~~Unit~~

Unit - II THEORY OF INFERENCE AND PREDICATE CALCULUS

Introduction:

Inference theory is concerned with the infering of a conclusion from certain hypothesis or basic assemptions called Premisses. When a conclusion derived from a set of premises by using rules of inference the process of search derivation is called a formal Proof:

The rules of inference are only meance used to draw a conclusion from a set of Premises in a finite sequence of steps called arguments.

Any Conclusion which is arrived at, by following these rules is called a valid conclusion. And the argument is called a Valid argument

Validity using Truth Table

Let A and B be two statement formula's we say that "B logically follows from A" or "B is a valid conclusion of the premises A". Iff $A \to B$ is a tautalogy.

From a set of Premises $\{H_1; H_2; H_3 \ldots H_m\}$ a Conclusion C follow logically iff $H_1 \wedge H_2 \wedge \dots \wedge H_m \Rightarrow C$

Problems.

Determined whether a conclusion follows logically from a premises H_1 and H_2.

a) $H_1 : P \rightarrow Q$, $H_2 : P$, $C : Q$

H_2 P	C Q	H_1 $P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

In first row H_1 and H_2 is True and also c is True.

$\therefore$ The conclusion is valid.

b) $H_1 : P \rightarrow Q$, $H_2 : \neg P$, $C : Q$

Soln:

P	C Q	H_1 $P \rightarrow Q$	H_2 $\neg P$
T	T	T	F
T	F	F	F
F	T	T	T
F	F	T	T

H_1 and H_2 are True in the third row but is not true in the forth row

The conclusion is Invalid

c) $H_1 : P \rightarrow Q$, $H_2 : Q$, $C : P$

Soln:

C	H_2	H_1
P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

H_1 and H_2 are true and the first row but is not true in the third row.

$\therefore$ The conclusion is invalid.

RULES OF INFERENCE

1. Rule P:

A Premises may be introduced at any point in the derivation.

2. Rule T:

A formula S may be introduce in a derivation if 'S' is tautalogically implied by any one or more of preseeding formula's in the derivation.

IMPLICATIONS:

$$\left.\begin{array}{l} I_1 : P \land Q \Rightarrow P \\ I_2 : P \land Q \Rightarrow Q \end{array}\right\} \text{(Simplification)}$$

$$\left.\begin{array}{l} I_3 : P \Rightarrow P \lor Q \\ I_4 : Q \Rightarrow P \lor Q \end{array}\right\} \text{(Addition)}$$

I_5 : $\neg P \Rightarrow P \to Q$

I_6 : $Q \Rightarrow P \to Q$

$I_7 : \neg(P \rightarrow Q) \Rightarrow P$

$I_8 : \neg(P \rightarrow Q) \Rightarrow \neg Q$

$I_9 : P, Q \Rightarrow P \wedge Q$

$I_{10} : \neg P, P \vee Q \Rightarrow Q$ (disjunctive Syllogism)

$I_{11} : P, P \rightarrow Q \Rightarrow Q$ (Modus Ponens)

$I_{12} : \neg Q, P \rightarrow Q \Rightarrow \neg P$ (Modus tollens)

$I_{13} : P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (Hypothetical Syllogism)

$I_{14} : P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (Dilemma)

EQUIVALENCES:

$E_1 : \neg\neg P \Leftrightarrow P$ (Double negation)

$E_2 : P \wedge Q \Leftrightarrow Q \wedge P$
$E_3 : P \vee Q \Leftrightarrow Q \vee P$ } (Commutative laws)

$E_4 : (P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
$E_5 : (P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ } (Associative laws)

$E_6 : P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
$E_7 : P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ } (Distributive laws)

$E_8 : \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
$E_9 : \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ } (Demorgan's Law)

$E_{10} : P \vee P \Leftrightarrow P$

$E_{11} : P \wedge P \Leftrightarrow P$

$E_{12} : R \vee (P \wedge \neg P) \Leftrightarrow R$

$E_{13}: R \wedge (P \vee \neg P) \Leftrightarrow R$

$E_{14}: R \vee (P \vee \neg P) \Leftrightarrow T$

$E_{15}: R \wedge (P \wedge \neg P) \Leftrightarrow F$

$E_{16}: P \to Q \Leftrightarrow \neg P \vee Q$

$E_{17}: \neg(P \to Q) \Leftrightarrow P \wedge \neg Q$

$E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$

$E_{19}: P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$

$E_{20}: \neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$

$E_{21}: P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

$E_{22}: (P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

PROBLEMS:

1. Demonstrate that R is a valid inference from premises $P \to Q$, $Q \to R$ and P

Soln:

