UNIT-V

CONE AND CYLINDER

SECTION: 1

The equation of a surface

An equation of first degree in x, y and z represents a plane.

If $F(x,y,z)=0$ is a homogeneous equation of degree $n (n>0)$ in x, y, z.

The origin O lies on the sphere.

$$F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x,y,z) = 0$$

for all values of λ. It follows that if the point $p(x,y,z)$ lies on the surface the point $Q(x,y,z)$ $Q(\lambda x, \lambda y, \lambda z)$ any point on the line OP also lies on the surface. The surface is therefore a cone with vertex at the origin.

The general homogeneous quadric in the three variables x, y, z

$$F(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \rightarrow (1)$$

Scanned by TapScanner

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fzy + 2gzx + 2hxy = 0 \longrightarrow (1)$$

in which a, b, c, f, g, h are constants, represents a cone with vertex at the origin.

If the quantities l, m, n satisfy the relation.

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$$

Then the line with direction cosines proportional to l, m, n is the generator of the cone.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

But in the equation (1) if the expression on the left splits into two linear rational factors, then (1) represents pair of planes. Since an equation of the first degree in x, y, z represents a plane

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

Scanned by TapScanner

splits in two linear factors if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Hence ① represents a cone if $\Delta \neq 0$ but a pair of planes if $\Delta = 0$.

SECTION: 2

Cone

If O is a fixed point and a variable point P moves along a fixed curve C, the surface generated by the straight line OP is a cone. O is the vertex and C base curve and the line OP a generator of the surface. If O is the origin, the equation of the cone is homogeneous.

If the base curve is known, the equation of the cone with the origin as the vertex can be found.

Let the base curve be the intersections of the surfaces $f(x,y,z)=0$, $F(x,y,z)=0$.

The line joining the origin to any point $Q(x,y,z)$ on the cone must pass through a point P on the curve. If $OP:OQ = \lambda:1$ the co-ordinates of P are $(\lambda x, \lambda y, \lambda z)$.

Since P lies on the curve and hence on the surfaces, we have

$$f(\lambda x, \lambda y, \lambda z)=0,$$

$$F(\lambda x, \lambda y, \lambda z)=0$$

Since the above relations are true for all values of λ, by eliminating λ between these two equations, the equation of the cone is obtained.

SECTION : 2.1

Right Circular Cone

A right circular cone is a surface generated by a line which passes through a fixed point and makes a constant angle with the fixed line through the fixed point.

The fixed point is called the vertex, the fixed line the axis and the constant angle the semi vertical angle of the cone.

It is easily seen that the section of this cone by any plane perpendicular to its axis is a circle.

PROBLEMS:

EXAMPLE 1:

Show that the equation of a right circular cone whose vertex is O, axis OZ and semi-vertical angle α is $x^2+y^2 = z^2 \tan^2\alpha$.

Solution:

Let the generator of the

cone be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

$\therefore$ The z-axis is $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

$$\therefore \cos\alpha = \frac{n}{\sqrt{l^2+m^2+n^2}}$$

$$\cos^2\alpha\,(l^2+m^2+n^2) = n^2$$

$$(l^2+m^2)\cos^2\alpha + n^2\cos^2\alpha = n^2$$

$$\cos^2\alpha\,(l^2+m^2) = n^2 - n^2\cos^2\alpha$$

$$= n^2(1-\cos^2\alpha)$$

$$(l^2+m^2)\cos^2\alpha = n^2\sin^2\alpha$$

$$l^2+m^2 = \frac{n^2\sin^2\alpha}{\cos^2\alpha}$$

$$l^2+m^2 = n^2\tan^2\alpha$$

Hence the equation of the cone is $x^2+y^2 = z^2\tan^2\alpha$.

EXAMPLE 2:

Find the equation of the cone with vertex O and base curve the conic in which the surface $ax^2+by^2+cz^2=1$ is cut by the plane $l_1x+m_1y+n_1z=p$.

