

UNIT-IV

LATTICES

A relation R on a set is any subset of $A \times A$

$$\text{ie } R = \{(x, y) \mid x \in A, y \in A\}$$

A relation R on a set A is said to be reflexive if $(x, x) \in R \forall x \in A$.

A relation R on a set A is said to be symmetric if $(x, y) \in R \Rightarrow (y, x) \in R$

A relation R on a set S is said to be transitive if $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$

A relation R on a set A is said to be an equivalence relation if it is reflexive symmetric and transitive.

A relation R on a set A is said to be anti symmetric if and only if

$$(x, y), (y, x) \Rightarrow x = y$$

A relation R on a set A is said to be a partial order relation if it is reflexive antisymmetric and transitive.

A set together with a partially ordered set (p.o. set)

Eg: 1

Let A be the set of numbers R be the relation "equal to" $"="$ is an equivalence relation.

(2) $"\leq"$ is a partial order relation.

(3) $"\geq"$ is a partial order relation.

(4) $"\subseteq"$ on set of sets is a partial order relation.

(5) $"\supseteq"$ on set of sets is a p.o. relation.

(6) The relation divides on the set of number is a p.o. relation.

a/a reflexive

$a/b, b/a \Rightarrow b/a$ antisymmetric

$a/b, b/c \Rightarrow a/c$ transitive

(7) Multiple of is p.o. relation.

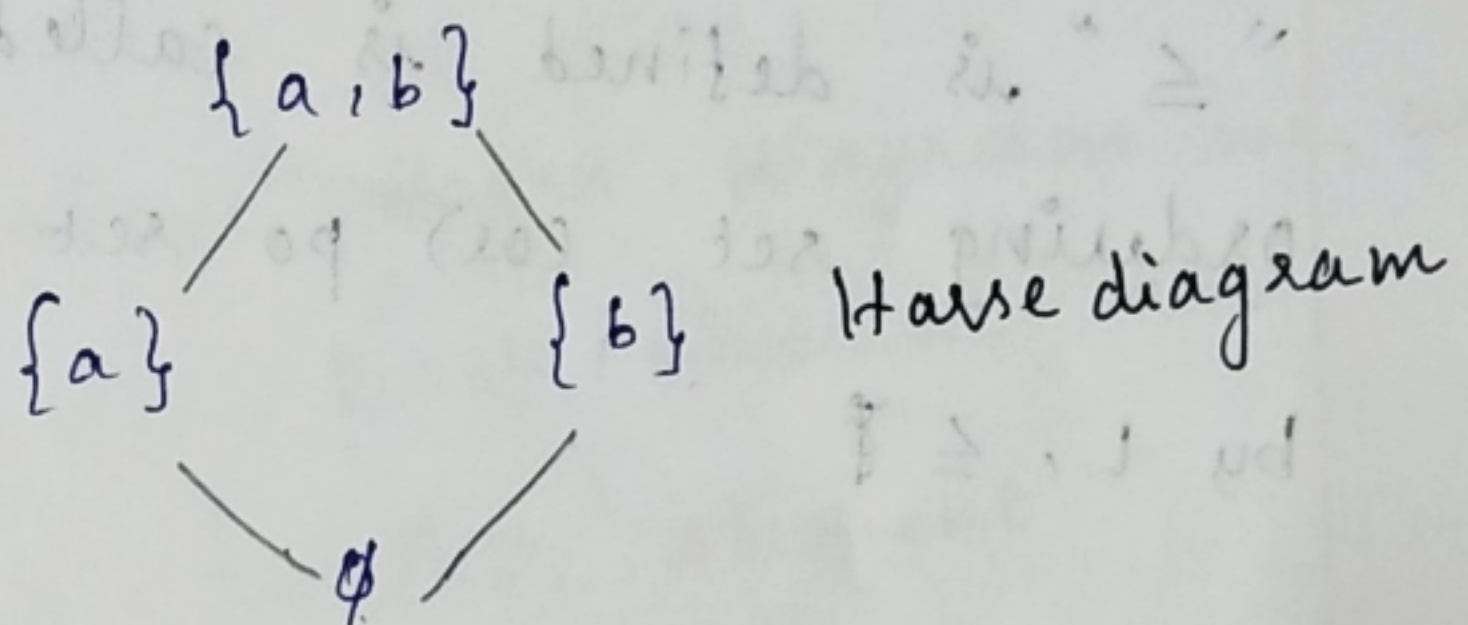
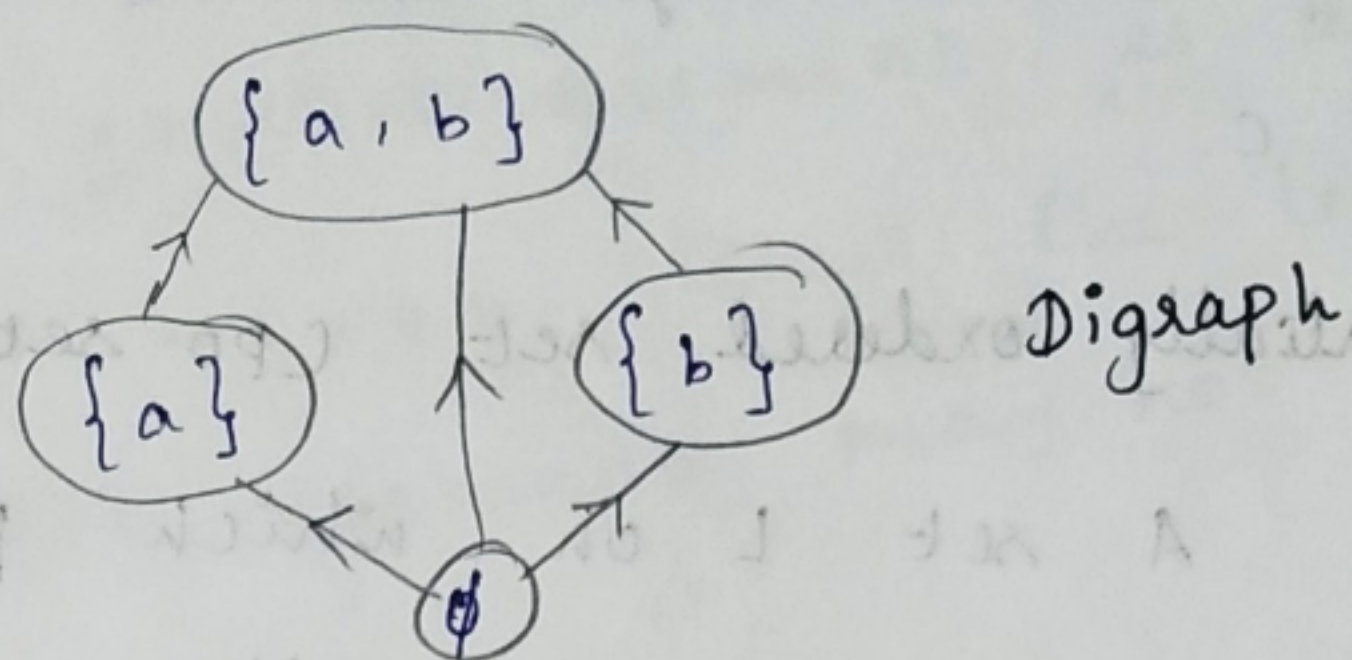
Let $B = \{a, b\}$ and $A =$ power set of B is

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Let R be the relation subset of A then

$$R = \{\emptyset, \emptyset\}, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\},$$

$(\emptyset, \{a, b\})$, $(\{a\}, \{a\})$, $(\{a\}, \{a, b\})$
 $(\{b\}, \{b\})$, $(\{b\}, \{a, b\})$, $(\{a, b\}, \{a, b\})$



It is easy to draw a hasse diagram (or) ordering diagram for a partial ordering relation.

