1+1}{-1} = \frac{z_1+1}{-2}$; $\because x=2, y=1, z=1$

$\therefore$ The point of contact is

$(4, -2, -3)$

---

2. Find the equation of a sphere which touches the sphere $x^2+y^2+z^2-6x+2z+1=0$ at the point $(2,-2,1)$ and passes through the origin.

Sol: The equation of the sphere

$$x^2+y^2+z^2-6x+2z+1=0$$

The tangent plane to the above sphere is

$xx_1 + yy_1 + zz_1 - \frac{6}{2}(x+x_1) + \frac{2}{2}(z+z_1) + 1 = 0$

at $(2,-2,1)$.

$$2x - 2y + z - 3(x+2) + (z+1) + 1 = 0$$

$$2x - 2y + z - 3x - 6 + z + 1 + 1 = 0$$

$$-x - 2y + 2z - 4 = 0$$

$$x + 2y - 2z + 4 = 0$$

$\therefore$ The required equation of the sphere is of the form

$$x^2 + y^2 + z^2 - 6x + 2z + 1 + k(x + 2y - 2z + 4) = 0$$

at (0,0,0).

$$1 + 4k = 0$$

$$k = \frac{-1}{4}$$

$$x^2 + y^2 + z^2 - 6x + 2z + 1 - \frac{1}{4}(x + 2y - 2z + 4) = 0$$

$$4(x^2 + y^2 + z^2) - 24x + 8z + 4 - x - 2y + 2z - 4 = 0$$

$$4(x^2 + y^2 + z^2) - 25x - 2y + 10z = 0$$

3. Find the condition that the line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ where $l^2 + m^2 + n^2 = 1$ should touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. Show that there are two spheres through the points $(0,0,0)$, $(2a,0,0)$, $(0,2b,0)$ which touch the above

line and that the distance between their centres is

$$\frac{2}{n^2}\left[c^2-(a^2+b^2+c^2)n^2\right]^{1/2}$$

Solution:

The co-ordinates of any point on the line is

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=r \longrightarrow (1)$$

are of the form

$$(a+lr,\ b+rm,\ c+rn)$$

Hence the points of intersection of this line and the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (2)$$

are given by the equation

$$(a+rl)^2+(b+rm)^2+(c+rn)^2+$$

$$2u(a+rl)+2v(b+rm)+2w(c+rn)$$
$$+d=0$$

$$a^2+r^2l^2+2arl+b^2+r^2m^2+2brm+c^2$$
$$+r^2n^2+2crn+2ua+2url+2vb$$
$$+2vrm+2wc+2wrn+d=0$$

$$r^2(l^2+m^2+n^2)+r(2al+2bm+2cn+2ul$$
$$+2vm+2wn)+a^2+b^2+c^2+2ua+2vb+$$
$$2wc+d=0$$

$$r^2(l^2+m^2+n^2)+r[2l(a+u)+2m(b+v)$$
$$+2n(c+w)]+a^2+b^2+c^2+2ua+2vb$$
$$+2wc+d=0$$

$$r^2(l^2+m^2+n^2)+r(2l(a+u)+2m(b+v)+$$
$$2n(c+w))+a^2+b^2+c^2+2ua+2vb+2wc+d=0$$

$$r^2+2r[(a+u)+(b+v)+(c+w)]+a^2+b^2+c^2+$$
$$2ua+2vb+2wc+d=0$$

Hence line ① touches the sphere
②, if $[l(a+u)+m(b+v)+n(c+w)]^2$
$$=a^2+b^2+c^2+2au+2bv+2cw+d$$

Let the equation of the sphere passing through the points $(0,0,0)$, $(2a,0,0)$, $(0,2b,0)$ at $(0,0,0)$.

$$d=0$$

At $(2a,0,0)$,

$$4a^2+4a.u=0$$

$$\therefore a=-u$$

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i.e $u = -a$

At $(0, 2b, 0)$

$$4b^2 + 4bv = 0$$

$$4b(b+v) = 0$$

$$b+v = 0$$

$$v = -b$$

$\therefore$ The equation of the sphere becomes

$$x^2+y^2+z^2-2ax-2by+2wz=0$$

If this sphere touches the line ① then at (a, b, c)

$$n^2(c+w)^2 = a^2+b^2+c^2-2a^2-2b^2+2wc$$

$$n^2(c^2+w^2+2wc) = a^2+b^2+c^2-2a^2-2b^2+2wc$$

$$n^2c^2+n^2w^2+2n^2wc = -a^2-b^2+c^2+2wc$$

$$n^2c^2+n^2w^2+2n^2wc+a^2+b^2-c^2-2wc=0$$

$$n^2w^2+2wc(n^2-1)+a^2+b^2-c^2+n^2c^2=0 \longrightarrow ③$$

It is a quadratic equation in w and so there are two values of w satisfy the equation. Hence there are two sphere touching the line ①.

∴ The centres of the two spheres are $(a, b, -w_1)$ and $(a, b, -w_2)$ where w_1 and w_2 are the roots of the equation ③

∴ Sum of the roots

$$w_1 + w_2 = \frac{-2(n^2-1)c}{n^2}$$

Product of the roots $w_1 w_2$.

$$w_1 w_2 = \frac{a^2+b^2-c^2+n^2c^2}{n^2}$$

Hence the square on the distance between the centres $= (w_1 - w_2)^2$

$$(w_1 - w_2)^2 = (w_1 + w_2)^2 - 4w_1 w_2$$

$$= \left(\frac{-2(n^2-1)c}{n^2}\right)^2 - 4\,\frac{(a^2+b^2-c^2+n^2c^2)}{n^2}$$

$$= \frac{4(n^2-1)^2c^2}{n^4} - 4\,\frac{(a^2+b^2-c^2+n^2c^2)}{n^2}$$

$$= \frac{4(n^2-1)^2c^2 - 4n^2(a^2+b^2-c^2+n^2c^2)}{n^4}$$

$$= \frac{4}{n^4}\left[(n^2-1)^2c^2 - n^2(a^2+b^2-c^2+n^2c^2)\right]$$

$$= \frac{4}{n^4}\left[(n^4+1-2n^2)c^2 - a^2n^2 - b^2n^2 + c^2n^2 - n^4c^2\right]$$

$$= \frac{4}{n^4}\left[n^4c^2 + c^2 - 2n^2c^2 - a^2n^2 - b^2n^2 + c^2n^2 - n^4c^2\right]$$

$$= \frac{4}{n^4}\left[c^2 - a^2n^2 - b^2n^2 - c^2n^2\right]$$

$$= \frac{4}{n^4}\left[c^2 - n^2(a^2+b^2+c^2)\right]$$

$\therefore$ Distance between the centres $= \frac{2}{n^2}\left[c^2 - n^2(a^2+b^2+c^2)\right]^{1/2}$

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