Solution:

Let the generator of the cone

$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

The co-ordinates of the point where the generator meets the base curve are of the form $(\lambda l, \lambda m, \lambda n)$ where λ is given by the equations.

$$\lambda^2(al^2+bm^2+cn^2)=1 \longrightarrow (1)$$

and $\lambda(ll_1+mm_1+nn_1)=p \longrightarrow (2)$

$(2) \Rightarrow \lambda = \frac{p}{ll_1+mm_1+nn_1}$

$(1) \Rightarrow al^2+bm^2+cn^2=\frac{(ll_1+mm_1+nn_1)^2}{p^2}$

EXAMPLE 3:

Find the condition for the equation $F(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$.

Solution:

Let the general equation of

the second degree be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \longrightarrow (1)$$

Let (x_1, y_1, z_1) be the vertex of the cone shift the origin to the point (x_1, y_1, z_1). Then

$$x = X + x_1$$

$$y = Y + y_1$$

$$z = Z + z_1$$

Then equation (1) becomes

$$a(X+x_1)^2 + b(Y+y_1)^2 + c(Z+z_1)^2 + 2f(Y+y_1)(Z+z_1) + 2g(Z+z_1)(X+x_1) + 2h(X+x_1)(Y+y_1) + 2u(X+x_1) + 2v(Y+y_1) + 2w(Z+z_1) + d = 0 \longrightarrow (2)$$

$$a(X^2+2Xx_1+x_1^2) + b(Y^2+2Yy_1+y_1^2) + c(Z^2+2Zz_1+z_1^2) + 2f(YZ+Yz_1+y_1Z+y_1z_1) + 2g(ZX+Zx_1+z_1X+z_1x_1) + 2h(XY+Xy_1+x_1Y+x_1y_1) + 2u(X+x_1) + 2v(Y+y_1) + 2w(Z+z_1) + d = 0$$

$aX^2 + 2aXx_1 + ax_1^2 + bY^2 + 2bYy_1$
$+ by_1^2 + cZ^2 + 2cZz_1 + cz_1^2 + 2fYZ +$
$2fYz_1 + 2fy_1Z + 2fy_1z_1 + 2gZX +$
$2gZx_1 + 2gz_1X + 2gz_1x_1 + 2hXY +$
$2hXy_1 + 2hx_1Y + 2hx_1y_1 + 2uX +$
$2ux_1 + 2vY + 2vy_1 + 2wZ + 2wz_1 + d = 0$

$aX^2 + bY^2 + cZ^2 + 2fYZ + 2gZX +$
$2hXY + 2ax_1X + 2gz_1X + 2hy_1X +$
$2uX + 2by_1Y + 2fz_1Y + 2hx_1Y + 2vY$
$+ 2cz_1Z + 2fy_1Z + 2gx_1Z + 2wZ$
$+ ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1$
$+ 2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$

$aX^2 + bY^2 + cZ^2 + 2fYZ + 2gZX +$
$2hXY + 2(ax_1 + gz_1 + hy_1 + u)X +$
$2(by_1 + fz_1 + hx_1 + v)Y + 2(cz_1 + fy_1$
$+ gx_1 + w)Z + (ax_1^2 + by_1^2 + cz_1^2 +$
$2fy_1z_1 + 2hy_1z_1 + 2gz_1x_1 + 2hx_1y_1$
$+ 2ux_1 + 2vy_1 + 2wz_1 + d = 0$

Scanned by TapScanner

Since this equation has to be a homogeneous equation in x, y and z.

