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Games and Strategies

"When individuals and groups do not exercise self-restraint, the constitution should have to tell them when to stop"

17:1. INTRODUCTION

Many practical problems require decision-making in a *competitive situation* where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent. For example, candidates for an election, advertising and marketing campaigns by competing business firms, countries involved in military battles, etc. have their conflicting interests. In a competitive situation the courses of action (alternatives) for each competitor may be either finite or infinite. A competitive situation will be called a 'Game', if it has the following properties :

(*i*) There are a finite number of competitors (participants) called *players.*

(*ii*) Each player has a finite number of strategies (alternatives) available to him.

(*iii*) A play of the game takes place when each player employs his strategy.

(*iv*) Every game results in an outcome, *e.g.*, loss or gain or a draw, usually called *payoff*, to some player.

17:2. TWO-PERSON ZERO-SUM GAMES

When there are two competitors playing a game, it is called a 'two-person game'. In case the number of competitors exceeds two, say n, then the game is termed as a 'n-person game'.

Games having the 'zero-sum' character that the algebraic sum of gains and losses of all the players is zero are called *zero-sum games*. The play does not add a single paisa to the total wealth of all the players; it merely results in a new distribution of initial money among them. Zero-sum games with *two* players are called *two-person zero-sum* games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non-zero sum character or simply a *non-zero sum game*.

17:3. SOME BASIC TERMS

1. *Player.* The competitors in the game are known as players. A player may be individual or group of individuals, or an organisation.

2. *Strategy.* A strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt. A strategy may be of two types :

(*a*) *Pure strategy*. If the players select the same strategy each time, then it is referred to as pure-strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

(*b*) *Mixed strategy*. When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize expected gains or to minimize expected losses.

3. *Optimum strategy*. A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

4. *Value of the game*. It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair, if it is non-zero.

5. *Payoff matrix*. When the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called the *payoff matrix*. Since the game is zero-sum, therefore gain of one player is equal to the loss of other and vice-versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff only for one of the players.

Let player A have m strategies $A_1, A_2, \ldots, A_m$ and player B have n strategies $B_1, B_2, \ldots, B_n$. Here, it is assumed that each player has his choices from amongst the pure strategies. Also it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to player A is :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player A} & \begin{array}{c|cccc} & B_1 & B_2 & \cdots & B_n \\ \hline A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \end{array}$$

The payoff matrix to player B is $(-a_{ij})$.

Example. Consider a two-person coin tossing game. Each player tosses an unbiased coin simultaneously. Player B pays Rs. 7 to A, if $\{H, H\}$ occurs and Rs. 4, if $\{T, T\}$ occurs; otherwise player A pays Rs. 3 to B. This two-person game is a zero-sum game, since the winnings of one player are the losses for the other. Each player has his choices from amongst two *pure* strategies H and T. If we agree conventionally to express the outcome of the game in terms of the payoffs to one player only, say A, then the above information yields the following payoff matrix in terms of the payoffs to the player A. Clearly, the entries in B's payoff matrix will be just the negative of the corresponding entries in A's payoff matrix so that the sum of payoff matrices for player A and player B is ultimately a null matrix. We generally display the payoff matrix of that player who is indicated on the left side of the matrix. For example, A's payoff matrix may be displayed as below :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player A} & \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{pmatrix} 7 & -3 \\ -3 & 4 \end{pmatrix} \end{array} \end{array}$$

17:4. THE MAXIMIN-MINIMAX PRINCIPLE

We shall now explain the so-called *Maximin-Minimax Principle* for the selection of the optimal strategies by the two players.

For player A, minimum value in each row represents the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the *maximin principle*, and the corresponding gain is called the *maximin value of the game*.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player B is called the *minimax principle*, and the corresponding loss is the *minimax value of the game*.

If the maximin value equals the minimax value, then the game is said to have a *saddle (equilibrium) point* and the corresponding strategies are called *optimum strategies*. The amount of payoff at an equilibrium point is known as the *value of the game*.

To illustrate the maximin-minimax principle, let us consider a two-person zero-sum game with the following 3×2 payoff matrix for player A :

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 9 & 2 \\ 8 & \mathbf{6}^{*\dagger} \\ 6 & 4 \end{pmatrix} \end{array}$$

Let the pure strategies of the two players be designated by

$$S_A = \{A_1, A_2, A_3\} \quad \text{and} \quad S_B = \{B_1, B_2\}.$$

Suppose that player A starts the game knowing fully well that whatever strategy he adopts, B will select that particular counter strategy which will minimize the payoff to A. Thus, if A selects the strategy A_1, then B will reply by selecting B_2, as this corresponds to the *minimum* payoff to A in the first row corresponding to A_1. Similarly, if A chooses the strategy A_2, he may gain 8 or 6 depending upon the strategy chosen by B. However, A can guarantee a gain of at least min. $\{8, 6\} = 6$ regardless of the strategy chosen by B. In other words, whatever strategy A may adopt he can guarantee only the minimum of the corresponding row payoffs. Naturally, A would like to maximise his minimum assured gain. In this example the selection of strategy A_2 gives the maximum of the minimum gains to A. We shall call this gain as the *maximin value* of the game and the corresponding strategy as the *maximin strategy*. The maximin value is indicated in bold type with a star.

On the other hand, player B wishes to minimize his losses. If he plays strategy B_1, his loss is at the most max. $\{9, 8, 6\} = 9$ regardless of what strategy A has selected. He can lose no more than max. $\{2, 6, 4\} = 6$ if he plays B_2. This minimum of the maximum losses will be called the *minimax value* of the game and the corresponding strategy the *minimax strategy*. The minimax value is indicated in bold type marked with [†]. We observe that in the present example the maximum of row minima is equal to the minimum of the column maxima. In symbols,

$$\max_i . \{r_i\} = 6 = \min_j . \{c_j\}$$

or

$$\max_i . [\min_j . \{a_{ij}\}] = 6 = \min_j . [\max_i . \{a_{ij}\}],$$

where $i = 1, 2, 3$ and $j = 1, 2$.

