II UG - OT - II : Unit : III

"Integer Programming
(முழு நிரலாக்கம்)"

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7.1 : Introduction (Defn IP ?)

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[7.1 to 7.6]

Integer Programming

"God made the integers, rest is the work of man"

7:1. INTRODUCTION

A linear programming problem in which all or some of the decision variables are constrained to assume non-negative integer values is called an **Integer Programming Problem.** This type of problem is of particular importance in business and industry, where quite often, the fractional solutions are unrealistic because the units are not divisible. For example, it is absurd to speak of 2.3 men working on a project or 8.7 machines in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum values of the variables to the nearest integer values. But, it is generally inaccurate to obtain an integer solution by rounding off in this manner, for there is no guarantee that the deviation from the 'exact' integer solution will not be too large to retain the feasibility.

The linear programming problem with the additional requirements that the variables can take on only, integer values may have the following mathematical form

Maximize or Minimize $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ subject to the constraints :

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i, \qquad i = 1, 2, \dots, m$$

and

$$x_j \ge 0, \qquad j = 1, 2, \dots, n$$

where x_j are integer valued for $j = 1, 2, \dots, p \ (p \le n)$.

7:2. PURE AND MIXED INTEGER PROGRAMMING PROBLEMS

An integer programming problem in which all variables are required to be integers is called a *pure, or all-integer programming problem.* For example, the LPP

Minimize $z = 9x_1 + 10x_2$ subject to the constraints :

$$4x_1 + 3x_2 \ge 4, \ x_1 \le 8, \ x_2 \le 8;$$

where $x_1 \ge 0$, $x_2 \ge 0$ and are integers

is a pure integer linear programming problem.

On the other hand, an integer linear programming problem in which only some of the variables are required to be integers, is known as a **mixed integer programming problem.** For example, the LPP

Maximize $z = -3x_1 + x_2 + 3x_3$ subject to the constraints :

$$-x_1 + 2x_2 + x_3 \le 4, \ 4x_2 - 3x_3 \le 2;$$

where $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$ and x_1 as well as x_3 are integers

is a mixed-integer linear programming problem. Note that x_2 is not required to be an integer.

An integer programming problem in which all the variables must have integer values only zero or unity, is called the **zero-one integer programming problem.**

7:3. GOMORY'S ALL-I.P.P. METHOD

A systematic procedure for obtaining an optimum integer solution to an all-integer programming problem was first suggested by R.E. Gomory. His method starts without taking into consideration the integer requirements. If the solution so obtained is integral then the current solution is optimum. However if some of the basic variables are not integer valued then an additional linear constraint called *Gomory's constraint (fractional cut)* is generated. After having generated a linear constraint (cutting plane), it is added as the last row, of the optimum simplex table indicating that the solution is no longer feasible. The modified problem is then solved by using dual simplex method. An optimum integer solution is obtained if all the variables in the solution are integer valued, otherwise another Gomory constraint is added and the procedure is repeated. The optimum integer solution will be reached eventually after necessary new constraints have been added to drive away all the superior non-integer solutions.

7:4. CONSTRUCTION OF GOMORY'S CONSTRAINTS

To illustrate construction of a Gomory's constraint, let us consider a linear programming problem for which an optimum *non-integer* basic feasible solution has been attained as displayed in the simplex table below :

$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
$\mathbf{y}_2$	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}
$\mathbf{y}_3$	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}
z	y_{00}	y_{01}	y_{02}	y_{03}	y_{04}

Clearly, the optimum basic feasible solution is given by

$$\mathbf{x}_B = [x_2, x_3] = [y_{10}, y_{20}]\ ;\ \text{max. } z = y_{00}.$$

Since $\mathbf{x}_B$ is a non-integer solution, we assume, for the sake of exposition only, that y_{10} is *fractional.*

Now, the constraint equation

$$y_{10} = y_{11}x_1 + y_{12}x_2 + y_{13}x_3 + y_{14}x_4$$

reduces to

$$y_{10} = y_{11}x_1 + x_2 + y_{14}x_4. \quad ...(1)$$

Because x_2 and x_3 are basic variables, we must have $\mathbf{B} = \mathbf{I}_2 = (\mathbf{y}_2\ \ \mathbf{y}_3)$. This implies that $y_{12} = 1$ and $y_{13} = 0$.