Coefficient of $x = 0$

Coefficient of $y = 0$

Coefficient of $z = 0$

Constant term $= 0$

$$ax_1 + hy_1 + gz_1 + u = 0 \longrightarrow (4)$$

$$hx_1 + by_1 + fz_1 + v = 0 \longrightarrow (5)$$

$$gx_1 + fy_1 + cz_1 + w = 0 \longrightarrow (6)$$

$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 +$$

$$2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$$ax_1 + by_1^2 + cz_1^2 + fy_1z_1 + fy_1z_1 +$$

$$gz_1x_1 + gz_1x_1 + hx_1y_1 + hx_1y_1 + ux_1$$

$$+ ux_1 + vy_1 + vy_1 + wz_1 + wz_1 + d = 0$$

$$x_1(ax_1 + hy_1 + gz_1 + u) + y_1(hx_1 + by_1 + fz_1 + v)$$

$$+ z_1(gx_1 + fy_1 + cz_1 + w) + ux_1 + vy_1 + wz_1$$

$$+ d = 0$$

$$0 + 0 + 0 + ux_1 + vy_1 + wz_1 + d = 0$$

$$ux_1 + vy_1 + wz_1 + d = 0 \rightarrow (7)$$

Eliminating x_1, y_1, z_1 from (4), (5), (6), (7) we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

NOTE:

The co-ordinates (α, β, γ) of the vertex of the cone is given by the equations

$$\frac{\partial F}{\partial \alpha} = 0, \quad \frac{\partial F}{\partial \beta} = 0, \quad \frac{\partial F}{\partial \gamma} = 0 \text{ and}$$

$$\frac{\partial F}{\partial t} = 0$$

wher t is made to used to make $F(\alpha, \beta, \gamma)$ homogeneous and is equated to unity after differentiation.

EXAMPLE 4:

Find the condition for the equation $ax^2+by^2+c^2+2fyz+2gzx+2hxy+0$ to represent a right circular cone obtain the equation of the axes and the verticle angle of the cone.

Solution:

Let the equation of the axis be $\frac{x}{p}=\frac{y}{q}=\frac{z}{r} \longrightarrow (1)$

and the semi-vertical angle be α. Let one of the generators of the cone be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \longrightarrow (2)$

The angle between (1) & (2) is α for all values of l, m, n.

$$\cos\alpha = \frac{lp+mq+nr}{\sqrt{l^2+m^2+n^2}\sqrt{p^2+q^2+r^2}} \longrightarrow (3)$$

Squaring on both sides, we get $\cos^2\alpha(l^2+m^2+n^2)(p^2+q^2+r^2)$

$$=(lp+mq+nr)^2$$

Scanned by TapScanner

$$(l^2+m^2+n^2)(p^2+q^2+r^2)\cos^2\alpha-(lp+mq+nr)^2=0$$

$$(l^2+m^2+n^2)(p^2+q^2+r^2)\cos^2\alpha-(l^2p^2+m^2q^2+n^2r^2+2lmpq+2mnqr+2nlpr)=0$$

$$l^2(p^2+q^2+r^2)\cos^2\alpha+m^2(p^2+q^2+r^2)\cos^2\alpha+n^2(p^2+q^2+r^2)\cos^2\alpha-l^2p^2-m^2q^2-n^2r^2-2pqlm-2qrmn-2prln=0$$

$$l^2[(p^2+q^2+r^2)\cos^2\alpha-p^2]+m^2[(p^2+q^2+r^2)\cos^2\alpha-q^2]+n^2[(p^2+q^2+r^2)\cos^2\alpha-r^2]-2pqlm-2qrmn-2prln=0$$

Hence the equation of the cone in (x,y,z) co-ordinates is

$$x^2[(p^2+q^2+r^2)\cos^2\alpha-p^2]+y^2[(p^2+q^2+r^2)\cos^2\alpha-q^2]+n^2[(p^2+q^2+r^2)\cos^2\alpha-r^2]-2pqlm-2qrmn-2prln=0$$

Hence the equation of the cone in (x,y,z) co-ordinates is

$$x^2\left[(p^2+q^2+r^2)\cos^2\alpha - p^2\right] + y^2\left[(p^2+q^2+r^2)\cos^2\alpha - q^2\right] + z^2\left[(p^2+q^2+r^2)\cos^2\alpha - r^2\right] - 2pqxy - 2qryz - 2prxz = 0 \longrightarrow (4)$$