Rules :

- (i) The vertices are denoted by points rather than circles.
- (ii) Each vertex must be related itself so the arrows from a vertex to itself are not necessary.
- (iii) If the vertex 'b' appears above vertex 'a' and if vertex 'a' is connected to vertex 'b' by an edge then Rb so direction arrows are not necessary.
 $\rightarrow$ Related to

(iv) If the vertex c is above a and if c is connected to a by a sequence of edges then $(a R c)$ a Related to c .

Partially ordered set (PO set)

A set L on which partial ordering " $\leq$ " is defined is called a partially ordering set (or) po set and is denoted by $L, \leq$

Let $(L, \leq)$ be po set. If an element $x \in L$ such that $x \leq y \forall y \in L$.

Then x is called the least member

(or) least element in L relative to the partial ordering $\leq$.

If there exists an element $x \in L$ $\rightarrow x \geq y \forall y \in L$. Then x is called

the greatest member (or) element in

' L ' relative to the partial ordered relation ' $\geq$ '.

The least element or the greatest element. If it exists is unique. The least element is denoted by 0 (zero) and the greatest element is denoted by 1 (one). Some times the least (or) greatest element may not exist at all.

From the above Hasse diagram we use that ' $\emptyset$ ' is the least element and $\{a, b\}$ is the greatest element.

Eg : 2.

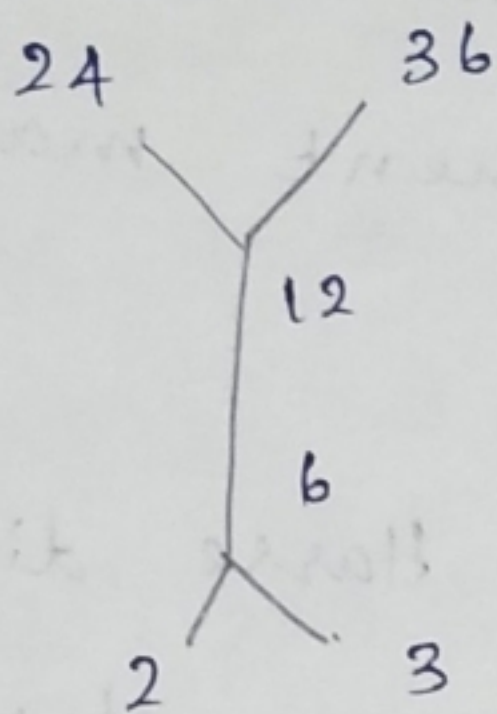
Let $A = \{2, 3, 6, 12, 24, 36\}$ be a relation on $A \Rightarrow$ set a aRb if a divides b .

From the Hasse diagram we see that there is no least (or) greatest element.

Sol

Let $(L, \leq)$ be a poset. An element $x \in L$ is called a minimal element or L relative to the po relation $\leq$ if for no $y \in L$, $y < x$. A minimal element

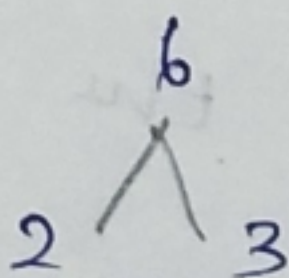
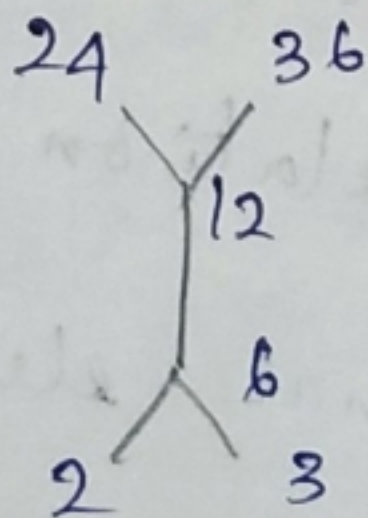
need not be unique. All these members which appear at the lowest level of a Hasse diagram of a poset are minimal elements.



An element $x \in L$ is called a maximal element of L relative to the poset relative $\leq$ if for no $y \in L, x < y$.

A maximal element need not be unique. All these members which appear at the highest level of a Hasse diagram poset are maximal elements.

Distinct minimal (or) maximal elements are not comparable.



Φ_n (i) 2, 3 are minimal element
and 24, 36 are maximal elements.

Φ_n (ii) 2, 3 are minimal and 6 is
maximal element.

(*) Defn Let $(P, \leq)$ be a poset, The partial
ordering $\leq$ is said to be simple ordering

(or) linear ordering on P if either (or)

linear ordering on P if either $x \leq y$ (or)

$y \leq x \forall x, y \in P$.

(*) Defn The poset $(P, \leq)$ is called a totally
ordered set (or) a chain if the partial
ordering $\leq$ is a linear ordering on P .

i.e. $\forall x, y \in P$ either $x \leq y$ (or) $y \leq x$.

Let $(P, \leq)$ be poset two elements x

and y are said to be comparable in P .

if either $x \leq y$ (or) $y \leq x$. Two elements

x and y are said to be non comparable

if neither $x \leq y$ (nor) $y \leq x$.

Let $(K, \leq)$ be a poset and $A \subseteq K$

any element $x \in K$ is called an upper

bound for A if $\forall a \in A, a \leq x$ any element $x \in K$ is

called a lower bound for A if $\forall a \in A, x \leq a$.

Ex: 3 $A = \{a, b, c\}$
 Let $B = \{\{b, c\}, \{b\}, \{c\}\}$

The upper bounds of B are $\{b, c\}$ and $\{a, b, c\}$. The lower bounds for B in ϕ

Let $C = \{\{c\}, \{a, c\}\}$

The upper bounds of C are $\{a, c\}$ and $\{a, b, c\}$. Two lower bounds of C are $\{c\}$

and ϕ .

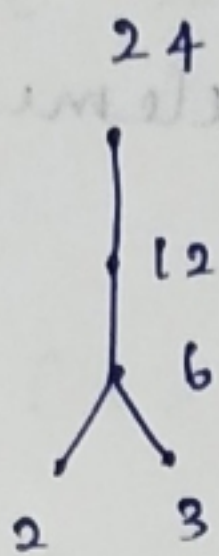
Least upper bound and greatest lower bound $[a, b]$.

Let $(L, \leq)$ be a poset and $A \subseteq L$. An element $x \in L$ is a least upper bound for A (or) ^{supremum} for A . If x is an upper bound for A and $x \leq y$ where y is any upper bound for A .

An element $x \in L$ is called the greatest lower bound (or) ~~infimum~~ for A . If x is a lower bound for A and $x \geq y$ where y any lower bound for A .