Now, since $y_{10} \geq 0$, the fractional part of y_{10} must also be non-negative. We split over each of the y_{1j} in (1) into an integral part I_{1j}, and a non-negative fractional part*, f_{1j}, for $j = 0, 1, 2, 3, 4$. After this decomposition, (1) may be written as

$$I_{10} + f_{10} = (I_{11} + f_{11})x_1 + x_2 + (I_{14} + f_{14})x_4$$

or

$$f_{10} - f_{11}x_1 - f_{14}x_4 = x_2 + I_{11}x_1 + I_{14}x_4 - I_{10} \quad ...(2)$$

A comparison between (1) and (2) suggests that if we add an additional constraint in such a way that the L.H.S. of (2) is an integer, then we shall be forcing the non-integer y_{10} towards an integer (and hence the corresponding non-integer solution towards an integer one). This is what we precisely want. The desired Gomory's constraint is $f_{10} - f_{11}x_1 - f_{14}x_4 \leq 0$. To see the truth, let it be possible to have $f_{10} - f_{11}x_1 - f_{14}x_4 = h$, where $h > 0$ is an integer. Then $f_{10} = h + f_{11}x_1 + f_{14}x_4$ is

*If a y_{1j} is negative, even then a non-negative fractional part of it may be separated. For example, $-5/8$ can be written as $-1 + 3/8$ thereby giving $3/8$ as a fractional part (non-negative) of it.

obviously greater than one. This contradicts the fact that $0 < f_{1j} < 1$ for $j = 0, 1, 2, 3, 4$. Thus, the fractional cut is

$$f_{11}x_1 + f_{14}x_4 \geq f_{10} \quad \text{or} \quad -f_{11}x_1 - f_{14}x_4 \leq -f_{10}$$

or
$$-f_{11}x_1 - f_{14}x_4 + G_1 = -f_{10},$$

where G_1 is a slack variable in the above first Gomory constraint or fractional cut.

This additional constraint is to be included in the given L.P.P. in order to move further towards obtaining an optimum all-integer solution. After the addition of this constraint, the optimum simplex table looks like as given below :

y_B	x_B	y_1	y_2	y_3	y_4	G_1
y_1	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	0
y_2	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}	0
G_1	$-f_{10}$	$-f_{11}$	0	0	$-f_{14}$	1
z	y_{00}	y_{01}	y_{02}	y_{03}	y_{04}	y_{05}

Since $-f_{10}$ is negative, the optimum solution is infeasible and thus the dual simplex method is to be applied for obtaining an optimum feasible solution. After obtaining this solution, the above referred procedure is applied for constructing second fractional cut. The process is to be continued so long as an all-integer solution has not been obtained.

7:5. FRACTIONAL CUT METHOD—ALL INTEGER LPP

An iterative procedure for the solution of an all integer linear programming problem by Gomory's fractional cut method (or cutting plane method) may be summarized in the following steps :

Step 1. Express the given linear programming problem into its standard form and determine an optimum solution by using simplex method ignoring the integer value restriction.

Step 2. Test the integrality of the optimum solution.

(*a*) If the optimum solution admits all-integer values, an optimum basic feasible solution is attained.

(*b*) If the optimum solution does not include all-integer values, then move on to next *step*.

Step 3. Choose the largest fractional value of the basic variables. Let it be f_{k0}.

Step 4. Express each of the negative fractions, if any, in the kth row of the optimum simplex table as the sum of a negative integer and a non-negative fraction.

Step 5. Generate the Gomorian constraint (fractional cut) in the form

$$G_1 = -f_{k0} + f_{k1}x_1 + f_{k2}x_2 + \dots + f_{kn}x_n,$$

where $0 \leq f_{kj} < 1$ and $0 < f_{k0} < 1$.

Step 6. Add the Gomorian constraint generated in *step 5* at the bottom of the optimum simplex table. Use dual simplex method to find an improved optimum solution.