Theorem 17-1. *Let (a_{ij}) be the $m \times n$ payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then $\overline{v} \geq \underline{v}$. That is,*

$$\min_{1 \leq j \leq n} . [\max_{1 \leq i \leq m} . \{a_{ij}\}] \geq \max_{1 \leq i \leq m} . [\min_{1 \leq j \leq n} . \{a_{ij}\}].$$

Proof. We have

$$\max_{1 \le i \le m} \{a_{ij}\} \ge a_{ij} \quad \text{for all } j = 1, 2, \ldots, n$$

and

$$\min_{1 \le j \le n} \{a_{ij}\} \le a_{ij} \quad \text{for all } i = 1, 2, \ldots, m$$

Let the above maximum be attained at $i = i'$ and the minimum be attained at $j = j'$, *i.e.*,

$$\max_{1 \le i \le m} \{a_{ij}\} = a_{i'j} \quad \text{and} \quad \min_{1 \le j \le n} \{a_{ij}\} = a_{ij'}$$

Then, we must have

$$a_{i'j} \ge a_{ij} \ge a_{ij'} \quad \text{for all } i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, n$$

From this, we get

$$\min_{1 \le j \le n} \{a_{i'j}\} \ge a_{ij} \ge \max_{1 \le i \le m} \{a_{ij'}\} \quad \text{for all } i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, m$$

$$\therefore \quad \min_{1 \le j \le n} \max_{1 \le i \le m} \{a_{ij}\}] \ge \max_{1 \le i \le m} [\min_{1 \le j \le n} \{a_{ij}\}]$$

or

$$\overline{v} \ge \underline{v}.$$

Remarks 1. A game is said to be fair, if $\underline{v} = 0 = \overline{v}$.

2. A game is said to be strictly determinable, if $\underline{v} = v = \overline{v}$.

Rule for determining a Saddle Point

We may now summarize the procedure of locating the saddle point of a payoff matrix as follows :

Step 1. Select the minimum element of each row of the payoff matrix and mark them [*].

Step 2. Select the greatest element of each column of the payoff matrix and mark them [†].

Step 3. If there appears an element in the payoff matrix marked [*] and [†] both, the position of that element is a saddle point,of the payoff matrix.

SAMPLE PROBLEMS

1701. *Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable games :*

$$(a) \quad \begin{array}{cc} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \end{array} \qquad (b) \quad \begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix} \end{array}$$

[Madurai M.Com. 1997]

Solution. (*a*) The payoff matrix for player *A* is

Player A	*Player B*		*Row minima*
	B_1	B_2	
A_1	$5^{\dagger}$	0^*	0
A_2	0^*	$2^{\dagger}$	0
Column maxima	5	2	

The payoffs marked with [*] represent the minimum payoff in each row and those marked with [†] represent the maximum payoff in each column of the payoff matrix. The largest component of row minima represents $\underline{v}$ (maximin value) and the smallest component of column maxima represents $\overline{v}$ (minimax value).

Thus obviously, we have

$$\underline{v} = 0 \text{ and } \overline{v} = 2.$$

Since $\underline{v} \ne \overline{v}$, the game is not strictly determinable.

(b) Here, the payoff matrix for player A is

Player A	Player B		Row minima
	B_1	B_2	
A_1	$0^{*\dagger}$	2	0
A_2	-1^{*}	$4^{\dagger}$	-1
Column maxima	0	4	

Since, the payoffs marked with [*] represent the minimum payoff in each row and those marked with [†] the maximum payoff in each column of the payoff matrix, we have

$$\underline{v}\text{ (maximin value)} = 0 \quad \text{and} \quad \overline{v}\text{ (minimax value)} = 0.$$

As $\underline{v} = \overline{v} = 0$, the game is strictly determinable and fair. Optimum strategies for players A and B are given by

$$S_0 = (A_1, B_1).$$

1702. *Solve the game whose payoff matrix is given by*

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

[Bharathidasan B.Com. 1999]

Solution. Consider the set of pure strategies

$$\alpha = \{A_1, A_2, A_3\} \text{ for player } A \quad \text{and} \quad \beta = \{B_1, B_2, B_3\} \text{ for player } B.$$

Assume that player B starts the game knowing fully well that whatever strategy he adopts, A will counter with a strategy that will minimize the payoff to B. Thus, if B selects B_1, then A will reply by selecting A_1 or A_2 as this corresponds to the minimum payoff to B in the first row corresponding to B_1. Similarly, if B chooses the strategy B_2, he may loose 4 or 3 or may neither loose nor gain depending upon the strategy chosen by A. However, B is assured of a gain of at least min. $\{0, -4, -3\}$; *i.e.*, -4 regardless of the strategy chosen by A. In other words, whatever strategy B may adopt, he can be assured of only the minimum of the corresponding row payoffs These corresponding to $B_i \in \beta$ are indicated by forming a column vector $r = \{1, -4, -1\}$ of the *row minima*. Naturally, B would like to maximize his minimum gain, which is just the largest component of r. Thus, maximum value of the game is maximum of $\{1, -4, -1\}$, *i.e.*, 1 which corresponds to B_1, the *maximin strategy*.

On the other hand, player A wishes to minimize his losses. If he plays strategy A_1, his loss is at the most maximum of $\{1, 0, 1\}$, *i.e.*, 1 regardless of what strategy B has adopted. He loses no more than max. $\{3, -4, 5\}$, if he plays A_2 and no more then max. $\{1\ -3, -1\}$ if he plays A_3. These maximum losses, corresponding to each $A_i \in \alpha$ are indicated by forming a row vector $c = (1, 5, 1)$ of the column maxima. The smallest component of c represents the minimum possible loss to A whatever strategy B may adopt. Thus, the minimax value of the game is min. (1, 5, 1), *i.e.*, which corresponds to A_1 and A_3, the *minimax strategies.*

The maximin value is generally marked by {*} and the minimax value by {†} as shown below :

$$\begin{array}{cccc|c} & A_1 & A_2 & A_3 & \mathbf{r} \\ B_1 & 1^{*\dagger} & 3 & 1^{*\dagger} & 1^{*} \\ B_2 & 0 & -4^{*} & -3 & -4 \\ B_3 & 1^{\dagger} & 5^{\dagger} & -1^{\dagger} & -1 \\ c: & 1^{\dagger} & 5 & 1^{\dagger} & \end{array}$$

We observe from the above that there exist two saddle points (having * and † both) at positions (1, 1) and (1, 3). Thus, the solution to the game is given by

(*i*) the optimum strategy for player B is B_1,

(*ii*) the optimum strategies for player A are A_1 and A_3,

(*iii*) the value of game is 1 for B and -1 for A.

Note : Since $v \neq 0$, the game is not fair, although it is strictly determinable.