Examples:

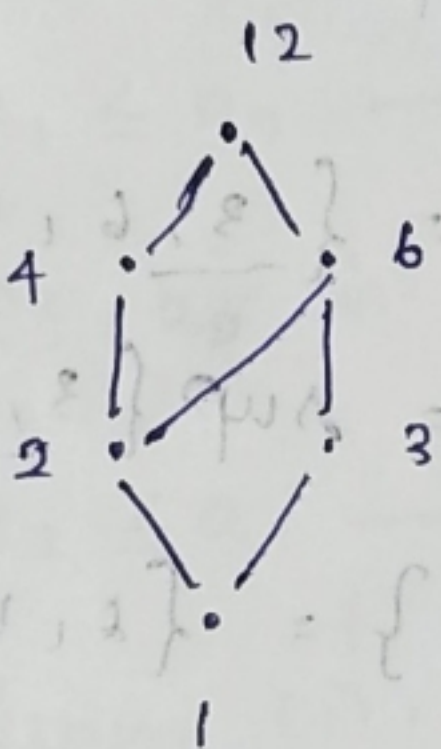
1. $A = \{2, 3, 6, 12, 24\}$ $x \leq y$ if $x|y$



minimal elements - 2, 3

maximal elements - 24

2. $A = \{1, 2, 3, 4, 6, 12\}$ $x < y$ if $x|y$

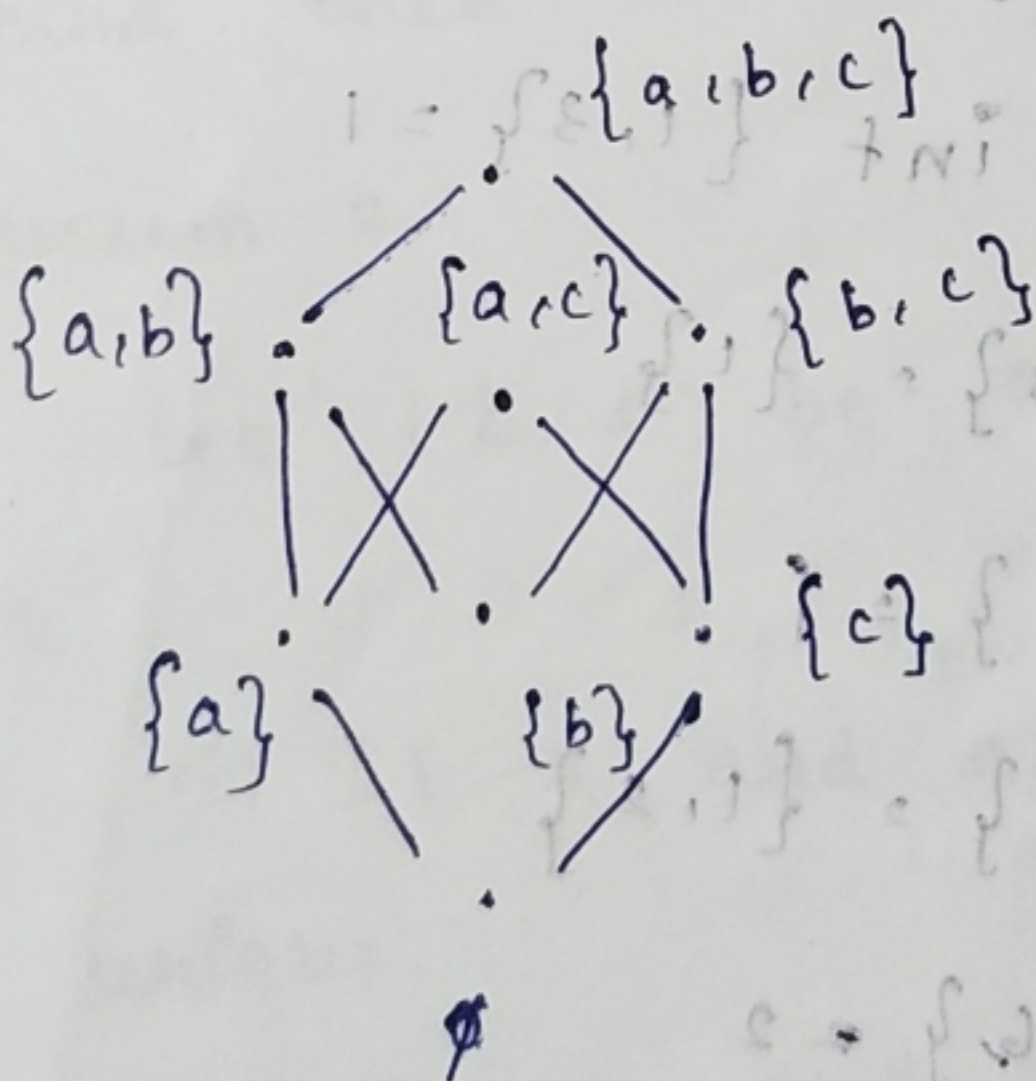


minimal element - 1

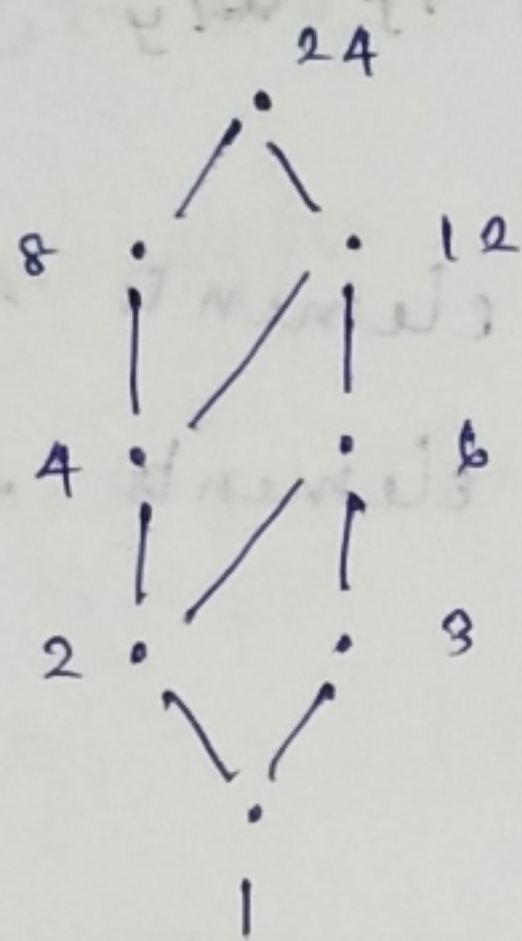
maximal element - 12

3. $A = \{a, b, c\}$, $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$x \leq y$ if $x \subseteq y$

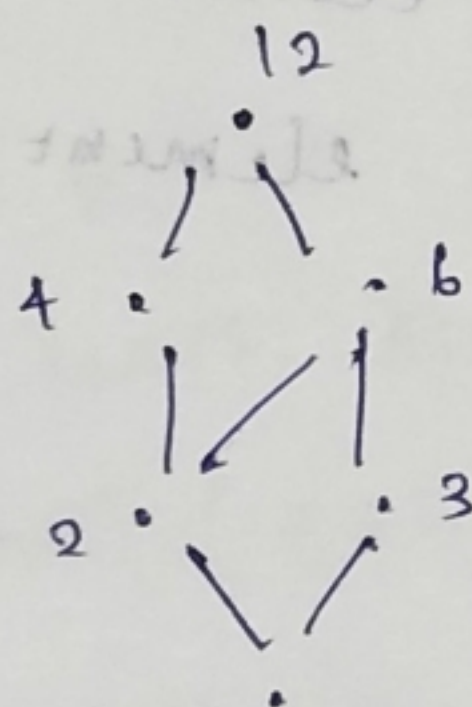


A. D_{24} Divisors of 24 $B = \{1, 2, 3, 4, 6, 8, 12, 24\}$



minimal element - 1
maximal element - 24

5.