Step 7. Go to *step 2* and repeat the procedure until an optimum basic feasible all-integer solution is obtained.

SAMPLE PROBLEM

701. *Find the optimum integer solution to the following L.P.P. :*

$$\text{Maximize } z = x_1 + 4x_2 \text{ subject to the constraints :}$$

$$2x_1 + 4x_2 \leq 7,\ 5x_1 + 3x_2 \leq 15;\ x_1, x_2 \geq 0 \text{ and are integers.}$$

Solution.

Step 1. Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$, an initial basic feasible solution is $s_1 = 7$ and $s_2 = 15$. Using simplex method an optimum non-integer solution is obtained and is given in the following simplex table :

Initial Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4
4	y_2	7/4	1/2	1	1/4	0
0	y_4	39/4	7/2	0	–3/4	1
		z (= 7)	1	0	1	0

Step 2. Since the optimum solution is not integer valued, we consider only the fractional parts of $x_{B1} = \frac{7}{4}\left(= 1 + \frac{3}{4}\right)$ and $x_{B2} = \frac{39}{4}\left(= 9 + \frac{3}{4}\right)$.

Step 3. Maximum $\{f_1, f_2\}$ = maximum $\left\{\frac{3}{4}, \frac{3}{4}\right\} = \frac{3}{4}$, *i.e.*, both f_1 and f_2 are equal. So, we arbitrarily select any one of these. Let us choose f_2.

Step 4. In the second row, since $y_{23} = -3/4$, we write $y_{23} = -1 + 1/4$.

Step 5. Let G_1 be the first Gomorian slack. Then, we write

$$G_1 = -f_{20} + f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 = -\frac{3}{4} + \frac{1}{2}x_1 + 0 \cdot x_2 + \frac{1}{4}x_3 + 0 \cdot x_4$$

Step 6. Adding this additional constraint in the optimum simplex table, we have

First Iteration. Drop G_1 and introduce $\mathbf{y}_1$.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1
4	y_2	7/4	1/2	1	1/4	0	0
0	y_4	39/4	7/2	0	–3/4	1	0
0	G_1	–3/4	–1/2	0	–1/4	0	1
		z (= 7)	1	0	1	0	0

As $\mathbf{x}_{B3} = -3/4$ only is negative, this basic variable leaves the basis. Further, since max. $\left\{\frac{(z_j - c_j)}{y_{3j}}, y_{3j} < 0\right\}$ = max. $\left\{\frac{1}{-1/2}, \frac{1}{-1/4}\right\} = -2$, $\mathbf{y}_1$ enters the basis, *i.e.*, x_1 becomes basic variable in place of $\mathbf{G}_1$.

Second Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1
4	y_2	1	0	1	0	0	1
0	y_4	9/2	0	0	–5/2	1	7
1	y_1	3/2	1	0	1/2	0	-2
		z (= 11/2)	0	0	1/2	0	2

Since, the optimum solution is still non-integral, we introduce the second Gomorian constraint. Now,
$$\mathbf{x}_{B2} = \frac{9}{2}\left(= 4 + \frac{1}{2}\right) \text{ and } \mathbf{x}_{B3} = \frac{3}{2}\left(= 1 + \frac{1}{2}\right)$$

$\therefore$ maximum $\{f_2, f_3\}$ = maximum $\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$, *i.e.*, both f_2 and f_3 are equal. So, let us choose $f_2 = 1/2$ and write $y_{23} = -6 + 1/2$.

$$\therefore \quad G_2 = -f_{20} + f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 = -\frac{1}{2} + 0 \cdot x_1 + 0 \cdot x_2 + \frac{1}{2}x_3 + 0 \cdot x_4$$

Adding this additional constraint in the second iterative table, we have

Third Iteration. Drop G_2 and introduce y_3.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1	G_2
4	y_2	1	0	1	0	0	1	0
0	y_4	9/2	0	0	–5/2	1	7	0
1	y_1	3/2	1	0	1/2	0	–2	0
0	G_2	–1/2	0	0	–1/2*	0	0	1
	z	11/2	0	0	1/2	0	2	0