Step no	Statement	Reason
$\{1\}$	(1) $P \to Q$	Rule P
$\{2\}$	(2) P	Rule P
$\{1,2\}$	(3) Q	Rule T (1), (2) and I_{11} (Modus Ponens)
$\{4\}$	(4) $Q \to R$	Rule P
$\{1,2,4\}$	(5) R	Rule T, (3), (4) and I_{11}

2. Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$

Soln:

Step no	Statement	Reason
{1}	(1) $(C \vee D) \to \neg H$	P
{2}	(2) $\neg H \to (A \wedge \neg B)$	P
{3}	(3) $(C \vee D) \to (A \wedge \neg B)$	T, (1), (2) and I_{13}
{4}	(4) $(A \wedge \neg B) \to (R \vee S)$	P
{1,2,4}	(5) $(C \vee D) \to (R \vee S)$	T, (3), (4) and I_{13}
{6}	(6) $C \vee D$	P
{1,2,4,6}	(7) $R \vee S$	T, (5), (6) and I_{11}

3) Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

Soln:

Step no	Statement	Reason
{1}	(1) $P \vee Q$	P
{1}	(2) $\neg P \to Q$	T, (1), E_1 and E_{16}
{3}	(3) $Q \to S$	P
{1,3}	(4) $\neg P \to S$	T, (2), (3) and I_{13}
{1, 3}	(5) $\neg S \to P$	T, (4), E_{18} and E_1

$\{6\}$	(6) $P \rightarrow R$	P
$\{1,3,6\}$	(7) $\neg S \rightarrow R$	T, (5) (6) I_{13}
$\{1,3,6\}$	(8) $S \vee R$	T, (7), E_{16} and E_1

4. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ and $\neg M$.

Soln:

Step no.	Statement	Reason
$\{1\}$	(1) $P \rightarrow M$	P
$\{2\}$	(2) $\neg M$	Rule P
$\{1,2\}$	(3) $\neg P$	Rule T, (1), (2) and I_{12}
$\{4\}$	(4) $P \vee Q$	Rule P
$\{1,2,4\}$	(5) Q	I_1, (3), (4) and I_{10}
$\{6\}$	(6) $Q \rightarrow R$	Rule P
$\{1,2,4,6\}$	(7) R	Rule T, (5) (6) and I_{11}
$\{1,2,4,6\}$	(8) $R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9

5. Show that $I_{12}: \neg Q, P \rightarrow Q \Rightarrow \neg P$

Solution:

Step no:	Statement	Reason
{1}	(1) $P \to Q$	Rule P
{1}	(2) $\neg Q \to \neg P$	T, (1) and E_{18}
{3}	(3) $\neg Q$	Rule P
{1, 3}	(4) $\neg P$	T, (2), (3) and I_{11}

6. RULES CP

It is also called the deduction theorem and it is generally used if the conclusion is of the form $R \to S$. In such cases R is taken as an <u>additional Premises</u> and S is derived from the given Premises and R. The rules CP follows from the equalance.

$$(P \wedge R) \to S \Leftrightarrow P \to (R \to S)$$

6. show that $(R \to S)$ can be derived from the Premises $P \to (Q \to S)$, $\neg R \vee P$ and Q

soln: single ($\to$) use CP rule)

Step no	Statement	Reason
{1}	(1) R	Rule P [additional Premises]
{2}	(2) $\neg R \vee P$	Rule P
{2}	(3) $R \to P$	Rule T, (2), E_{16}
{1, 2}	(4) P	Rule T, (1), (3) $P, P \to Q \Rightarrow Q$

{5}	(5) $P \to (Q \to S)$	Rule P
{6}	(6) $Q \to S$	Rule T, (4), (5) $P, P \to Q \Leftrightarrow Q$
{7}	(7) Q	Rule P
{8}	(8) S	Rule T, (6), (7)
{9}	(9) $R \to S$	Rule T, (1), (8) CP Rule

18. Derive $P \to (Q \to S)$ using the CP rule (if necessary) from the premises $P \to (Q \to R)$ and $Q \to (R \to S)$

Soln:

Step no	statement	Reason
{1}	(1) P	P (additional)
{2}	(2) $P \to (Q \to R)$	Rule P
{1, 2}	(3) $Q \to R$	Rule T, (1), (2)
{4}	(4) $Q \to (R \to S)$	Rule P
{4}	(5) $(Q \wedge R) \to S$	Rule T, (4), E_{19} $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$
{2}	(6) $(P \wedge Q) \to R$	Rule T, (2)
{4}	(7) $R \to S$	Rule T, (5), I_1, $Q \wedge R \Rightarrow R$
{5}	(8) $Q \to S$	Rule T, (3), (7) I_{13}
{6}	(9) $P \to (Q \to S)$	Rule T, (1), (8)

8. "If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game". Show that these statements constitute a valid argument.