$$ub - \{1, 3\} = \{3, 6, 12\}$$

$$lub \{1, 3\} = \sup \{3, 6, 12\} = 12$$

$$ub \{1, 2, 3\} = \{6, 12\}$$

$$lub \{1, 2, 3\} = \sup \{6, 12\} = 12$$

$$upper \text{ bound } \{4, 6\} = \{12\}$$

$$lb \{1, 3\} = \{1\} \quad lower \text{ bound}$$

$$glb \{1, 3\} = \inf \{1, 3\} = 1$$

$$lb \{1, 2, 3\} = \{1\}$$

$$glb \{1, 2, 3\} = 1$$

$$lb \{4, 6\} = \{2\}$$

$$glb \{4, 6\} = 2$$

Theorem 1 :

If a poset has a least element then it is different unique.

Proof :

Let $(L, \leq)$ be a poset if possible let a_1 and a_2 be two least element if a_1 is a least element by definition

$$a_1 \leq a_2 \rightarrow (1)$$

If a_2 is the least element by definition

$$a_2 \leq a_1 \rightarrow (2)$$

from (1), (2) $a_1 = a_2$

$\therefore$ The least element is unique.

Now if a poset has a greatest lower bound then it is unique.

Theorem 2 :

Let $(L, \leq)$ be a poset and $a_1, a_2 \in L$

(i) If a_1 and a_2 have glb then it is unique.

(ii) If a_1 and a_2 have lub then it

is unique.

Proof :

Let l_1 and l_2 be 2 different glb's of $\{a_1, a_2\}$ in L . By the definition of lower bound.

$$l_1 \leq a_1, \quad l_1 \leq a_2$$

$$l_2 \leq a_1, \quad l_2 \leq a_2$$

Since l_1 is a glb of a_1 and a_2 , $l_1 \geq l_2$ for any lower bound l_2 of a_1 and a_2 .

Since l_2 is a glb of a_1 and a_2 , $l_2 \geq l_1$ for any lower bound l_1 of a_1 and a_2 .

Now $l_1 \geq l_2$ and $l_2 \geq l_1$ (anti symmetric)

$\therefore$ GLB of any 2 elements a_1 and a_2

of a poset is unique.

Similarly we can know that the

LUB of any 2 elements a_1 and a_2 of

a poset is unique.

Note : Notation

LUB of a and b $a \oplus b$ and $a + b$ or $a \vee b$ or $a \cup b$.

(supremum of a and b) (join) (sum)
GLB of a and b is $a \times b$ or $a \cdot b$ or $a \wedge b$ or $a \cap b$

Definition :

Lattice

A Lattice is a poset $(L, \leq)$ in which every pair of elements has a glb and lub.

From the definition of lub and glb

$$a \wedge b \leq a, \quad a \leq a \oplus b$$

$$a \leq b, \quad a \leq c \Rightarrow a \leq b \wedge c, \quad a \leq b \oplus c$$

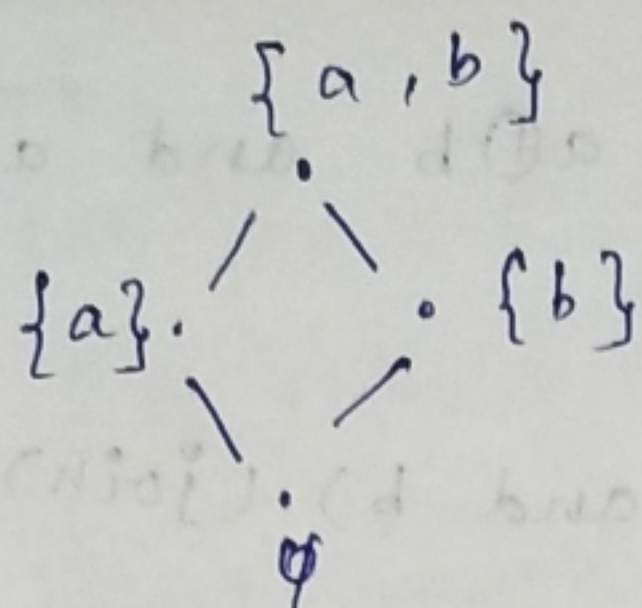
$$a \geq b, \quad a \geq c \Rightarrow a \geq b \wedge c, \quad a \geq b \oplus c$$

Example 1 :

Let A be any set, $P(A)$ be its power set then $(P(A), \subseteq)$ is a lattice

Meet and join are $\cap$ and $\cup$ let

$$A = \{a, b\} \quad P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$



Here $(P(A), \subseteq)$ is a lattice.

Example 2 :

Let $\mathbb{Z}^+$ be the set of integers
 $\leq$ relation $x \leq y$ if and only if $x|y$.

Then $(\mathbb{Z}^+, \leq)$ is a lattice.

Meet of x and $y = x * y$ is the

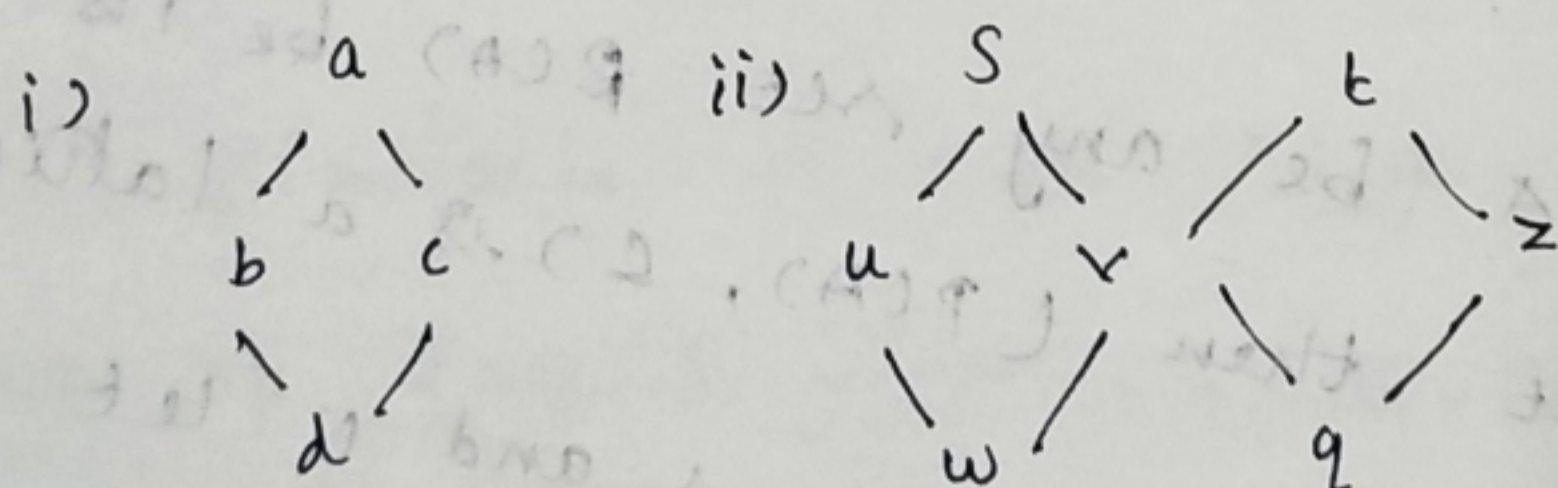
GCD of x and y .