1703. *Determine the range of value of p and q that will make the payoff element a_{22}, a saddle point for the game whose payoff matrix (a_{ij}) is given below :*

$$\begin{array}{cc} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 2 & 4 & 7 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \end{array}$$

Solution. Let us first of all ignore the values of p and q and determine the maximin and minimax values of the payoff matrix. For this, we have

$$\begin{array}{cccccc} & B_1 & B_2 & B_3 & & \textit{Row minima} \\ A_1 & 2 & 4 & 5 & & 2 \\ A_2 & 10 & 7 & q & & 7 \\ A_3 & 4 & p & 8 & & 4 \\ \textit{Column maxima} & 10 & 7 & 8 & & \end{array}$$

Obviously, the maximin value ($\underline{v}$) is 7 and the minimax value ($\overline{v}$) is also 7. Thus, there exists a saddle point at position (2, 2).

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$.

Hence, the required range of values of p and q is

$$7 \leq q, \quad p \leq 7.$$

PROBLEMS

1704. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games :

(*a*)
$$\begin{array}{ccc} & & \textit{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(*b*)
$$\begin{array}{ccc} & & \textit{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 10 & 6 \\ 8 & 2 \end{bmatrix} \end{array}$$

[Madurai M.Com. 1993]

1705. Consider the game G with the following payoff matrix :

$$\begin{array}{ccc} & & \textit{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

(*a*) Show that G is strictly determinable whatever μ may be.

(*b*) Determine the value of G. ***[Amravathi B.E. (Rul.) 1994]***

1706. For the game with payoff matrix :

$$\begin{array}{cc} & \textit{Player } A \\ \text{Player } B & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

determine the best strategies for players A and B and also the values of the game for them. Is this game (*i*) fair? (*ii*) strictly determinable?

1707. For what value of λ, the game with following payoff matrix is strictly determinable?

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

1708. Solve the following 2-person zero-sum game :

(a)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix} \end{array}$$

[Madurai M.Com. 2003]

(b)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} -2 & 12 & -4 \\ 1 & 4 & 8 \\ -5 & 2 & 3 \end{bmatrix} \end{array}$$

[Delhi M.B.A. 2011]

1709. Solve the game whose pay-off matrix is given by

(a)

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} & \begin{bmatrix} 20 & 15 & 12 & 35 \\ 25 & 14 & 5 & 10 \\ 40 & 2 & 8 & 5 \\ -5 & 4 & 0 & 0 \end{bmatrix} \end{array}$$

[Delhi PG Dip. in Glob. Bus. Oper. 2011]

(b)

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 8 & 4 & 1 \\ 10 & 6 & 2 \\ 12 & 8 & 3 \\ 4 & 4 & 4 \end{bmatrix} \end{array}$$

[Gujarat M.B.A. 2009]

1710. Solve the game whose payoff matrix is given below :

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

[Madras B.E. (Mech.) 2000; Delhi B.Sc. (Stat.) 1995]

1711. Solve the following game :

$$\begin{bmatrix} 7 & 8 & 4 & 6 & 8 \\ -8 & 6 & 1 & 9 & 6 \end{bmatrix}$$

[Madras M.C.A. (Oct.) 2002]

1712. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following payoff matrix describes the increase in market share for Firm *A* and the decrease in market share for Firm *B*. Determine the optimum strategies for each firm.

		Firm B		
		No promotion	*Moderate promotion*	*Much promotion*
Firm A	No promotion	5	0	−10
	Moderate promotion	10	6	2
	Much promotion	20	15	10

(*i*) Which firm would be the winner, in terms of market share?

(*ii*) Would the solution strategies necessarily maximize profits for either of the firms? *[Delhi M.B.A. (April) 1999]*

1713. Two competitive manufacturers are producing a new toy under licence from a patent holder. In order to meet the demand, they have the option of running the plant for 8, 16 or 24 hours a day. As the length of production increases, so does the cost. One of the manufacturers say *A*, has set up the matrix given below, in which he estimates the percentage of the market that he could capture and maintain the different production schedules :

Manufacturer A	*Manufacturer B*		
	C_1 : 8 *hrs.*	C_2 : 16 *hrs.*	C_3 : 24 *hrs.*
S_1 : 8 hrs.	60%	56%	34%
S_2 : 16 hrs.	63%	60%	55%
S_3 : 24 hrs.	83%	72%	60%

(*i*) At which level should each produce?

(*ii*) What percentage of the market will manufacturer *B* have? *[Delhi PG Dip. in Glob. Bus. Oper. 2011]*

17:5. GAMES WITHOUT SADDLE POINTS—MIXED STRATEGIES

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position—the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called *mixed* strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents which least player A can expect to win and the least which player B can lose. The expected payoff to a player in a game with arbitrary payoff matrix (a_{ij}) of order $m \times n$ is defined as :

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i\, a_{ij}\, q_j = \mathbf{p}^T \mathbf{A} \mathbf{q}$$

where $\mathbf{p}$ and $\mathbf{q}$ denote the mixed strategies for players A and B respectively.

Maximin-Minimax Criterion. Consider an $m \times n$ game (a_{ij}) without any saddle point, *i.e.*, strategies are mixed. Let $p_1, p_2, ..., p_m$ be the probabilities with which player A will play his moves $A_1, A_2, ..., A_m$ respectively; and let $q_1, q_2, ..., q_n$ be the probabilities with which player B will play his moves $B_1, B_2, ..., B_n$ respectively. Obviously, $p_i \geq 0$ $(i = 1, 2, ..., m)$, $q_j \geq 0$ $(j = 1, 2, ..., n)$, and $p_1 + p_2 + ... + p_m = 1$; $q_1 + q_2 + ... + q_n = 1$.

The expected payoff function for player A, therefore, will be given by

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i\, a_{ij}\, q_j$$

Making use of maximin-minimax criterion, we have

For Player A.

$$\underline{v} = \max_{\mathbf{p}}.\ \min_{\mathbf{q}}.\ E(\mathbf{p}, \mathbf{q}) = \max_{\mathbf{p}}. \left[\min_{j}. \left\{ \sum_{i=1}^{m} p_i\, a_{ij} \right\} \right]$$

$$= \max_{\mathbf{p}}. \left[\min_{j}. \left\{ \sum_{i=1}^{m} p_i\, a_{i1}, \sum_{i=1}^{m} p_i\, a_{i2}, \ ..., \sum_{i=1}^{m} p_i\, a_{in} \right\} \right]$$

Here, $\min. \left\{ \sum_{i=1}^{m} p_i\, a_{ij} \right\}$ denotes the expected gain to player A, when player B uses his jth pure strategy.