Soln:

Let, P : There was a ball game

Q : Travelling was difficult

R : They arrived on time.

∴ The Premises are,

$P \to Q$, $R \to \neg Q$, R

The Conclusion is $\neg P$

Step no	statement	Reason
{1}	(1) $P \to Q$	Rule P
{1}	(2) $\neg P \vee Q$	Rule T, (1), E_{16}
{2}	(3) $R \to \neg Q$	Rule P
{3}	(4) R	Rule P
{2,3}	(5) $\neg Q$	Rule T, (3), (4) I_{11} [Modus Ponens]
{6}	(6) $\neg P$	Rule T, (2), (5) [disjunctive Syllogism]

CONSISTANCY OF PREMISES:

A set of formula's $H_1, H_2, \ldots H_m$ is said to be inconsistent if their conjuction implies a contradiction.

$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R$ for some formula R.

A set of formula's $H_1, H_2, \ldots H_n$ is said to be Consistant if it is not inconsistant.

INDIRECT METHOD OF PROOF:

Step 1: Introduce the negation of the decide conclusion as a new Premise

Step 2: Using this new Premise together with a given Premise derived a Contradiction.

Step 3: Assert the decide Conclusion as a Logical inference from the Premises.

DEFINITION:

The analysis of the validity of a formula from a given set of premises by using derivation is called theory of Inference.

PROBLEMS: Use Indirect Method

1. Show that $\neg(P \wedge Q)$ follow from $\neg P \wedge \neg Q$.

Soln:

Step no	Statement	Reason
{1}	(1) $\neg\neg(P \wedge Q)$	P [assumed]
{1}	(2) $P \wedge Q$	T, (1) and E_1
{1}	(3) P	T, (2) and I_1
{4}	(4) $\neg P \wedge \neg Q$	Rule P
{4}	(5) $\neg P$	Rule T, (4), I_1

{1,4}	(6) $P \wedge \neg P$	Rule T, (3), (5), I_9
{7}	(7) F	Rule T, (6); $P \wedge \neg P \Leftrightarrow F$

2. Show that the following Premises are inconsistant - Indirect method (Contradiction)

i) If Jack misses many classes through illness then he fails high school.

ii) If Jack fails high school, then he is uneducated

iii) If Jack reads a lot of books, then he is not uneducated.

iv) Jack misses many classes through illness and reads a lot of books

Soln:

E : Jack misses many classes

S : Jack fails high school.

A : Jack reads a lot of books

H : Jack is uneducated

The Premises are $E \to S$, $S \to H$, $A \to \neg H$ and $E \wedge A$

Step no	Statement	Reason
{1}	(1) $E \to S$	Rule P
{2}	(2) $S \to H$	Rule P
{1,2}	(3) $E \to H$	Rule T, (1), (2) and I_{13}

{4}	(4) $A \to \neg H$	Rule P
{4}	(5) $H \to \neg A$	Rule T, (4), E_{18}
{1,2,4}	(6) $E \to \neg A$	Rule T, (3), (5), I_{13}
{1,2,4}	(7) $\neg E \vee \neg A$	Rule T, (6), E_{16}
{1,2,4}	(8) $\neg(E \wedge A)$	Rule T, (7), E_8
{9}	(9) $E \wedge A$	Rule P
{1,2,4,9}	(10) $(E \wedge A) \wedge (E \wedge A)$	Rule T, (8), (9), I_9
{1,2,4,9}	(11) F	Rule T, (10) $P \wedge \neg P \Leftrightarrow F$

3) Use the indirect method.

$r \to \neg q,\ r \vee s,\ s \to \neg q,\ p \to q \Rightarrow \neg p$

Soln:

Step no		Statement	Reason
{1}	(1)	p	P (additional Premise)
{2}	(2)	$p \to q$	Rule P
{1,2}	(3)	q	Rule T, (1), (2) and Modus Ponens
{4}	(4)	$r \to \neg q$	Rule P
{5}	(5)	$s \to \neg q$	Rule P
{4,5}	(6)	$r \vee s \to \neg q$	Rule T, (4), (5) and equalence
{7}	(7)	$r \vee s$	Rule P

{1,5,7}	(8)	$\neg q$	Rule T, (6), (7) and Modus Ponens
{1,2,4,5,7}	(9)	$q \wedge \neg q$	Rule T, (9) and ~~Ne~~ conjunction
{1,2,4,5,7}	(10)	F	Rule T, (9) and Negation law

4. Show that 'b' can be derived from the Premises $a \to b$, $c \to b$, $d \to (a \vee c)$, d by the indirect method.

Soln:

Step	no	Statement	Reason
{1}	(1)	$a \to b$	Rule P
{2}	(2)	$c \to b$	Rule P
{1,2}	(3)	$(a \vee c) \to b$	T, (1), (2) and Equalence
{4}	(4)	$d \to (a \vee c)$	Rule P
{5}	(5)	$d \to b$	T, (3), (4) and Hypothetical Syllogism
{6}	(6)	d	Rule P
{5,6}	(7)	b	T, (5), (6) and Modus Ponens
{8}	(8)	$\neg b$	P (additional)
{9}	(9)	$b \wedge \neg b$	T, (7), (8) and Conjunction
{10}	(10)	F	T, (9) and Negation Laws

PREDICATE CALCULUS.
(or)
PREDICATE LOGIC

Introduction:

In Mathematics and computer programs, we encounter statements involving variables such as "$x>10$", "$x=y+5$" and "$x+y=z$". These statement are neither true nor false, when the values of the variable are not specified.

The statement "x is greater than 10" has two parts the first part the variable "x", is the subject of the statement. The Second "is greater that 10". which refers to a property that the subject can have is called the predicate.

We can denote the statement "x is greater that 10" by the notation $P(x)$, where B denotes the predicate "is greater that 10" and this "x is variable $P(x)$ is called propositional function at "x".

Consider the statement.

John is a bachelor.

Here, "is a bachelor" is the predicate and "John" is the subject. This can be denoted by $B(J)$, where 'B' is the predicate and J is the subject.

Let 'R' denote "this painting". Then then statement "This painting is red can be denoted R(P)

Using connectives, we can form compound statement such as "John is a bachelor and this painting is red". This can be written as $B(J) \wedge R(P)$

Consider the statement "Jack is taller than Jill" the predicate "is taller than" is 2 place predicate, because names of two objects are needed to complete a statement involving there predicates.

If 'G' symbolize "is taller than", J_1 denotes Jack and J_2 denotes "Jill". then the statement can be written as $G(J_1, J_2)$ Note that the order is which the names appear is the statement as well as is the predicate is important.

QUANTIFIERS:

Certain Statements involve words such as 'All', 'Some', 'none' or 'one'. Such words indicate quantity and they are called quantifiers.

The quantifier 'All'. is the universal qualifier and is denoted by (x) or $(\forall x)$

The quantifier 'Some' is the existential quantifier denoted by $(\exists x)$

Ex:

1. All men are rich

$M(x)$: x is a men

$R(x)$: x is rich

$(\forall x)(M(x) \rightarrow R(x))$.

2. Some men are clever

Ans:

$M(x)$: x is a Men

$C(x)$: x is clever

$(\exists x)(M(x) \wedge C(x))$

$(\forall x : \rightarrow)$

$(\exists x : \wedge)$

3. All Men are mortal

Soln:

$M(x)$: x is a Men

$H(x)$: H is a Mortal.

$\therefore (\forall x)(M(x) \rightarrow H(x))$

4. Every apple is red.

Soln:

$A(x)$: x is a apple

$R(x)$: x is red.

$\therefore (\forall x)(A(x) \rightarrow R(x))$.

5. Any integer is either positive or negative.

Soln:

$I(x)$: x is an integer.

$P(x)$: x is positive

$N(x)$: x is negative.

$\therefore (\forall x)(I(x) \rightarrow (P(x) \vee N(x)))$

6. Some real numbers are rational.

Soln:

$R_1(x)$: x is a real number.

$R_2(x)$: x is a rational number.

$\therefore (\exists x)(R_1(x) \wedge R_2(x))$

FREE AND BOUND VARIABLES:

The scope of a quantifiere in a formula immediatly following the quantifier. If the scope is an atomic formula, no paranthesis is used otherwise Paranthesis are needed.

An occurence of a variable in a formula is said to be a bound occurance If this occurence is within the scope of a quantifier using this variable.

An occurence of a variable is called Free occurence if this occurence of the variable is not a bound occurence.

A variable is called a bound or free variable according as in a formula, if alleast one occurence of that variable is bound or free occurence.

(or)

Given a formula containing a part of from $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurence of x is an x-bound part of a formula is called a bound occurence of x. Any occurence of x or any variable that is not a bound occurence is called a free occurence.

Example:

1. $(x)P(x,y)$: Scope is $P(x,y)$
   x is bound occurence
   y is free occurence

2. $(x)(P(x) \rightarrow Q(x))$: Scope is $P(x) \rightarrow Q(x)$
   x is bound occurence.

3. $(\exists x)(P(x) \wedge Q(x))$: Scope is $P(x) \wedge Q(x)$
   x is bound occurence.

4. $P(x) \wedge Q(x)$

   Scope is $P(x)$
   x in $P(x)$ is bound occurence.
   x in $Q(x)$ is free occurence.

5. $(\forall x)(A(x) \rightarrow (\exists y)B(x,y))$
   Scope of $\forall x$ is $A(x) \rightarrow (\exists y(B(x,y))$
   Scope of $\exists y$ is $B(x,y)$.