Join of x and $y = x + y$ is the

LCM of x and y .

Example: 3

Consider the following Hasse Diagram.



i) $\text{lub}\{a, b\} = a$

$\text{lub}\{a, c\} = a$

$\text{lub}\{a, d\} = a$

$\text{glb}\{a, b\} = b$

$\text{glb}\{a, c\} = c$

$\text{glb}\{a, d\} = d$

$$\text{lub} \{b, c\} = a$$

$$g/b \{b, c\} = d$$

$$\text{lub} \{b, d\} = b$$

$$g/b \{b, d\} = d$$

$$\text{lub} \{c, d\} = c$$

$$g/b \{c, d\} = d$$

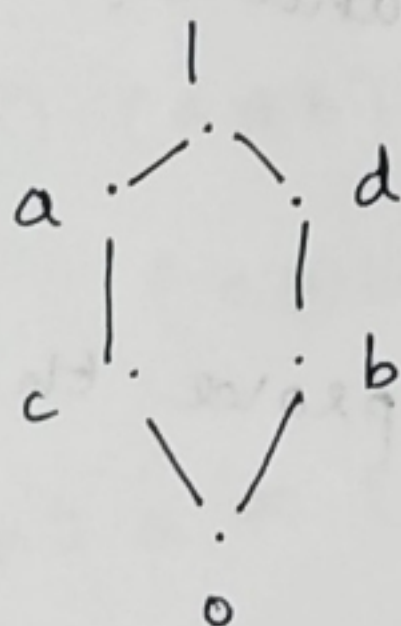
It is a lattice because every pair of element g/b and lub .

ii) $\text{lub} \{s, t\} = \text{does not exist}$

$$g/b \{a, z\} = g/b \{w, q\}$$

$\therefore$ The given poset is not a lattice.

iii)



$$\text{lub} \{a, b\} = \text{does not exist}; g/b \{a, b\} = 0$$

$$\text{lub} \{c, d\} = 1; g/b \{c, d\} = \text{no } g/b$$

so the given poset is not a lattice.

Theorem 3 :

Every chain is a lattice.

Proof :

let $(L, \leq)$ be a chain and $a, b \in L$.

then $a \leq b$ (or) $b \leq a$.

Case (i): $a \leq b$

Then b is upper bound for a and b
if u is any other upper bound for a
and b then $a \leq u, b \leq u$.

Thus $b \leq u$ for any upper bound u
of a and b .

$\therefore b$ is the lub of a and b .

$a \leq b \Rightarrow a$ is lower bound of a and b .

If l is any other lower bound for a

and b we have $l \leq a, l \leq b$.

$\therefore a$ is the glb of a and b .

Case (ii).

Similarly if $b \leq a$ we can prove that

$$a \vee b = a, a \wedge b = b.$$

Thus lub and glb exists $\forall a, b \in L$

Hence $(L, \leq)$ is a lattice.

Principle of duality:

Any statement about lattice
involving the operation $*$ and $\oplus$ and
the relations $\leq$ and $\geq$ remains true if
 $*$ is replaced by $\oplus$, $\oplus$ by $*$, $\leq$ by $\geq$ and

$\geq$ by $\leq$.

The operation $*$ and $\oplus$ are called the duals of each other.

similarly $\leq$ and $\geq$ are dual operators. The lattice $(L, \leq)$ and $(L, \geq)$ are called duals of each other.

Properties of lattices:

For any $a, b, c \in L$, we have

(i) $a * a = a$, $a \oplus a = a$ (Idempotent)

(ii) $a * b = b * a$, $a \oplus b = b \oplus a$ (commutative)

(iii) $(a * b) * c = a * (b * c)$
 $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ } Associative

(iv) $a * (a \oplus b) = a$ } Absorption

$a \oplus (a * b) = a$

Proof:

(i) $a * a = \text{glb } \{a, a\}$

$= a$

$a \oplus a = \text{lub } \{a, a\}$

(ii) $a * b = \text{glb } \{a, b\}$

$a * b = \text{glb } \{b, a\}$

$a * b = b * a$

$a \oplus b = \text{lub } \{a, b\}$

$a \oplus b = \text{lub } \{b, a\}$

$a \oplus b = b \oplus a$

$$(iii) (a * b) * c$$

$$g/b \{ (a * b), c \}$$

$$g/b \{ a, b, c \} \text{ --- (1)}$$

$$a * (b * c) = g/b \{ a, b * c \}$$

$$g/b \{ a, b, c \}$$

$$(a * b) * c = a * (b * c)$$

$$(iv) \text{ for any } a \in L, a \leq a \text{ --- (2)}$$

$$(\leq \text{ is reflexive) and } a \leq a \oplus b \text{ --- (3)}$$

$$\text{by ② defin of } \oplus$$

$$a \leq a * (a \oplus b) \text{ --- (4)}$$

By similarly

$$a \oplus (a * b) = a$$

By principle of duality we have $a \geq a * (a \oplus b)$ --- (5)

$$(4), (5) \rightarrow a * (a \oplus b) = a \text{ and } a \oplus (a * b) = a$$

By duality $a \oplus (a * b) = a$

Theorem - 4 :

Let $(L, \leq)$ be a lattice in which $*$ and $\oplus$ denote the operator of meet and join for any $a, b \in L$, $a \leq b$ if $a * b = a$
 $\Leftrightarrow a \oplus b = b$.

(for any $a, b \in L$: $a \leq b \Leftrightarrow a * b = a \Rightarrow a \oplus b = b$)

Proof :

Let $a \leq b$ — (1)

Since $\leq$ is reflexive

$a \leq a * b$ — (2)

$\therefore a \leq a * b$ — (3)

$a \geq a * b$ — (4)

$a * b \leq a$ by definition of $*$ — (4)

(3) and (4) gives $a * b = a$

Thus if $a \leq b$ then $a * b = a$ — (5)

$a * b = a$

Let $a * b = a$ this is possible only

if $a \leq b$ if $a * b = a$ then $a \leq b$ — (6)

from (5) and (6)

$a \leq b \Leftrightarrow a * b = a$

$\therefore a \leq b$ if and only if $a * b = a$

$$\text{let } a * b = a \text{ — (1)}$$

now $a \oplus b = b \oplus a$ (commutative law)

$$= b \oplus (a * b) \text{ by (1)}$$

$$= b \oplus (b * a) \text{ (commutative)}$$

$$a \oplus b = b \text{ by absorption law}$$

$$\text{Hence } a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$

Theorem: 5 state and prove isotonicity law

Let $(L, \leq)$ be a lattice and $a, b, c \in L$
then the following property called isotonicity holds.