Final Iteration. Optimum Solution in Integers.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1	G_2
4	y_2	1	0	1	0	0	1	0
0	y_4	7	0	0	0	1	7	–5
1	y_1	1	1	0	0	0	–2	1
0	y_3	1	0	0	1	0	0	–2
	z	5	0	0	0	0	2	1

The table shows that an optimum basic feasible integer solution has been obtained. Hence, the optimum solution is

$$x_1 = 1,\ x_2 = 1 \text{ and maximum } z = 5.$$

702. *Solve the following integer linear programming problem using the cutting-plane algorithm :*

$$\text{Maximize} \quad z = 3x_1 + x_2 + 3x_3 \quad \text{subject to the constraints :}$$

$$-x_1 + 2x_2 + x_3 \le 4,\quad 4x_2 - 3x_3 \le 2,\quad x_1 - 3x_2 + 2x_3 \le 3\,;$$

$$x_1,\ x_2 \text{ and } x_3 \text{ all are non-negative integers.}$$

Solution. Introducing slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective constraints, an initial basic feasible solution is $s_1 = 4$, $s_2 = 2$ and $s_3 = 3$. Using, now, simplex method an optimum non-integer solution is obtained as given in the following simplex table :

Initial Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
3	y_3	10/3	0	0	1	4/9	1/9	4/9
1	y_2	3	0	1	0	1/3	1/3	1/3
3	y_1	16/3	1	0	0	1/9	7/9	10/9
	z (= 87/3)		0	0	0	2	3	5

Since, $x_{B1} = \frac{10}{3}\left(= 3 + \frac{1}{3}\right)$ and $x_{B3} = \frac{16}{3}\left(= 5 + \frac{1}{3}\right)$, we choose maximum $\{f_1, f_3\} = \frac{1}{3}$, *i.e.*, both f_1 and f_3 are equal. So, let us choose $f_1 = \frac{1}{3}$.

Let G_1 be the first Gomorian slack. Then, we write

$$G_1 = -f_{10} + f_{11}x_1 + f_{12}x_2 + f_{13}x_3 + f_{14}s_1 + f_{15}s_2 + f_{15}s_3$$

$$= -\frac{1}{3} + 0.x_1 + 0.x_2 + 0.x_3 + \frac{4}{9}s_1 + \frac{1}{9}s_2 + \frac{4}{9}s_3.$$

By the addition of this constraint in the above optimum simplex table, we get

First Iteration. Drop G_1 and introduce $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
						4/9	1/9	4/9	0
3	$\mathbf{y}_3$	(3 + 1/3)	0	0	0	1/3	1/3	1/3	0
1	$\mathbf{y}_2$	3	0	1	0	1/9	7/9	10/9	0
3	$\mathbf{y}_1$	(5 + 1/3)	1	0	1	–4/9	–1/9	–4/9	1
0	$\mathbf{G}_1$	–1/3	0	0	0	2	3	5	0
		z (= 87/3)	0	0	0				

Since, $\mathbf{x}_{B4}\left(=-\frac{1}{3}\right)$ only is negative, G_1 leaves the basis. Further, max. $\left\{\frac{2}{-4/9}, \frac{3}{-1/9}, \frac{5}{-4/9}\right\} = -\frac{9}{2}$. Therefore, $\mathbf{y}_4$ enters the basis, thereby, giving $y_{44}\left(=-\frac{4}{9}\right)$ as the leading element.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
					1	0	0	0	1
3	$\mathbf{y}_3$	3	0	0	0	0	1/4	0	3/4
1	$\mathbf{y}_2$	2 + 3/4	0	1	0	0	3/4	1	1/4
3	$\mathbf{y}_1$	5 + 1/4	1	0	0	1	1/4	1	–9/4
0	$\mathbf{y}_4$	3/4	0	0	0	0	5/2	3	9/2
		z (= 55/2)	0	0	0				