   All occurrence of x, y are bound occurence.

PROBLEMS:

1. Symbalozied the expression "All the world loves a lover".

2m & 5m

Soln:

Let $P(x)$: x is a person.

$L(y)$: y is a Lover.

$R(x,y)$: x loves y.

For a given statement, we can written as for all x, if x is a person then for all y, if y is a person and y is a Lover then x loves y,

$\therefore (x)\,(P(x) \rightarrow (y)\,(P(y) \wedge L(y) \rightarrow R(x,y)))$

* 2m.

2. Let $P(x)$: x is a person

$F(x,y)$: x is the father of y.

$M(x,y)$: x is the mother of y.

Write the predicate "x is the father of the mother of y".

Soln:

Let z be the mother of y. Now we need, x is the father of z and z is mother of y. It is assumed that

$(\exists z)\,(P(z) \wedge F(x,z) \wedge M(z,y))$

3. For any given positive integer, there is a greater positive integer.

Soln:

$P(x)$: x is a positive integer. there is

$G(x,y)$: x is greater then y.

$\therefore (\forall x)(P(x) \rightarrow (\exists y)\ (P(y) \wedge G(x,y)))$

INFERENCE THEORY OF THE PREDICATE CALCULUS.

RULES:

1) Universal Specification (US)

$$(\forall x)\, A(x) \Rightarrow A(y).$$

2. Universal Generalization (UG)

$$A(x) \Rightarrow (\forall y)\ (\text{or})\ A(y) \Rightarrow (\forall x)\, A(x))$$

3. Existential Specification (ES)

$$(\exists x)\, P(x) \Rightarrow P(y)$$

4. Existential Generalization (EG)

$$A(y) \Rightarrow (\exists x)\, A(x)$$

FORMULAS:

E_{23}: $(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)(A(x)) \vee (\exists x) B(x).$

E_{24}: $(x)(A(x) \wedge B(x)) \Leftrightarrow (x) A(x) \wedge (x) B(x)$

E_{25}: $\neg (\exists x) A(x) \Leftrightarrow (x) \neg A(x)$

E_{26}: $\neg (x) A(x) \Leftrightarrow (\exists x) \neg A(x).$

I_{15}: $(x) A(x) \vee (x) B(x) \Rightarrow (x) (A(x) \vee B(x))$

I_{16}: $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x) A(x) \wedge (\exists x) B(x)$

E_{27}: $(x)(A \vee B(x)) \Leftrightarrow A \vee (x) B(x).$

E_{28}: $(\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x) B(x).$

E_{29}: $(\exists x)(A(x) \rightarrow B) \Leftrightarrow (\exists x)(A(x) \rightarrow B)$

E_{31}: $A \rightarrow (x) B(x) \Leftrightarrow (x)(A \rightarrow B(x))$

E_{30}: $(\exists x)\, A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

E_{32}: $A \rightarrow (\exists x) B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$

PROBLEMS:

Show that $(x)(H(x) \to M(x)) \wedge H(S) \Rightarrow M(S)$

Note that this problem is a symbolic translation of a well-known argument known as the "Socrate argument" which is given by.

All men are Mortal.

Socrate is a men.

Therefore Socrates is a mortal. If we denote $H(x)$: x is a men, $M(x)$: x is a Mortal and s : Socrates, we can put the argument in the above form.

Solution:

Step no	statement	Reason
$\{1\}$	(1) $x(H(x) \to M(x))$	Rule P
$\{1\}$	(2) $H(S) \to M(S)$	US, (1)
$\{3\}$	(3) $H(S)$	Rule P
$\{1,3\}$	(4) $M(S)$	T, (2), (3), I_{11}

1.2 show that

$(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$

Soln:

Step no	statement	Reason
$\{1\}$	(1) $(x)(P(x) \to Q(x))$	Rule p
$\{1\}$	(2) $P(y) \to Q(y)$	US, (1)
$\{3\}$	(3) $(x)(Q(x) \to R(x))$	Rule P
$\{3\}$	(4) $Q(y) \to R(y)$	US, (3)

$\{1,3\}$	(5) $P(y) \rightarrow R(y)$	Rule T, (2), (4), I_{13}
$\{1,3\}$	(6) $(x)(P(x) \rightarrow R(x))$	UG (5)

3. Show that $(\exists x) M(x)$ follows logically from the premises $(x)(H(x) \rightarrow M(x))$ and $(\exists x) H(x)$.

soln:

Step no	Statement	Reason
$\{1\}$	(1) $(\exists x) H(x)$	Rule P
$\{1\}$	(2) $H(y)$	ES, (1)
$\{3\}$	(3) $(x)(H(x) \rightarrow M(x))$	Rule P
$\{3\}$	(4) $H(y) \rightarrow M(y)$	US, (3)
$\{1,3\}$	(5) $M(y)$	Rule T, (2) (4), I_{11}
$\{1,3\}$	(6) $(\exists x) M(x)$	EG, (5).