$$b \leq c \Rightarrow a * b \leq a * c$$

$$b \leq c \Rightarrow a \oplus b \leq a \oplus c$$

Proof:

By theorem (4)

$$a \leq b \Leftrightarrow a * b = a \text{ — (1)}$$

$$b \leq c \Leftrightarrow b * c = b \text{ — (2)}$$

To prove that

$a * b \leq a * c$ it is enough to prove

$$\text{that } (a * b) * (a * c) = a * b$$

$$\text{know } (a * b) * (a * c)$$

$$= a * (b * a) * c \quad (\text{Associative})$$

$$= a * (a * b) * c \quad (\text{Commutative})$$

$$= (a * a) * (b * c) \quad (\text{Associative})$$

$$= a * (b * c) \text{ by (1)} \quad (\text{Idempotent})$$

$$= a * b \text{ (by 2)}$$

$$b \leq c \Rightarrow a * b \leq a * c$$

By theorem (4) we know that

$$a \leq b \Leftrightarrow a \oplus b = b \quad \text{--- (3)}$$

$$b \leq c \Leftrightarrow b \oplus c = c \quad \text{--- (4)}$$

To prove that $a \oplus b \leq a \oplus c$ it is enough

To prove that $(a \oplus b) \oplus (a \oplus c) = a \oplus c$

$$(a \oplus b) \oplus (a \oplus c) = a \oplus (b \oplus a) \oplus c \quad (\text{Associative})$$

$$= a \oplus (a \oplus b) \oplus c \quad (\text{Commutative})$$

$$= (a \oplus a) \oplus (b \oplus c) \quad (\text{Associative})$$

$$= a \oplus (b \oplus c) \quad (\text{Idempotent})$$

$$= a \oplus c \text{ (by eqn 4)}$$

$$b \leq c \Rightarrow (a \oplus b) \leq a \oplus c$$

$$\text{Hence, } b \leq c \Rightarrow a * b \leq a * c,$$

$$a \oplus b \leq a \oplus c.$$

Theorem 6: state and prove distributive inequalities

Let $(L, \leq)$ be a lattice for any $a, b, c \in L$ the following inequalities hold.

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

$$a * (b \oplus c) \geq (a * b) \oplus (a * c)$$

Proof:

We know that

$$a \leq a \oplus b, a \leq a \oplus c$$

$$a \leq (a \oplus b) * (a \oplus c) \quad \text{--- (1)}$$

By definition of $*$ and $\oplus$

$$a * b \leq b \leq a \oplus b$$

$$a * c \leq c \leq a \oplus c$$

$$b * c \leq (a \oplus b) * (a \oplus c) \quad \text{--- (2)}$$

From (1) and (2)

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

By duality $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

Theorem 7: state and prove modular inequality

Let $(L, \leq)$ be a lattice for any

$a, b, c \in L$ the following hold,

$$a \leq c \Leftrightarrow \{a \oplus (b * c) \leq (a \oplus b) * c$$

Proof:

By theorem (b) we have $a \oplus (b * c) \leq (a \oplus b) * c$
 $* (a \oplus c) \text{ --- (1)}$

By theorem (a) $a \leq c \Leftrightarrow a \oplus c = c \text{ --- (2)}$

$$(1) \Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

$$\text{Thus } a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c.$$

Lattice as an algebraic system:

Definition:

A lattice is an algebraic system $(L, *, \oplus)$ with two binary operations $*$ and $\oplus$ on L which are both commutative, associative and satisfy absorption laws.

We show that $\leq$ relation on L .

follows as consequence of the properties of the operators $*$ and $\oplus$.

Define a relation R on L such that

$$\text{for } a, b \in L, a R b \Rightarrow a * b = a$$

We show that R is a $\leq$ relation

$$i) a * a = a \quad [\text{Idempotent Law}]$$

$a R a$. R is reflexive.

(ii) Suppose for $a, b \in L$ $a R b$
 and $b R a$ $a * b = a$, $b * a = b$
 but $a * b = b * a$ commutative
 $a = b$

Thus $a R b, b R a \Rightarrow a = b$

R is antisymmetric

(iii) for $a, b \in L$, let $a R b, b R c$

Now $a R b, b R c \Rightarrow a * b = a \xrightarrow{(1)}$

$$b * c = b \xrightarrow{(2)}$$

$$a * c = (a * b) * c$$

$$= a * (b * c)$$

$$= a * b \text{ by (2)}$$

$$= a \text{ by (1)}$$

$$a R c$$

R is transitive

Since R is reflexive, antisymmetric and

transitive R is a po relation Law

$a \leq b$ if $a * b = a$ also $a * (a \oplus b) = a$

(absorption law)

$$a \oplus b = b$$

further the glb and lub of 2 elements a and b are $a * b$ and $a \oplus b$.

Thus on a lattice $(L, *, \oplus)$ it is possible to define a po relation R such that for

$$a, b \in L, a R b \iff a * b = a$$

$$a \oplus b = b$$

and

$$\text{lub } \{a, b\} = a \oplus b \text{ and}$$

$$\text{glb } \{a, b\} = a * b$$

Wrt $(L, *, \oplus)$ with R on L .

Sub lattice :

Definition :

let $(L, *, \oplus)$ be a lattice and $S \subseteq L$ be a subset of L . The algebra

$(S, *, \oplus)$ is closed under

both operations $*$ and $\oplus$.

Definition :

let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be 2 lattice of mapping $g: L \rightarrow S$ is called

a lattice homomorphism? from the lattice L to S is for $a, b \in L$,

$$g(a * b) = g(a) \wedge g(b)$$

$$g(a \oplus b) = g(a) \vee g(b)$$

We have $a \leq b \Leftrightarrow a * b = a$

$$g(a * b) = g(a) * g(b) = g(a)$$

$$\Leftrightarrow g(a) \leq g(b)$$

$a \leq b \Rightarrow g(a) \leq g(b)$ if g is a homomorphism.

If a homomorphism $g: L \rightarrow S$ of 2 lattices $(L, *, \oplus)$ and $(S, \wedge, \vee)$ is a bijection (one-one & onto) then g is called an Isomorphism.

If there exists an isomorphism b/w 2 lattices then the lattices are said to be isomorphic.

A homomorphism $g: L \rightarrow L$ whose $(L, *, \oplus)$ is a lattice is called an endomorphism.

If $g: L \rightarrow L$ is an isomorphism then g is called an automorphism.

Definition : Complete

A lattice is called complete if each of its non-empty subset has a lub and glb.

every finite lattice is complete lattice must have a least element and greatest element.

The least and greatest elements of a lattice if they exist are called the bounds denoted by 0 and 1.

A lattice which has both the elements 0 and 1 is called a bounded lattice. For the lattice $(L, *, \oplus)$ with.

$$L = \{a_1, a_2, \dots, a_n\}$$

$$* a_i = 0 \text{ and } i = 1$$

$$i = 1$$

$$\bigoplus_{i=1}^n a_i = 1$$

$$i = 1$$

The bounds 0 and 1 of a lattice $(L, *, \oplus, 0, 1)$ satisfy the conditions for any $a \in L$.