This is equivalent to

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \longrightarrow (5)$$

Comparing the co-efficients of $x^2, y^2, z^2, yz, zx, xy$ in (4) and (5), we get

$$\frac{(p^2+q^2+r^2)\cos^2\alpha - p^2}{a} = \frac{(p^2+q^2+r^2)\cos^2\alpha - q^2}{b}$$

$$= \frac{(p^2+q^2+r^2)\cos^2\alpha - r^2}{c} = \frac{-qr}{f} = \frac{-rp}{g} = \frac{-pq}{h}$$

Let each ratio be $\frac{-pqr}{k}$

Then $\frac{-qr}{f} = \frac{-pqr}{k} \Rightarrow p = \frac{k}{f}$

$$\frac{-rp}{g} = \frac{-pqr}{k} \Rightarrow q = \frac{k}{g}$$

$$\frac{-pq}{h} = \frac{-pqr}{k} \Rightarrow r = \frac{k}{h}$$

$$\frac{(p^2+q^2+r^2)\cos^2\alpha - p^2}{a} = \frac{-pqr}{k}$$

Substitute the values of p, q, r, we get

$$\left(\frac{k^2}{f^2}+\frac{k^2}{g^2}+\frac{k^2}{h^2}\right)\cos^2\alpha - \frac{k^2}{f^2} = \frac{-k}{f}\cdot\frac{k}{g}\cdot\frac{k}{h}\cdot\frac{1}{k}$$

$$\frac{k^2\left[\left(\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right)\cos^2\alpha - \frac{1}{f^2}\right]}{a} = \frac{-k^2}{fgh}$$

$$\left(\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right)\cos^2\alpha - \frac{1}{f^2} = \frac{-a}{fgh}$$

$$\left(\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right)\cos^2\alpha = \frac{1}{f^2} - \frac{a}{fgh}$$

$$\left(\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right)\cos^2\alpha = \frac{fgh - af^2}{f^2 fgh}$$

$$\left(\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right)\cos^2\alpha = \frac{1}{f^2}\cdot f\left[\frac{gh-af}{fgh}\right]$$

$$fgh\left[\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right]\cos^2\alpha = \frac{gh-af}{f} \longrightarrow (6)$$

Similarly, for the other two expressions, we get

$$fgh\left[\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right]\cos^2\alpha = \frac{hf-bg}{g} \rightarrow (7)$$

and $$fgh\left[\frac{1}{f^2}+\frac{1}{g^2}+\frac{1}{h^2}\right]\cos^2\alpha = \frac{fg-ch}{h} \rightarrow (8)$$

from (6), (7), (8), we get

$$\frac{gh-af}{f} = \frac{hf-bg}{g} = \frac{fg-ch}{h}$$

when these conditions are satisfied, the axis of the cone is the line $fx = gy = hz$.

EXERCISES 15

1. Find the equation of the cone of the second degree which passes through the axes.

Solution:

The cone passes through the axes the vertex of the cone is the origin.

Scanned by TapScanner

The general equation of the cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \longrightarrow (1)$$

We know that if x-axis is a generator then at $(a,0,0)$

$(1) \Rightarrow a=0$

at $(0,b,0)$

$(1) \Rightarrow b=0$

at $(0,0,c)$

$(1) \Rightarrow c=0.$

Hence the equation of the cone is given by

$$0x^2+0y^2+0z^2+2fyz+2gzx+2hxy=0$$

$$2[fyz+gzx+hxy]=0$$

$$fgz+gzx+hxy=0$$

SECTION: 8

Cylinder

The surface generated by a variable line which remains parallel to a fixed line and intersects a given curve (or touches a given surface) is called a cylinder.