4. Prove that,

$(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$.

Proof:

Step no	Statement	Reason
$\{1\}$	(1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
$\{1\}$	(2) $P(y) \wedge Q(y)$	ES, (1)
$\{1\}$	(3) $P(y)$	T, (2), I_1
$\{1\}$	(4) $Q(y)$	T, (2), I_2
$\{1\}$	(5) $(\exists x) P(x)$	EG, (3)
$\{1\}$	(6) $(\exists x) Q(x)$	EG, (4)
$\{1\}$	(7) $(\exists x) P(x) \wedge (\exists x) Q(x)$	T, (5), (6), I_9

5. Show that from

a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$

b) $(\exists y)(M(y) \wedge \neg W(y))$ the conclusion

$(x)(F(x) \rightarrow \neg S(x))$ follows.

soln:

Step no	Statement	Reason
{1}	(1) $(\exists y)(M(y) \wedge \neg W(y))$	Rule P
{1}	(2) $M(z) \wedge \neg W(z)$	ES, (1)
{1}	(3) $\neg(M(z) \rightarrow W(z))$	Rule T, (2), E_1 $P \wedge \neg Q \Leftrightarrow \neg(P \rightarrow Q)$
{1}	(4) $(\exists y)\neg(M(y) \rightarrow W(y))$	EG, (3)
{1}	(5) $\neg(y)(M(y) \rightarrow W(y))$	E_{26}, (4) $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A$
{6}	(6) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$	Rule P
{1, 6}	(7) $\neg(\exists x)(F(x) \wedge S(x))$	Rule T, (5), (6), I_{12}
{1, 6}	(8) $(x)\neg(F(x) \wedge S(x))$	T (7), E_{26}
{1, 6}	(9) $\neg(F(y) \wedge S(y))$	US, (8)
{1, 6}	(10) $F(y) \rightarrow \neg S(y)$	T, (9), E_8, E_{16}.
{1, 6}	(11) $(x)(F(x) \rightarrow \neg S(x))$	UG, (10)

6. Show that

$(x)(P(x) \vee Q(x)) \Rightarrow (x) P(x) \vee (\exists x) Q(x)$

Soln:

We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional Premise.

Step no	Statement	Reason
{1}	(1) $\neg((x)\, P(x) \vee (\exists x) Q(x))$	P(assumed).
{1}	(2) $\neg(x)P(x) \wedge \neg(\exists x) Q(x)$	T, (1), Eq
{1}	(3) $\neg(x) P(x)$	T, (2), I_1
{1}	(4) $(\exists x) \neg P(x)$	T, (3), E_{26}
{1}	(5) $\neg(\exists x) Q(x)$	T, (2), I
{1}	(6) $(x) \neg Q(x)$	T, (5), E_{25}
{1}	(7) $\neg P(y)$	E(S)(4)
{1}	(8) $\neg Q(y)$	US (6)
{1}	(9) $\neg P(y) \wedge \neg Q(y)$	T, (7), (8), I_9
{1}	(10) $\neg(P(y) \vee Q(y))$	T, (9), E_9
{11}	(11) $(x)(P(x) \vee Q(x))$	Rule P
{11}	(12) $P(y) \vee Q(y)$	US, (11)
{1, 11}	(13) $\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	T, (10), (12), I_9 Contradiction.

7. Show that $\neg P(a,b)$ follows logically from $(x)(y)(P(x,y) \rightarrow W(x,y)) \rightarrow W(x,y)$ and $\neg W(a,b)$

Soln:

Step no	Statement	Reason
$\{1\}$	(1) $(x)(y)(P(x,y) \rightarrow W(x,y))$	P
$\{1\}$	(2) $(y)(P(a,y) \rightarrow W(a,y))$	US, (1)
$\{1\}$	(3) $P(a,b) \rightarrow W(a,b)$	US, (2)
$\{4\}$	(4) $\neg W(a,b)$	P
$\{1,4\}$	(5) $\neg P(a,b)$	T, (3), (4), I_{12}

EXERCISE:

1. $H_1: \neg P$ $H_2: P \rightleftarrows Q$ $C: \neg(P \wedge Q)$

Soln:

P	Q	$\neg P$	$P \wedge Q$	$\neg(P \wedge Q)$	$P \rightleftarrows Q$
T	T	F	T	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	T	T

H_1 and H_2 are true 4th row and C is also true.

∴ The conclusion is valid.