For player B.

$$\overline{v} = \min_{\mathbf{q}}. \left[\max_{i}. \left\{ \sum_{j=1}^{n} q_j\, a_{1j}, \sum_{j=1}^{n} q_j\, a_{2j}, ..., \sum_{j=1}^{n} q_j\, a_{mj} \right\} \right].$$

Here $\max. \left\{ \sum_{j=1}^{n} q_j\, a_{ij} \right\}$ denotes the expected loss to player B when player A uses his ith strategy.

The relationship $\underline{v} \leq \overline{v}$ holds good in general and when p_i and q_j correspond to the optimal strategies the relation holds in 'equality' sense and the expected value for both the players becomes equal to the optimum expected value of the game.

Definition. *A pair of strategies* $(\mathbf{p}, \mathbf{q})$ *for which* $\underline{v} = \overline{v} = v$ *is called a **saddle point** of* $E(\mathbf{p}, \mathbf{q})$.

Theorem 17-2. *For any* 2×2 *two-person zero-sum game without any saddle point having the payoff matrix for player A*

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \end{array}$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix},$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

where $p_1 + p_2 = 1$ *and* $q_1 + q_2 = 1$. *The value* v *of the game to A is given by*

$$v = \frac{a_{11} a_{22} - a_{21} a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Proof. Let a mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, where $p_1 + p_2 = 1$. Thus, if player B moves B_1 the net expected gain of A will be

$$E_1(p) = a_{11} p_1 + a_{21} p_2$$

and if B moves B_2, the net expected gain of A will be

$$E_2(p) = a_{12} p_1 + a_{22} p_2.$$

Similarly, if B plays his mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$, where $q_1 + q_2 = 1$, then B's net expected loss will be

$$E_1(q) = a_{11} q_1 + a_{12} q_2$$

if A plays A_1, and

$$E_2(q) = a_{21} q_1 + a_{22} q_2$$

if A plays A_2.

The expected gain of player A, when B mixes his moves with probabilities q_1 and q_2 is, therefore, given by

$$E(\mathbf{p}, \mathbf{q}) = q_1 [a_{11} p_1 + a_{21} p_2] + q_2 [a_{12} p_1 + a_{22} p_2].$$

Player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

Now,
$$\begin{aligned} E(\mathbf{p}, \mathbf{q}) &= q_1 [a_{11} p_1 + a_{21}(1 - p_1)] + (1 - q_1)[a_{12} p_1 + a_{22}(1 - p_1)] \\ &= [a_{11} + a_{22} - (a_{12} + a_{21})] p_1 q_1 + (a_{12} - a_{22}) p_1 + (a_{21} - a_{22}) q_1 + a_{22} \\ &= \lambda \left(p_1 - \frac{a_{22} - a_{21}}{\lambda} \right) \left(q_1 - \frac{a_{22} - a_{12}}{\lambda} \right) + \frac{a_{11} a_{22} - a_{12} a_{21}}{\lambda}, \end{aligned}$$

where $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$.

We see that if A chooses $p_1 = \frac{a_{22} - a_{21}}{\lambda}$, he ensures an expected gain of at least $(a_{11} a_{22} - a_{12} a_{21})/\lambda$. Similarly, if B chooses $q_1 = \frac{a_{22} - a_{12}}{\lambda}$, then B will limit his expected loss to at most $(a_{11} a_{22} - a_{12} a_{21})/\lambda$. These choices of p_1 and q_1 will thus be optimal to the two players.

Thus, we get

$$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \text{and} \quad p_2 = 1 - p_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \text{and} \quad q_2 = 1 - q_1 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

and
$$v = \frac{a_{11} a_{22} - a_{12} a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Hence, we have

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \ \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}; \quad \text{and} \quad v = \frac{a_{11}\,a_{22} - a_{12}\,a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Note : The above formulae for p_1, p_2, q_1, q_2 and v are valid only for 2×2 games without saddle points.

SAMPLE PROBLEMS

1714. *For the game with the following payoff matrix, determine the optimum strategies and the value of the game :*

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

[ICSI (June) 1996; Madurai M.Com. (Nov.) 2002]

Solution. Clearly, the given matrix is without a saddle point. So, the mixed strategies of P_1 and P_2 are :

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix}, \ S_{P_2} = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix}; \quad p_1 + p_2 = 1 \ \text{ and } \ q_1 + q_2 = 1$$

If $E(p, q)$ denotes the expected payoff function, then

$$E(p, q) = 5p_1q_1 + 3(1-p_1)q_1 + p_1(1-q_1) + 4(1-p_1)(1-q_1)$$
$$= 5p_1q_1 - 3p_1 - q_1 + 4 = 5(p_1 - 1/5)(q_1 - 3/5) + 17/5.$$

If P_1 chooses $p_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $q_1 = 3/5$, P_2 can keep $E(p_1, q_1)$ down to 17/5. So P_1 might as well settle for 17/5 and P_2 reconcile to 17/5. Hence, the optimum strategies for P_1 and P_2 are

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ 1/5 & 4/5 \end{bmatrix}, \ S_{P_2} = \begin{bmatrix} 1 & 2 \\ 3/5 & 2/5 \end{bmatrix}$$

and the value of the game is $v = 17/5$.

1715. *Consider a "modified" form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Re. 1.00 if the coins turn both tails. The non-matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?*

[Delhi M.B.A. 1999, 2007]

Solution. The payoff matrix for the matching player is given by

$$\begin{array}{ccc} & & \textit{Non-matching Player} \\ & & \begin{array}{cc} H & T \end{array} \\ \text{Matching Player} & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Clearly, the payoff matrix does not possess any saddle point. The players will use mixed strategies. The optimum mixed strategy for matching player is determined by

$$p_1 = \frac{1 - (-3)}{8 + 1 - (-3 - 3)} = \frac{4}{15}, \quad p_2 = \frac{11}{15}$$

and for the non-matching player, by

$$q_1 = \frac{1 - (-3)}{8 + 1 - (-3 - 3)} = \frac{4}{15}, \quad q_2 = \frac{11}{15}.$$

The expected value of the game (corresponding to the above strategies) is given by

$$v = \frac{8 - 3(-3)(-3)}{8 + 1 - 1(-3 - 3)} = -\frac{1}{15}.$$

Thus, the optimum mixed strategies for matching player and non-matching player are given by

$$S_{match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix} \text{ and } S_{non\text{-}match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix}.$$

Clearly, we would like to be the non-matching player.