The variable line is called the generator. The fixed straight line is called the axis of the cylinder and the given curve is called the guiding curve of the cylinder.

SECTION: 8.1

Equation of a circle whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and whose guiding curve is

$$f(x, y, z) = 0, \quad ax + by + cz + d = 0.$$

Let (x_1, y_1, z_1) be any point on the cylinder. Then the equation of the generator passing through (x_1, y_1, z_1) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \longrightarrow (1)$$

Any point on this generator is $(x_1 + kl, y_1 + km, z_1 + nk)$

If this point lies on the guiding curve, we get

$$f(x_1, kl\, y_1 + km z_1 + kn) = 0 \longrightarrow (2)$$

and $a(x_1 + kl) + b(y_1 + km) + c(z_1 + kn) + d = 0$

$$\longrightarrow (3)$$

Eliminate k from (2) and (3) and suppose we get the equation $\phi(x_1, y_1, z_1) = 0$. Hence the locus of (x_1, y_1, z_1) is $\phi(x, y, z) = 0$.

This is the equation of the required cylinder.

PROBLEMS.

EXAMPLE

Find the equation of the cylinder whose generations are parallel to the z-axis and the guiding curve is $ax^2+by^2=cz$, $lx+my+nz=p$.

Solution:

The equation of the z-axis is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$

Let (x_1, y_1, z_1) be any point on the cylinder. The equation of the generator passing through (x_1, y_1, z_1) is

$$\frac{x-x_1}{0}=\frac{y-y_1}{0}=\frac{z-z_1}{1}$$

Hence any point on this line is of the form (x_1, y_1, z_1+k).

If this point lies on the guiding ~~centre~~ curve

$$ax_1^2+by_1^2-c(z_1+k)=0 \longrightarrow (1)$$

$$lx_1+my_1+n(z_1+k)=p \longrightarrow (2)$$

$$(2) \Rightarrow lx_1+my_1+nz_1+nk=p$$

$$k=\frac{p-lx_1-my_1-nz_1}{n}.$$

Substitute the value of k in (1), we get

$$ax_1^2 + by_1^2 - cz_1 = ck$$

$$ax_1^2 + by_1^2 - cz_1 = \frac{c}{n}(p - lx_1 - my_1 - nz_1)$$

i.e $n(ax_1^2 + by_1^2) + c(lx_1 + my_1 - p) = 0$

Hence the equation of the required cylinder is $n(ax^2 + by^2) + c(lx + my - p) = 0$

SECTION 8.2 :

The equation of the right circular cylinder with axis $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and radius of the guiding circle λ.

Let $p(x_1, y_1, z_1)$ be any point on the cylinder. The axis is the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Hence it passes through the point $Q(\alpha, \beta, \gamma)$ and the direction cosines of the axis are

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$$

Let the axis meet the normal section of the cylinder through p at M.

Then $PQ^2 = QM^2 + MP^2$

QM is the projection of the line QP on the axis

$$QM = \frac{(x_1-\alpha)l}{\sqrt{l^2+m^2+n^2}} + \frac{(y_1-\beta)m}{\sqrt{l^2+m^2+n^2}} + \frac{(z_1-\gamma)n}{\sqrt{l^2+m^2+n^2}}$$

$$PQ = \sqrt{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2}$$

$MP = \lambda$

Hence $$\left\{\frac{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n}{\sqrt{l^2+m^2+n^2}}\right\}^2 + \lambda^2$$

$$= (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-n)^2$$

Hence the equation of the cylinder

(i.e) the locus of (x_1, y_1, z_1) is

Scanned by TapScanner

$$\lambda^2 + \frac{[(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2}{l^2+m^2+n^2}$$

$$= (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$

EXAMPLE : 1

Find the equation of a right circular cylinder of radius 3 with axis $\frac{x+2}{3} = \frac{y-4}{6} = \frac{z-1}{2}$

Solution :

Let the equation of the line be

$$\frac{x+2}{3} = \frac{y-4}{6} = \frac{z-1}{2}$$

∴ The axis passes through the point $Q(-2, 4, 1)$.