2. $H_1: P \rightarrow Q$, $H_2: Q$, $C: P$

Soln:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

H_1 and H_2 are true in 1^{st} and 3^{rd} row but 'C' is not true in C.

$\therefore$ The Conclusio is Invalid.

3. Show that $A \vee B$ follows logically from the premises. $P \vee Q$, $(P \vee Q) \to \neg R$, $\neg R \to (S \wedge \neg T)$ and $(S \wedge \neg T) \to (A \vee B)$

Soln:

Step no	Statement	Reason
1	$P \vee Q$	Rule P
2	$P \vee Q \to \neg R$	Rule P
3	$\neg R$	Rule T, (1), (2)
4	$\neg R \to (S \wedge \neg T)$	Rule P
5	$S \wedge \neg T$	Rule T, (3), (4) $P, P \to Q \Rightarrow Q$
6	$(S \wedge \neg T) \to (A \vee B)$	Rule P
7	$A \vee B$	Rul T, (5), (6) $P, P \to Q \Rightarrow Q$

4. Using Indirect Method of Proof, derive $R \to \neg S$ from the premises $P \to (Q \vee R)$, $Q \to \neg P$, $S \to \neg R$, P.

Soln:

Step no	Statement	Reason
1	$P \to (Q \vee R)$	Rule P
2	P	Rule P
3.	$Q \vee R$	T, (1), (2) and Modus Ponens
4	$P \wedge S$	P (additional)
5	S	T, (1) and Simplification
6	$S \to \neg R$	Rule P
7	$\neg R$	T, (5), (6) and Modus Ponens
8.	Q	T, (3), (7) and Disjunctive Syllogism
9.	$Q \to \neg P$	P
10.	$\neg P$	T, (8), (9) and Modus Ponens
11	$P \wedge \neg P$	T, (2)(10) and Conjunction
12.	F	T, (11) and Negation law

5. Prove that the premises $P \to Q$, $Q \to R$, $S \to \neg Q$ and $P \wedge S$ are inconsistent.

Soln:

Step no	Statement	Reason
1	$P \to Q$	Rule P
2	$Q \to R$	Rule P
3	$P \to R$	T, (1), (2)
4	$S \to \neg R$	Rule P
5	$R \to \neg S$	T, (4)
6.	$P \to \neg S$	T, (3)(5)
7	$\neg P \vee \neg S$	T, (6)
8	$\neg(P \wedge S)$	T, (7)
9	$P \wedge S$	Rule P
10	$(P \wedge S) \wedge \neg(P \wedge S)$	Rule P
11	F	T, (10)

6. Prove that the premises $A \to (B \to C)$, $D \to (B \wedge \neg C)$, and $(A \wedge D)$ are inconsistant

Soln:

Step no	Statement	Reason
1	$A \to (B \to C)$	Rule P
2	$A \to (\neg B \vee C)$	T, E_{16}
3	$\neg(\neg B \vee C) \to \neg A$	T, E_{18}
4	$(B \wedge \neg C) \to \neg A$	T, (3)
5	$D \to (B \wedge \neg C)$	Rule P
6	$D \to \neg A$	T, (5)(4)
7.	$\neg D \vee \neg A$	T, (6)

8	$\neg(D \wedge A)$	T, (7)
9	$A \wedge D$	P
10.	$(A \wedge D) \wedge \neg(A \wedge D)$	T, (9) (8)
11.	F	T, (10)
		$P \wedge \neg P \Leftrightarrow F$.

7. Show that $\neg Q, P \to Q, P \vee R \Rightarrow R$.

Soln:

Step no	Statement	Reason
1	$P \vee R$	Rule P
2.	$\neg R$	Negation of the Conclusion.
3.	$\neg R \to P$	(1) $P \to Q \Leftrightarrow \neg P \vee Q$
4.	P	(3), (2) $P, P \to Q \Rightarrow Q$
5	$P \to Q$	Rule P
6.	Q	(4) (5) $P, P \to Q \Rightarrow Q$
7.	$\neg Q$	Rule P
8.	$Q \wedge \neg Q$	(6), (7) $P, Q \Rightarrow P \wedge Q$
9	F	(8) $Q \wedge \neg Q \Rightarrow F$

8. Show that $P \to S$ follows logically from the premises $\neg P \vee Q$, $\neg Q \vee R$, $R \to S$

Soln:

Step no	Statement	Reason
1	$\neg P \vee Q$	Rule P
2.	$P \to Q$	(1) $P \to Q \Leftrightarrow \neg P \vee$
3.	$\neg Q \vee R$	Rule P

4	$Q \to R$	T, (3)
5.	$P \to R$	(2),(4) chain rule
6.	$R \to S$	Rule P.
7	$P \to S$	(5),(6) chain rule.