PROBLEMS

1716. Solve the following game and determine the value of the game :

(a)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array},$$

[Madras B.E. (Mech.) 1999]

(b)

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

[Allahabad M.B.A. 1998]

1717. In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A. *[Amravathi B.E. (Rul.) 1994]*

1718. Two players A and B match coins. If the coins match, then A wins two units of value, if the coins do not match, then B wins 2 units of value. Determine the optimum strategies for the players and the value of the game. *[Madras M.B.A. (Nov.) 2006; Delhi M.Com. 2008]*

1719. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players. *[Jodhpur M.Sc. (Math.) 1994]*

17:6. GRAPHIC SOLUTION OF $2 \times n$ AND $m \times 2$ GAMES

The procedure described in the last section will generally be applicable for any game with 2×2 payoff matrix unless it possesses a saddle point. Moreover, the procedure can be extended to any square payoff matrix of any order. But it will not work for the game whose payoff matrix happens to be a *rectangular* one, say $m \times n$. In such cases a very simple graphical method is available if either m or n is two. The graphic short-cut enables us to reduce the original $2 \times n$ or $m \times 2$ game to a much simpler 2×2 game. Consider the following $2 \times n$ game :

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \end{array}.$$

It is assumed that the game does not have a saddle point. Let the optimum mixed strategy for A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$. The average (expected) payoff for A when he plays S_A against B's pure moves $B_1, B_2, \ldots, B_n$ is given by

B's pure move	*A's expected payoff E (p)*
B_1	$E_1(p_1) = a_{11}p_1 + a_{21}p_2 = a_{11}p_1 + a_{21}(1 - p_1)$
B_2	$E_2(p_1) = a_{12}p_1 + a_{22}p_2 = a_{12}p_1 + a_{22}(1 - p_1)$
$\vdots$	$\vdots$
B_n	$E_n(p_1) = a_{1n}p_1 + a_{2n}p_2 = a_{1n}p_1 + a_{2n}(1 - p_1).$

According to the maximin criterion for mixed strategy games, player A should select the values of p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the expected payoff lines :

$$E_j(p_1) = (a_{1j} - a_{2j})\, p_1 + a_{2j} \qquad (j = 1, 2, \ldots, n).$$

The highest point on the *lower envelope* of these lines will give maximum of the minimum (*i.e.*, maximin) expected payoffs to player A as also the maximum value of p_i.

The two lines* passing through the *maximin* point identify the two *critical* moves of B which, combined with two of A, yield the 2×2 matrix that can be used to determine the optimum strategies of the two players, for the original game, using the results of the previous section.

The $(m \times 2)$ games are also treated in the same way where the upper envelope of the straight lines corresponding to B's expected payoffs will give the maximum expected payoff to player B and the lowest point on this then gives the minimum expected payoff (minimax value) and the optimum value of q_1.

SAMPLE PROBLEMS

1720. *Solve the following 2×2 game graphically :*

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player } A \quad \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Math.) 1996]

Solution. Clearly, the problem does not possess a saddle point. Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_2 = 1 - p_1$, against B. Then A's expected payoffs against B's pure moves are given by

B's pure move	*A's expected payoff $E(p_1)$*
B_1	$E_1(p_1) = p_1 + 1$
B_2	$E_2(p_1) = p_1$
B_3	$E_3(p_1) = -3p_1 + 3$
B_4	$E_4(p_1) = -4p_1 + 2$

These expected payoff equations are then plotted as functions of p_1 as shown in Fig. 17.1 which shows the payoffs of each column represented as points on two vertical axis *1* and 2, unit distance apart. Thus line B_1 joins the first payoff element 2 in the first column represented by +2 on axis 2, and the second payoff element 1 in the first column represented by +1 on axis *1*. Similarly, lines B_2, B_3 and B_4 join the corresponding representation of payoff elements in the second, third and fourth columns. Since the player A wishes to maximize his minimum expected payoff we consider the highest point of intersection H on the *lower envelope* of the A's expected payoff equations. This point H represents the *maximin* expected value of the game for A. The lines B_2 and B_4, passing through H, define the two relevant moves B_2 and B_4 that alone B needs to play. The solution to the original 2×4 game, therefore, boils down that of the simpler game with the 2×2 payoff matrix :

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \end{array}$$

*If there are more than two lines passing through the maximin point, there are ties for the optimum mixed strategies for player B. Thus any two such lines with *opposite sign* slopes will define an *alternative optimum* for B.

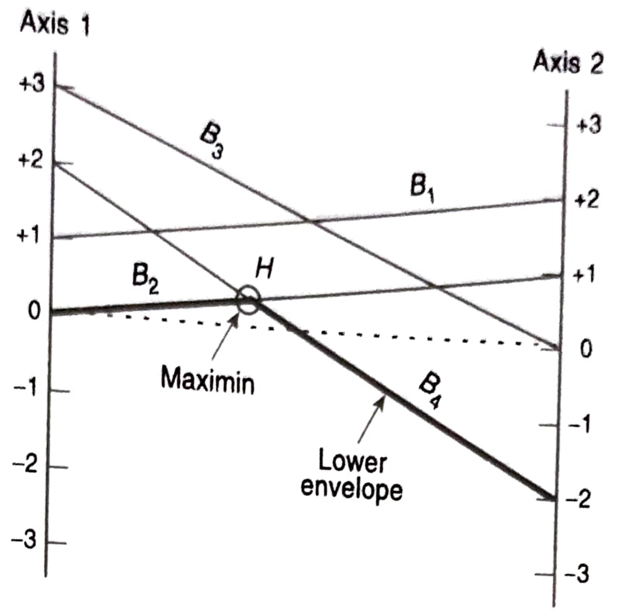

Fig. 17.1. The maximin value

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_2 & B_4 \\ q_2 & q_4 \end{bmatrix}$

be the optimum strategies for A and B, then we have

$$p_1 = \frac{2-0}{1+2-(-2)} = 2/5, \quad p_2 = 1 - p_1 = 3/5,$$

$$q_2 = \frac{2-(-2)}{1+2-(-2)} = 4/5, \quad q_4 = 1 - q_1 = 1/5.$$

Hence, the solution to the game is

(*i*) the optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$,

(*ii*) the optimum strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$

and (*iii*) the expected value of the game is $v = \dfrac{2\times 1 - 0\times(-2)}{1+2-(0-2)} = \dfrac{2}{5}$.