$a + a = a$	$a * a = a$
$a \oplus 1 = 1$	$a * 0 = 0$

Definition:

In a bounded lattice $(L, *, \oplus, 0, 1)$ an element $b \in L$ is called a complement of an element $a \in L$ if

$$a * b = 0, a \oplus b = 1$$

Any element $a \in L$ may not have a complement. An element of L may have more than one complement in L .

Definition:

A lattice $(L, *, \oplus, 0, 1)$ is said to be a complemental lattice if every element of L has at least one complement.

Definition:

A lattice $(L, *, \oplus)$ is said to be distributive lattice if for any a, b, c

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

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The bounded 0 and 1 of a lattice $(L, *, \oplus, 0, 1)$ satisfy the conditions for any $a \in L$.

$a \oplus a = a$	$a * a = a$
$a \oplus 1 = 1$	$a * 0 = 0$



Definition:

In a bounded lattice $(L, *, \oplus, 0, 1)$ an element $b \in L$ is called a complement of an element $a \in L$ if

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Any element $a \in L$ may not have a complement. An element of L may

have more than one complement
is L .

Definition :

A lattice $(L, *, \oplus, 0, 1)$ is said to be a complemented lattice if every element of L has at least one complement.

Definition :

A lattice $(L, *, \oplus)$ is said to be distributive lattice if for any $a, b, c \in L$

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Theorem : 8

Every chain is a distributive lattice.

Proof :

Let $(L, \leq)$ be a (lattice) chain

$a, b, c \in L$.

Case (i) $a \leq b : a \leq c$

by definition of $\oplus$

$$a \leq b \oplus c$$

by theorem ④ $a * (b \oplus c) = a$ — (1)

$a \leq b \Rightarrow a * b = a$ by theorem ④

$a \leq c \Rightarrow a * c = a$

$$\therefore (a * b) \oplus (a * c) = a \oplus a$$

$$(a * b) \oplus (a * c) = a \quad (\text{Idempotent}) \text{ — (2)}$$

from ① and ②

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

By duality

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

$\therefore (L, \leq)$ is a distributive lattice.

Case (ii)

$$a \geq b, a \geq c$$

by definition of $\oplus$, $a \geq b \oplus c$

$$a \geq b \Rightarrow a * b = b \quad (\text{by theorem 4})$$

$$a * (b \oplus c) = b \oplus c \text{ — (3)}$$

$$a \geq b \Rightarrow a * b = b \quad \left. \vphantom{a \geq b} \right\} \text{ by theorem ④}$$

$$a \geq c \Rightarrow a * c = c$$

$$(a * b) \oplus (a * c) = b \oplus c \text{ — (4)}$$

from ③ and ④

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

by duality $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

$\therefore (L, \leq)$ is a distributive lattice

Theorem: 9

Let $(L, *, \oplus)$ be a distributive lattice for any $a, b, c \in L$.

$$(a * b = a * c) \rightarrow (1)$$

$$(a \oplus b = a \oplus c) \Rightarrow (b = c) \rightarrow (2)$$

Proof:

$$(a * b = a * c) \wedge (a \oplus b = a \oplus c)$$

$$(a * b) \oplus c = (a * c) \oplus c$$

$$= c \oplus (a * c) \text{ commutative}$$

$$= c \oplus (c * a) \text{ commutative}$$

$$= c \text{ Absorption} \rightarrow (3)$$

$$(a * b) \oplus c = (a \oplus c) * (b \oplus c) \text{ distributive law}$$

$$= (a \oplus b) * (b \oplus c) \text{ from (2)}$$

$$= (b \oplus a) * (b \oplus c) \text{ commutative}$$

$$= b \oplus (a * c) \text{ distributive}$$

$$= b \oplus (a * b) \text{ by (1)}$$

$$= b \oplus (b * a) \text{ by commutative}$$

$$= b \rightarrow (1) \text{ absorption}$$

$$\text{from } (3) \text{ \& } (4) \quad b = c$$

$$\therefore (a * b) = (a * c) \wedge (a \oplus b) = (a \oplus c)$$

Theorem 10:

In a distributive lattice if an element $a \in L$ has a complement, then it must be unique.

Proof:

Let $a \in L$ and b, c are two complements of a since b is the complement of a .

$$a * b = 0 \quad a \oplus b = 1 \rightarrow (1)$$

c is a complement of a

$$a * c = 0, \quad a \oplus c = 1 \rightarrow (2)$$

from (1) and (2) $a * b = a * c$

$$a \oplus b = a \oplus c$$

$\therefore$ By the previous theorem

$$b = c$$

Hence the complement of a is unique.

MODULAR LATTICE

Definition :

A lattice is said to be modular if $a \leq c \Rightarrow a \oplus (b * c) = (a \oplus b) * c$

Theorem 11 :

Every distributive lattice is modular.

Proof :

Let $(L, \leq)$ be a distributive lattice then by definition.

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c) \quad \text{--- (1)}$$

Let $a \leq c$ by theorem (4)

$$a \leq c \Leftrightarrow a \oplus c = c \quad \text{--- (2)}$$

$$\textcircled{1} \text{ in } \textcircled{2} \Rightarrow a \oplus (b * c) = (a \oplus b) * c$$

$\therefore (L, \leq)$ is a modular lattice.

The converse is not true.

Problems:

1. Show that 0 and 1 are complement of each other (or) show that 0 is only the complement of 1.

$$\left. \begin{array}{l} a * b = 0 \\ a \oplus b = 1 \end{array} \right\} \text{complement}$$

Let 0 and 1 be the bounds of lattice L .

By the properties of bounds

$$a \oplus 1 = 1 \quad a * 0 = 0$$

$$a \oplus 0 = 0 \quad a * 1 = 1$$

We have $0 * 1 = 0$ and $0 \oplus 1 = 1$

$\therefore$ 0 and 1 are complements of each other.

Now we show 1 is only complement of 0.

If possible let $c \neq 1$ is complement of 0. $c \in L$.

by the definition of complement

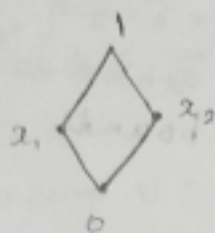
$$0 * c = 0, \quad 0 \oplus c = 1 \quad \text{by but}$$

$$0 \oplus c = c \quad c = 1$$

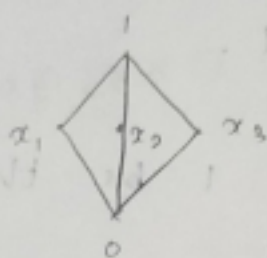
which is contradiction of $c \neq 1$.

Hence 1 is only complement of zero
may 0 is only complement of one.

2. An element $a \in L$ may or may not have a complement further an element of L may have more than one complement in L .