Its direction cosines are

$\frac{3}{7}, \frac{6}{7}, \frac{2}{7}$.

Let $p(x_1, y_1, z_1)$ be any point on the cylinder and let the axis of the cylinder meet the plane perpendicular to the axis through the point $P(x_1, y_1, z_1)$ be M.

Scanned by TapScanner

Then $MP = 3$ (Radius of the cylinder)

$$PQ^2 = (x_1+2)^2 + (y_1-4)^2 + (z_1-1)^2$$

$$QM = \frac{(x_1+2)3 + (y_1-4)6 + (z_1-2)1}{7}$$

$$PQ^2 = QM^2 + MP^2$$

$$(x_1+2)^2 + (y_1-4)^2 + (z_1-1)^2$$

$$= 9 + \frac{[3(x_1+2) + 6(y_1-4) + (z_1-2)]^2}{49}$$

Hence the locus of the cylinder is

$$(x+2)^2 + (y-4)^2 + (z-1)^2$$

$$= \frac{[3(x+2) + 6(y-4) + (z-2)]^2}{49} + 9$$

$$x^2 + 4 + 4x + y^2 - 8y + 16 + z^2 - 2z + 1$$

$$= 3^2(x+2)^2 + 6^2(y-4)^2 + (z-2)^2 +$$

$$2(3)(x+2) \times 6(y-4) +$$

$$\frac{2(6)(y-4)(z-2) + 2(z-2)3(x+2)}{49} + 9$$

$$49(x^2+y^2+z^2-4x-8y-2z+21)$$
$$= 9(x+2)^2 + 36(y-4)^2 + (z-2)^2 +$$
$$36(x+2)(y-4) + 12(y-4)(z-2) +$$
$$6(z-2)(x+2) + 9(49).$$

Simplifying, we get

$$40x^2 + 13y^2 + 48z^2 + 12yz + 6zx +$$
$$36xy + 316x - 152y - 60z + 192 = 0$$

EXAMPLE 2:

Find the equation of the right circular cone described on the circle through the points $(a,0,0)$, $(0,a,0)$, $(0,0,a)$ as a guiding curve.

Solution:

Let the guiding curve is the intersection of the plane passing through $A(a,0,0)$, $B(0,a,0)$, $C(0,0,a)$ and the sphere.

Through the points A, B, C and the origin O.

Scanned by TapScanner

The equation of the sphere OABC is

$$x^2+y^2+z^2-ax-ay-az=0$$

Its centre is the point $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$ and its radius is $\frac{a\sqrt{3}}{2}$.

The equation of the plane ABC is $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$

i.e., $x+y+z=a$

The axis of the cylinder is perpendicular to this plane and passes through the centre of the circle ABC.

Since the triangle ABC is equilateral, its centroid $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$ is the centre of the circle

Its radius can be found as $a\left(\frac{2}{3}\right)^{1/2}$. Hence the axis of the cylinder is

$$\frac{x-\frac{a}{3}}{1}=\frac{y-\frac{a}{3}}{1}=\frac{z-\frac{a}{3}}{1}$$

Hence the direction cosines of this line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Let $P(x_1, y_1, z_1)$ be any point on the cylinder. The axis passes through the point $Q\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$. If the axis meets the plane perpendicular to the axis at M, we get

$$MP = a\left(\frac{2}{3}\right)^{1/2}$$

$$PQ^2 = \left(x_1 - \frac{a}{3}\right)^2 + \left(y_1 - \frac{a}{3}\right)^2 + \left(z_1 - \frac{a}{3}\right)^2$$