1721. *Obtain the optimal strategies for both-persons and the value of the game for zero-sum two-person game whose payoff matrix is as follows :*

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

[Guru Nanak Dev Univ. B.Com. 2006]

Solution. Clearly, the given problem does not possess any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then B's expected payoffs against A's pure moves are given by

A's pure move	*B's expected payoff* $E(q_1)$
A_1	$E_1(q_1) = 4q_1 - 3$
A_2	$E_2(q_1) = -2q_1 + 5$

A_3 $E_3(q_1) = -7q_1 + 6$

A_4 $E_4(q_1) = 3q_1 + 1$

A_5 $E_5(q_1) = 2$

A_6 $E_6(q_1) = -5q_1$

The expected payoff equations are then plotted as functions of q_1 as shown in Fig. 17.2 :

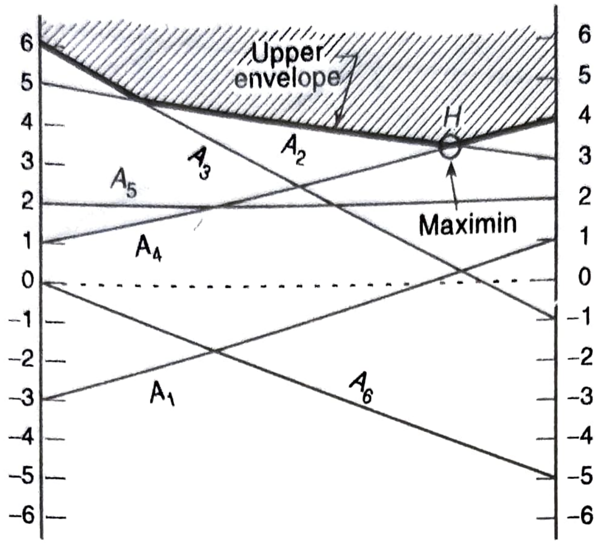

Fig. 17.2. The minimax value

Since, the player B wishes to minimize his maximum expected payoff, we consider the lowest point of intersection H on the *upper envelope* of B's expected payoff equations. This point H represents the minimax expected value of the game for player B. The lines A_2 and A_4 passing through H, define the two relevant moves A_2 and A_4 that alone the player A needs to play. The solution to the original 6×2 game, therefore, reduces to that of the simpler game with 2×2 payoff matrix :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

If we now, let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix}, \; p_1 + p_2 = 1 \; ; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \; q_1 + q_2 = 1$$

then using the usual method of solution for 2×2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}$$

and the value of the game as $v = 17/5$.

PROBLEMS

1722. Solve the following problem graphically :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

[Jodhpur M.Sc. (Math. 1993]

1723. Use graphical method in solving the following game :

$$\begin{array}{cc} & \textit{Player A} \\ \text{Player } B & \begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

[Panjab Tech. Univ. M.B.A. (Dec.) 2010]

Solve the following games graphically :

1724.

$$\begin{array}{c c} & \textit{Player B} \\ \text{Player A} & \begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix} \end{array}$$

[Jammu M.B.A. 2008]

1725.

$$\begin{array}{c c} & \textit{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} \end{array}$$

1726.

$$\begin{array}{c c c} & & \textit{B's strategy} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{A's strategy} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{bmatrix} \end{array}$$

[Gujarat M.B.A. 1998]

1727. (*a*)

$$\begin{array}{c c} & \textit{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{array}{c c} & B \\ A & \begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix} \end{array}$$

[Madras M.E. (Struct.) 2000]

1728.

$$\begin{array}{c c} & \textit{Player B} \\ \text{Player A} & \begin{bmatrix} -4 & 3 \\ -7 & 1 \\ -2 & -4 \\ -5 & -2 \\ -1 & -6 \end{bmatrix} \end{array}$$

[Karnataka B.E. (Ind.) 1994]

1729. Two firms *A* and *B* make colour and black & white television sets. Firm *A* can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs. 400 per colour set and Rs. 300 per black & white set. Firm *B* can, on the other hand, make either 300 colour sets, or 150 colour and 150 black & white sets, or 300 black & white sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set.

Write the payoff matrix of *A* per week. Obtain graphically *A*'s and *B*'s optimum strategies and value of the game. *[Bombay M.M.S. 1997; Delhi M.B.A. (Nov.) 2009]*

1730. A soft drink company calculated the market share of two products against its major competitor having three products and found out the impact of additional advertisement in any one of its products against the other :

$$\begin{array}{c c c} & & \textit{Competitor B} \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Company A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 & 7 & 15 \\ 20 & 12 & 10 \end{bmatrix} \end{array}$$

What is the best strategy for the company as well as the competitor? What is the payoff obtained by the company and the competitor in the long run? Use graphical method to obtain the solution. *[Meerut M.Sc. (Math.) 1999; Delhi M.B.A. (April) 1998]*

17:7. DOMINANCE PROPERTY

Sometimes, it is observed that one of the pure strategies of either player is always *inferior* to at least one of the remaining ones. The superior strategies are said to *dominate* the inferior ones. Clearly, a player would have no incentive to use inferior strategies which are dominated by the superior ones. In such cases of dominance, we can reduce the size of the payoff matrix by deleting those strategies which are dominated by the others. Thus if each element in one row, say kth of the payoff matrix (a_{ij}) is less than or equal to the corresponding elements in some other row, say rth, then player A will never choose kth strategy. In other words, probability $p_k = P$ (choosing the kth strategy) is zero, if $a_{kj} \leq a_{rj}$ for all $j = 1, \ldots, n$.

The value of the game and the non-zero choice of probabilities remain unchanged even after the deletion of kth row from the payoff matrix. In such a case the kth strategy is said to be dominated by the rth one.

General rules for dominance are :

(a) If all the elements of a row, say kth, are less than or equal to the corresponding elements of any other row, say rth, then kth row is *dominated* by rth row.

(b) If all the elements of a column, say kth are greater than or equal to the corresponding elements of any other column, say rth, then kth column is dominated by the rth column.

(c) Dominated rows or columns may be deleted to reduce the size of payoff matrix, as the optimal strategies will remain unaffected.

The *Modified Dominance Property*. The dominance property is not always based on the superiority of pure strategies only. A given strategy can also be said to be dominated if it is inferior to an average of two or more other pure strategies. More generally, if some convex linear combination of some rows dominates the ith row, then ith row will be deleted. Similar arguments follow for columns.