9. show that $J \wedge S$ follows logically from the premises $P \to Q$, $Q \to \neg R$, R and $P \vee (J \wedge S)$

Soln :

Step no	Statement	Reason
1	$P \to Q$	P
2.	$Q \to \neg R$	P
3.	$P \to \neg R$	(1),(2) chain rule
4	$R \to \neg P$	(3) Contra Positive $(P \to Q \Leftrightarrow \neg Q \to \neg P)$
5.	R	P
6	$\neg P$	(4),(5) $P, P \to Q \Leftrightarrow Q$
7	$P \vee (J \wedge S)$	P
8	$\neg P \to (J \wedge S)$	(7) $P \to Q \Rightarrow \neg P \vee Q$
9	$J \wedge S$	(6),(8) $P, P \to Q \Rightarrow Q$.

10. show that the premises "One student in this class knows programs in JAVA" and "every one who knows how to write programs in JAVA can get a high-paying Job" imply the conclusion "Someone in this class get a high paying job".

Soln:

Let $C(x)$: x is in this class

$J(x)$: x knows JAVA Programming

$H(x)$: x can get a high paying job.

Then the premises are : $\exists x\,(C(x) \wedge J(x))$ and $\forall x\,(J(x) \rightarrow H(x))$

$\therefore$ The conclusion is $\exists x\,(C(x) \wedge H(x))$

Step no	statement	Reason
$\{1\}$ 1	$\exists x(C(x) \wedge J(x))$	Rule P
$\{1\}$ 2	$C(a) \wedge J(a)$	ES and (1)
$\{1\}$ 3	$C(a)$	T, (2) and Simplification
$\{1\}$ 4	$J(a)$	T, (2) and Simplification
$\{5\}$ 5	$\forall x(J(x) \rightarrow H(x))$	Rule P
$\{5\}$ 6	$J(a) \rightarrow H(a)$	US and (5)
$\{1,5\}$ 7	$H(a)$	T, (4), (6) Modus ponens
$\{1,5\}$ 8	$C(a) \wedge H(a)$	T, (3), (4), and Conjunction
$\{1,5\}$ 9	$\exists x(C(x) \wedge H(x))$	EG, and (8)

11. Prove that $(x)(P(x) \to Q(x)), (x)(R(x) \to \neg Q(x))$
$\Rightarrow (x)(R(x) \to \neg P(x))$

Soln:

Step no	Statement	Reason
1	$(x)(P(x) \to Q(x))$	P
2	$P(y) \to Q(y)$	ES.
3.	$\neg Q(y) \to \neg P(y)$	T, (2) $P \to Q \Rightarrow \neg Q \to \neg P$
4.	$(x)(R(x) \to \neg Q(x))$	Rule P
5.	$R(y) \to \neg Q(y)$	ES
6.	$R(y) \to \neg P(y)$	T, (5), (1) $P \to Q, Q \to R \Rightarrow P \to R$.
7.	$(x)(R(x) \to \neg P(x))$	UG, (6).

12. Are the following conclusions validly derivable from the premises given
$(x)(P(x) \to Q(x)), (\exists y) P(y), C: (\exists x) Q(z)$

Soln:

Step no	Statement	Reason
1	$(x)(P(x) \to Q(x))$	P
2	$P(W) \to Q(W)$	US
3	$(\exists y)\ P(y)$	P
4	$P(W)$	ES
5	$Q(W)$	$P, P \to Q \Rightarrow Q$

13 $(\exists x)\ (P(x) \wedge Q(x))$, C: $(x)\ P(x)$

Soln:

Step no	Statement	Reason
1	$(\exists x)(P(x) \wedge Q(x))$	P
2.	$P(y) \wedge Q(y)$	ES
3.	$P(y)$	T, (2), $P \wedge Q \Rightarrow P$
4.	$(x)\ P(x)$	UG.

14. $(\exists x)\ P(x)$, $(\exists x)\ Q(x)$ C: $(\exists x)\ (P(x) \wedge Q(x))$

Soln:

Step no	Statement	Reason
1	$(\exists x)(P(x))$	P
2	$P(y)$	EG
3.	$(\exists x)\ (Q(x))$	P
4.	$Q(y)$	EG
5.	$P(y) \wedge Q(y)$	T, (1), (4)
6.	$(\exists x)(P(x) \wedge Q(x))$	EG.

15. $(x)\ (P(x) \rightarrow Q(x))$, $\neg Q(x)$, C: $(x)\ \neg P(x)$

Soln:

Step no	Statement	Reason
1	$(x)\ (P(x) \rightarrow Q(x))$	P
2.	$P(a) \rightarrow Q(a)$	US
3.	$\neg Q(a)$	P
4.	$\neg P(a)$	T, (3), (2) $\neg Q, P \rightarrow Q \Rightarrow \neg P$
5.	$(x)\ \neg P(x)$	UG.