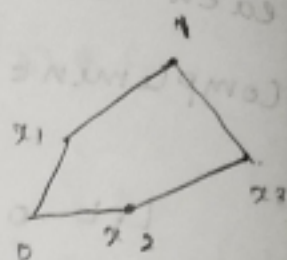
Complement of x_1 is x_2



complement of x_1 are x_2, x_3

complement of x_2 are x_1, x_3

complement of x_3 are x_1, x_2



complement of x_1 are x_2, x_3

complement of x_2 is x_3

complement of x_3 is x_1

1. Let $L = \{1, 2, 3, 4, 6, 12\}$ is the lattice $(L, \leq)$ where $\leq$ divides a complemented lattice.

least element = 1 (0)

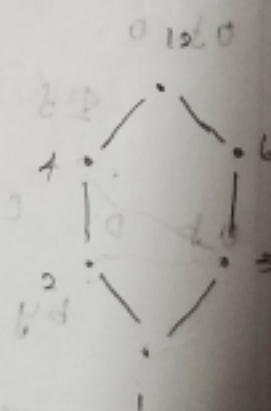
Greatest element = 12 (1)

$$3 * 4 = 1$$

$$3 \oplus 4 = 12$$

$$3 * 4 = 0$$

$$3 \oplus 4 = 1$$



ii

3, 2, 4 are complements of each other.

Now

$$2 * 3 = 1$$

$$2 \oplus 3 = 6 \neq 12$$

$$2 * 3 = 0$$

$$2 \oplus 3 \neq 1$$

$\therefore 3$ is not the complement of 2 .

$$2 * 4 = 2 \neq 1$$

$$2 \oplus 4 \neq 12$$

$$2 * 4 \neq 0$$

$$2 \oplus 4 \neq 1$$

A is not a complemented lattice.

2. Verify whether $(L, \leq)$ is a distributive lattice where $L = \{1, 2, 3, 5, 30\}$ and $\leq$ is divides.

$$2 \oplus (5 * 3) = 2 \oplus 1 = 2 \quad \text{--- (1)}$$

$$2 \oplus 5 = 30, \quad 2 \oplus 3 = 30$$

$$(2 \oplus 5) * (2 \oplus 3) = 30 * 30 = 30 \quad \text{--- (2)}$$

$$2 \oplus (5 * 3) \neq (2 \oplus 5) * (2 \oplus 3)$$

$\therefore (L, \leq)$ is not a distributive lattice.

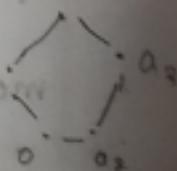
3. Verify whether the lattice is distributive.

$$a_3 * (a_1 \oplus a_2) = a_3 * 1 = a_3 \quad \text{--- (1)}$$

$$a_3 * a_1 = 0, \quad a_3 * a_2 = a_2$$

$$(a_3 * a_1) \oplus (a_3 * a_2) = 0 \oplus a_2 = a_2 \quad \text{--- (2)}$$

$$\therefore a_3 * (a_1 \oplus a_2) \neq (a_3 * a_1) \oplus (a_3 * a_2)$$



$$a_1 * (a_2 \oplus a_3) = a_1 * a_3 = 0 \quad \text{--- (3)}$$

$$(a_1 * a_2) \oplus (a_1 * a_3) = 0 \oplus 0 = 0 \quad \text{--- (4)}$$

$$a_1 * (a_2 \oplus a_3) = (a_1 * a_2) \oplus (a_1 * a_3)$$

$$a_2 * (a_1 \oplus a_3) = a_2 * 1 = a_2 \quad \text{--- (5)}$$

$$(a_2 * a_1) \oplus (a_2 * a_3) = 0 \oplus a_3 = a_3 \quad \text{--- (6)}$$

$$\therefore a_2 * (a_1 \oplus a_3) \neq (a_2 * a_1) \oplus (a_2 * a_3)$$

$\therefore$ The given lattice is not a distributive.

4. Verify whether the given lattice is distributive.

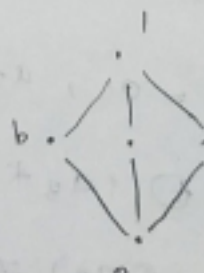
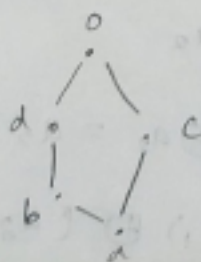
$$b_1 * (b_2 \oplus b_3) = b_1 * 1 = b_1 \quad \text{--- (1)}$$

$$(b_1 * b_2) \oplus (b_1 * b_3) = 0 \oplus 0 = 0 \quad \text{--- (2)}$$

$$\therefore b_1 * (b_2 \oplus b_3) \neq (b_1 * b_2) \oplus (b_1 * b_3)$$

$\therefore L$ is not a distributive lattice.

5. Find which of the following are modular lattice.



is (i) $b \leq d$, $b \oplus (c * d) = b \oplus a = b$ — (1)

$$(b \oplus c) * d = e * d = d \quad \text{--- (2)}$$

$$b \oplus (c * d) \neq (b \oplus c) * d$$

(ii) is not a modular lattice.

is (iii) $0 \leq d$, $0 \oplus (c * b) = 0 \oplus 0 = 0$

$$(0 \oplus c) * (0 \oplus b) = c \oplus b = 0$$

$$0 \leq d, 0 \oplus (c * d) = 0 \oplus 0 = 0$$

$$(0 \oplus c) * d = c * d = 0$$

Thus for x, y, z is the given lattice.

$$x \leq y \Rightarrow x \oplus (y * z) = z \oplus y$$

(ii) is modular lattice.

3) b. Given an example for a modular lattice which is not distributive.

$$a \oplus (b * c) = a \oplus c = 0$$

$$(a \oplus b) * (a \oplus c) = 1 * 1 = 1$$

$$\therefore a \oplus (b * c) \neq (a \oplus b) * (a \oplus c)$$

$\therefore$ This is not a distributive lattice. But it is a modular lattice $\Rightarrow a \oplus (b * c) = (a \oplus b) * c$

Complement	Bound	bounded lattice.	
$a * b = 0$	$a \oplus 0 = a$	$a \oplus a = a$	$a * a = a$
$a \oplus b = 1$	$a \oplus a = a$	$a \oplus 1 = 1$	$a * b = 0$

7. Given an example for a distributive lattice let x be any set and $P(x)$ be its power set defined $*$ and $\oplus$ by $\cap$ and $\cup$. Then $(P(x), \cap, \cup)$ is a distributive lattice since

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

8. Show that the De Morgan's laws given by $(a * b)' = a' \oplus b'$, $(a \oplus b)' = a' * b'$ hold in a complemented distributive lattice.

WkE

b is the complement of a

$$\text{If } a * b = 0, a \oplus b = 1$$

To show that

$a' \oplus b'$ is the complement of $a * b$

It is enough to show that

$$(a * b) * (a' \oplus b') = 0 \text{ and}$$

$$(a * b) \oplus (a' \oplus b') = 1$$

Now

$$(a * b) * (a' \oplus b')$$

$$= (a * b * a') \oplus (a * b * b') \text{ distributive}$$

$$= (a * a' * b) \oplus (a * b * b') \text{ commutative}$$

$$= (0 * b) \oplus (a * b)$$

$$= 0 \oplus 0$$

$$= 0$$

$$(a * b) \oplus (a' \oplus b')$$

$$= (a' \oplus b') \oplus a * b \text{ commutative}$$

$$= (a' \oplus b' \oplus a) * (a' \oplus b' \oplus b)$$

$$= (a' \oplus a \oplus b') * (a' \oplus b' \oplus b)$$

$$= (1 \oplus b') * (a' \oplus 1)$$

$$= 1 * 1$$

$$= 1$$

$$(a * b)' = a' \oplus b'$$

By duality $(a \oplus b)' = a' * b'$