SAMPLE PROBLEMS

1731. *Two firms are competing for business under the condition so that one firm's gain is another firm's loss. Firm A's payoff matrix is given below :*

$$\begin{array}{lll} & & \textit{Firm B} \\ & & \begin{array}{ccc} \textit{No ad} & \textit{Medium ad} & \textit{Heavy ad} \end{array} \\ \textit{Firm A} & \begin{array}{l} \textit{No advertising} \\ \textit{Medium advertising} \\ \textit{Heavy advertising} \end{array} & \begin{bmatrix} 10 & 5 & -2 \\ 13 & 12 & 15 \\ 16 & 14 & 10 \end{bmatrix} \end{array}$$

Suggest optimum strategies for the two firms and the net outcome thereof. **[Delhi MIB 2011]**

Solution. Clearly, the first column is dominated by the second column as all the elements of the first column are greater than elements of second column. Thus eliminating first column, we get

$$\begin{array}{llll} & & & \textit{Firm B} \\ & & & \begin{array}{cc} \textit{Medium ad} & \textit{Heavy ad} \\ B_2 & B_3 \end{array} \\ \textit{Firm A} & \begin{array}{l} \text{No advertising} \\ \text{Medium advertising} \\ \text{Heavy advertising} \end{array} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 5 & -2 \\ 12 & 15 \\ 14 & 10 \end{bmatrix} \end{array}$$

Again, first row is dominated by second and third row as all the elements of first row are less than the respective elements of second, and third row. Hence eliminating first row, we obtain the following 2×2 payoff matrix.

$$\begin{array}{llll} & & & \textit{Firm B} \\ & & & \begin{array}{cc} \textit{Medium ad} & \textit{Heavy ad} \\ B_2 & B_3 \end{array} \\ \textit{Firm A} & \begin{array}{l} \text{Medium advertising} \\ \text{Heavy advertising} \end{array} & \begin{array}{c} A_2 \\ A_3 \end{array} & \begin{bmatrix} 12 & 15 \\ 14 & 10 \end{bmatrix} \end{array}$$

Since the reduced payoff matrix do not have any saddle point, the strategies are mixed.

So, let

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}, \quad p_1+p_2+p_3 = q_1+q_2+q_3 = 1$$

Using the usual method for the solution of 2×2 payoff matrices, the optimum strategies for the two players and the value of the game can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 4/7 & 3/7 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 5/7 & 2/7 \end{bmatrix} \text{ and } v = 90/7.$$

Hence, firm A should adopt strategy A_2 and A_3 with 57% of time and 43% of time respectively (or with 57% and 43% probability on any one play of the game respectively). Similarly, firm B should adopt strategy B_2 and B_3 with 71% of time and 29% of time respectively (or with 71% and 29% probability on any one play of the game respectively).

1732. *Solve the following game :*

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{array}{c|cccc} & I & II & III & IV \\ \hline I & 3 & 2 & 4 & 0 \\ II & 3 & 4 & 2 & 4 \\ III & 4 & 2 & 4 & 0 \\ IV & 0 & 4 & 0 & 8 \end{array} \end{array}$$

[Madurai M.Com. 2003; Panjab M.C.A. 2007]

Solution. From the above payoff matrix, we observe that first row is dominated by third row and first column is dominated by third column. The reduced payoff matrix is

$$\begin{array}{c|ccc} & II & III & IV \\ \hline II & 4 & 2 & 4 \\ III & 2 & 4 & 0 \\ IV & 4 & 0 & 8 \end{array}$$

Now, none of the pure strategies of player B is inferior to any of his other strategies. However, a *convex linear combination* (average) of strategies *III* and *IV* dominates strategy *II* of player B, yielding the reduced payoff matrix

$$\begin{array}{c|cc} & III & IV \\ \hline II & 2 & 4 \\ III & 4 & 0 \\ IV & 0 & 8 \end{array}$$

Again, we observe that none of the pure strategies of player A is inferior to any of his other strategies. However, a convex linear combination of strategies *III* and *IV* dominates strategy *II* of player A, yielding the reduced payoff matrix

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{array}{c|cc} & III & IV \\ \hline III & 4 & 0 \\ IV & 0 & 8 \end{array} \end{array}$$

Now, letting

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & p_1 & p_2 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_1 & q_2 \end{bmatrix},$$

where $p_1 + p_2 = 1$, $q_1 + q_2 = 1$, and then using the method of solving 2×2 games, we can easily obtain the optimum strategies as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

and the value of game as $v = 8/3$.

1732. (*a*) *Solve the game whose payoff matrix is given below :*

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \\ -2 & 1 & -2 & 0 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2006]

Solution. Clearly A_3 is dominated by A_2. Thus, the deletion of third row yields the reduced payoff matrix

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} B \\ \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

Now, let A play the strategies A_1 and A_2 with respective probabilities p_1 and $1-p_1$. Then, the expected payoff for A, when B chooses to play B_1, shall be $4p_1 - (1-p_1)$, *i.e.*, $5p_1 - 1$. Similarly, payoffs in respect of other strategy plays can be determined. These are presented graphically in Fig. 17.3. Point H represents the maximin expected value of game to player A :

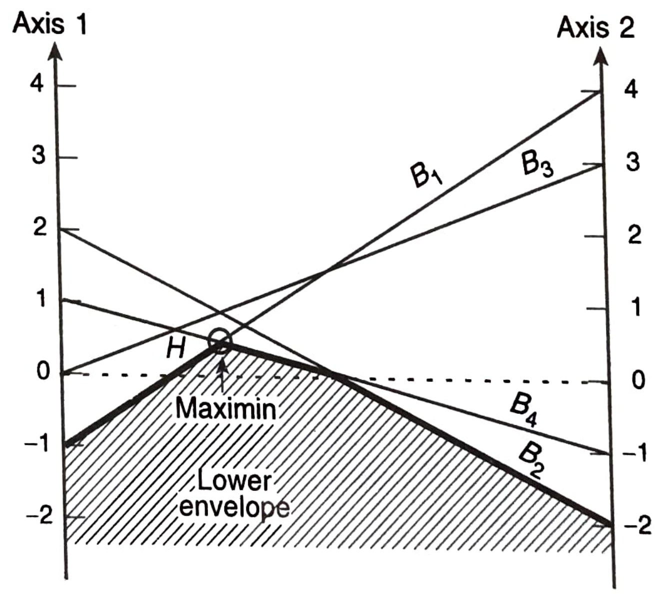

Fig. 17.3

Alternatively, the solution to the original game reduces to that of the simpler game with 2×2 payoff matrix as

$$\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

If, we let

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}, \quad p_1 + p_2 = 1 \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \quad q_1 + q_2 = 1;$$

then the optimum strategies can easily be determined as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/7 & 5/7 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 2/7 & 0 & 0 & 5/7 \end{bmatrix}; \text{ and } v = 3/7.$$

1733. *Two firms A and B have for years been selling a competitive product which forms a part of both firms' total sales. The marketing executive of firm A raised the question. "What should be the firm's strategies in terms of advertising for the product in question?" The market research team of firm A developed the following data for varying degrees of advertising :*

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm A with no advertising : 40% of the market with medium advertising by firm B and 28% of the market with large advertising by firm B.