QM = Projection of PQ on the axis of the cylinder

$$= \left(x_1 - \frac{a}{3}\right)\frac{1}{\sqrt{3}} + \left(y_1 - \frac{a}{3}\right)\frac{1}{\sqrt{3}} + \left(z_1 - \frac{a}{3}\right)\frac{1}{\sqrt{3}}$$

$$\therefore \left(x_1 - \frac{a}{3}\right)^2 + \left(y_1 - \frac{a}{3}\right)^2 + \left(z_1 - \frac{a}{3}\right)^2$$
$$= \left[\frac{x_1 - \frac{a}{3} + y_1 - \frac{a}{3} + z_1 - \frac{a}{3}}{\sqrt{3}}\right]^2$$

Simplifying we get

$$x_1^2 + y_1^2 + z_1^2 - (x_1y_1 + y_1z_1 + z_1x_1) = a^2$$

Hence the equation of the

cylinder is

$$x^2+y^2+z^2-(xy+yz+zx)=a^2$$

SECTION : 8.3

Enveloping cylinder

The locus of a line which is parallel to a given line which touches a given surface is called an enveloping cylinder.

Equation of the enveloping cylinder of the surface $ax^2+by^2+cz^2=1$ having the generator parallel to

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Let (x_1,y_1,z_1) be any point on the cylinder. Then the equation of the generator passing through the point (x_1,y_1,z_1) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

This line should touch

Scanned by TapScanner

the surface $ax^2+by^2+cz^2=1$

(i-e) the line should not meet the surface in two coincident points

Any point on this line is of the form

(x_1+rl, y_1+m, z_1+rn)

If it lies on the surface

$a(x_1+rl)^2+b(y_1+rm)^2+c(z_1+rn)^2=1$

(i-e) $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+ax_1^2+by_1^2+cz_1^2-1=0$

This equation must give two equal roots $(alx_1+bmy_1+cnz_1)^2$

$= (al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1)$

EXERCISE 17

2. Find the equation to the cylinder whose generators are parallel to the line

$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and guiding curve $x^2+y^2=1$, $z=3$.

Solution:

The equation of the guiding curve is

$$x^2+2y^2=1,\ z=3$$

Let (x_1, y_1, z_1) be a point on the cylinder. Then the equation of the generator through $P(x_1, y_1, z_1)$ are

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$$

When this line meets the plane $z=3$, we have

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{3-z_1}{3}$$

$$x = x_1 + \frac{3-z_1}{3} = \frac{3x_1-z_1+3}{3}$$

$$y = y_1 - \frac{2(3-z_1)}{3} = \frac{3y_1+2z_1-6}{3}$$

This point lies on the curve $x^2+2y^2=1$.

$$\therefore \left(\frac{3x_1 - z_1 + 3}{3}\right)^2 + 2\left(\frac{3y_1 + 2z_1 - 6}{3}\right)^2 = 1$$

$$(3x_1 - z_1 + 3)^2 + 2(3y_1 + 2z_1 - 6)^2 = 9$$

$$9x_1^2 + z_1^2 + 9 - 6x_1z_1 - 6z_1 + 18x_1$$
$$+ 2(9y_1^2 + 4z_1^2 + 36 + 12y_1z_1 - 24z_1 - 36y_1) = 9$$

$$9x_1^2 + z_1^2 + 9 - 6x_1z_1 - 6z_1 + 18x_1 + 18y_1^2$$
$$+ 8z_1^2 + 72 + 24y_1z_1 - 48z_1 - 72y_1 - 9 = 0$$

$$9x_1^2 + 18y_1^2 + 9z_1^2 + 24y_1z_1 - 6z_1x_1$$
$$+ 18x_1 - 72y_1 - 54z_1 + 72 = 0$$

$\therefore$ The locus of (x_1, y_1, z_1) is

$$9x^2 + 18y^2 + 9z^2 + 24yz - 6zx + 18x$$
$$- 72y - 54z + 72 = 0$$

(i.e)

$$3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x$$
$$- 24y - 18z + 24 = 0$$

Scanned by TapScanner