(iii) Firm A using medium advertising : 70% of market with no advertising by firm B and 45% of the market with large advertising by firm B.

(iv) Firm A using large advertising : 75% of the market with no advertising by firm B and 47.5% of the market with medium advertising by firm B.

(a) Based upon the foregoing information, answer the marketing executive's questions.

(b) What advertising policy should firm A pursue when consideration is given to the above factors : selling price Rs. 4.00 per unit; variable cost of product Rs. 2.50 per unit; annual volume of 30,000 units for firm A; cost of annual medium advertising Rs. 5,000 and cost of annual large advertising Rs. 15,000? What contribution, before other fixed costs, is available to the firm?

[Delhi M.B.A. 2000]

Solution. (*a*) Using the given information, the pay-off matrix of the game between firms *A* and *B* is as follows :

Firm A	*Firm B*		
	No advertising	*Medium advertising*	*Large advertising*
No Advertising	50	40	28
Medium Advertising	70	50	45
Large Advertising	75	47.5	50

Clearly, the first row is dominated by the second row and first column is dominated by the second column. Eliminating first row and first column, the reduced pay-off matrix, therefore, is

$$\begin{array}{c c} & \begin{array}{cc} \textit{Medium Adv.} & \textit{Large Adv.} \end{array} \\ \begin{array}{c} \textit{Medium Adv.} \\ \textit{Large Adv.} \end{array} & \left[\begin{array}{cc} 50 & 45 \\ 47.5 & 50 \end{array}\right] \end{array}$$

Since, the reduced pay-off matrix do not have any saddle point, the strategies are mixed. So, let

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

and
$$p_1 + p_2 + p_3 = 1, \quad q_1 + q_2 + q_3 = 1.$$

Using the usual method for solving a 2×2 pay-off matrices, the optimum strategies for the two players and the value of the game are easily obtained :

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 1/3 & 2/3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 2/3 & 1/3 \end{bmatrix} \text{ and } v = 145/3.$$

Hence, the optimum strategy for firm *A* is to apply strategy A_2 (medium strategy) with probability 0.33 and strategy A_3 (large strategy) with probability 0.67 on any one play of the game. With this policy, the firm may expect to gain 145/3 (= 48.3) per cent of the market share.

(*b*) When the annual value of firm *A* is of 30,000 units, its market share is :

		Firm B		
		No Adv.	*Medium Adv.*	*Large Adv.*
	No Adv.	$0.50 \times 30{,}000 = 15{,}000$	$0.40 \times 30{,}000 = 12{,}000$	$0.28 \times 30{,}000 = 8{,}400$
Firm A	*Medium Adv.*	$0.70 \times 30{,}000 = 21{,}000$	$0.50 \times 30{,}000 = 15{,}000$	$0.45 \times 30{,}000 = 13{,}500$
	Large Adv.	$0.75 \times 30{,}000 = 22{,}500$	$0.475 \times 30{,}000 = 14{,}250$	$0.50 \times 30{,}000 = 15{,}000$

Given the expenditure on medium and large advertisements as Rs. 5,000 and Rs. 15,000 respectively, net profit to firm *A* can be calculated as follows :

Net profit = (Sales price − Cost price) × Sales volume − Advertising expenditure
= (4 − 2.50) × Sales volume − Advertising expenditure

The net profit to firm *A* is shown below :

		Firm B		
		No Advvertising	*Medium Advertising*	*Large Advertising*
Firm A	*No Adv.*	1.5 × 15,000 − 0 = 22,500	1.5 × 12,000 − 0 = 18,000	1.5 × 8,400 − 0 = 12,600
	Medium Adv.	1.5 × 21,000 − 5,000 = 26,500	1.5 × 15,000 − 5,000 = 17,500	1.5 × 13,500 − 5,000 = 15,250
	Large Adv.	1.5 × 22,500 − 15,000 = 18,750	1.5 × 14,250 − 15,000 = 6,375	1.5 × 15,000 − 15,000 = 7,500

From the above pay-off matrix, we observe the following :

(*i*) If firm *A* chooses the strategy of 'No advertising', then minimum profit is of Rs. 12,600, because firm *B* can adopt its strategy 'Large advertising'.

(*ii*) If firm *A* chooses the strategy of 'Medium advertising', then minimum profit is of Rs. 15,250 because firm *B* can again adopt its strategy 'Large advertising'.

(*iii*) If firm *A* chooses the strategy of 'Large advertising', then minimum profit is of Rs. 6,375 because firm *B* can adopt its strategy 'Medium advertising'.

Based on these observations, the firm *A* must adopt the policy of 'Medium advertising' to gain maximum profit of Rs. 15,250 among these three alternatives and must spend Rs. 5,000 for advertising.

PROBLEMS

1734. *A* and *B* play game in which each has three coins, a 5 p., 10 p. and a 20 p. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, *A* wins *B*'s coin. If the sum is even, *B* wins *A*'s coin. Find the best strategy for each player and the value of the game. *[Annamalai M.B.A. 2009]*

1735. Two competitors are competing for the market share of the similar product. The pay-off matrix in terms of their advertising plan is given below :

		Competitor B		
		No Advertising	*Medium Advertising*	*Large Advertising*
Competitor A	*No Advertising*	10	5	−2
	Medium Advertising	13	12	13
	Large Advertising	16	14	10

Suggest the optimal strategies for the two firms and the value of game for both the firms. *[Delhi M.B.A. 2009]*

1736. Find optimal strategies for two companies competing with each other on the basis of the following pay-off matrix showing gain or loss of customers :

Company A	*Company B*		
	Quantity discounts	*CSR initiatives*	*Events sponsorship*
Quantity Discounts	20	60	30
CSR Initiative	−10	30	25
Event Sponsorship	40	50	−30

(*i*) State the required assumptions for finding optimal strategies for two companies,

(*ii*) Find out value of the game. Use dominance rule.

[Delhi M.Com. 2010]