Here, max. $\left\{\frac{3}{4}, \frac{1}{4}\right\} = \frac{3}{4}$. This suggests that the second fractional cut (Gomorian constraint) will be :

$$G_2 = -\frac{3}{4} + \frac{1}{4}s_1 + \frac{3}{4}G_1$$

By adding this constraint in the optimum simplex table, we have

Second Iteration. Drop G_2 and introduce G_1.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$	$\mathbf{G}_2$
3	$\mathbf{y}_3$	3	0	0	1	0	0	0	1	0
1	$\mathbf{y}_2$	11/4	0	1	0	0	1/4	0	3/4	0
3	$\mathbf{y}_1$	21/4	1	0	0	0	3/4	1	1/4	0
0	$\mathbf{y}_4$	3/4	0	0	0	1	1/4	1	–9/4	0
0	$\mathbf{G}_2$	–3/4	0	0	0	0	–1/4	0	–3/4	1
		z (= 55/2)	0	0	0	0	5/2	3	9/2	0

Since $y_{57}\left(=-\frac{3}{4}\right)$ is the leading element, the next iterative simplex table is :

Final Iteration. Optimum Solution in Integers.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$	$\mathbf{G}_2$
3	$\mathbf{y}_3$	2	0	0	1	0	–1/3	0	0	4/3
1	$\mathbf{y}_2$	2	0	1	0	0	0	0	0	1
3	$\mathbf{y}_1$	5	1	0	0	0	2/3	1/9	0	1/3
0	$\mathbf{y}_4$	3	0	0	0	1	1	1	0	–3
0	$\mathbf{G}_1$	1	0	0	0	0	1/3	0	1	–4/3
		z (= 23)	0	0	0	0	1	1/3	0	6

This table shows that an optimum basic feasible integer solution has been reached. Hence, the optimum integer valued solution is :

$$x_1 = 5,\ x_2 = 2 \text{ and } x_3 = 2, \text{ with maximum } z = 23.$$

703. *A company manufactures three products P_1, P_2 and P_3 which yield per unit profit of Rs. 200, Rs. 400 and Rs. 300 respectively. Each of these products is processed on three different machines. The time required on each machine per unit of the product is given below :*

Product	*Time required (hours per unit)*		
	Machine 1	*Machine 2*	*Machine 3*
P_1	30	20	10
P_2	40	10	30
P_3	20	20	20
Time available (hours)	600	400	800

How many products of each type should be produced to maximise profit ? Fractional units of product are not permissible.

Solution. Let x_1, x_2 and x_3 be the number of units of products P_1, P_2 and P_3 respectively. Then, the appropriate mathematical formulation of the given problem is :

Maximize $z = 200x_1 + 400x_2 + 300x_3$ subject to the constraints :

$$30x_1 + 40x_2 + 20x_3 \le 600 \quad \textit{(Time constraint on } M_1)$$

$$20x_1 + 10x_2 + 20x_3 \le 400 \quad \textit{(Time constraint on } M_2)$$

$$10x_1 + 30x_2 + 20x_3 \le 800 \quad \textit{(Time constraint on } M_3)$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \text{ and are integers.}$$

Introducing the slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective constraints and then using simplex method, a non-integer optimum solution is displayed below :

Initial Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
400	y_2	20/3	1/3	1	0	1/3	–1/3	0
300	y_3	50/3	5/6	0	1	–1/6	2/3	0
0	y_6	80/3	–5/3	0	0	–2/3	–1/3	1
		z (= 23,000/3)	550/3	0	0	250/3	200/3	0

Here, $x_{B_1} = \frac{20}{3} = 6 + \frac{2}{3}$, $x_{B_2} = \frac{50}{3} = 16 + \frac{2}{3}$ and $x_{B_3} = \frac{80}{3} = 26 + \frac{2}{3}$. Thus, all the three fractional parts are same, *i.e.*, $f_1 = f_2 = f_3 = \frac{2}{3}$. So, let us choose $f_2 = \frac{2}{3}$ for the construction of Gomorian constraint. Therefore, from the above table, second row is written as :

$$16 + \frac{2}{3} = \frac{5}{6}x_1 + x_3 - \left(\frac{1}{6}\right)s_1 + \frac{2}{3}s_2$$

or
$$\left(16 + \frac{2}{3}\right) = \left(0 + \frac{5}{6}\right)x_1 + x_3 + \left(-1 + \frac{5}{6}\right)s_1 + \left(0 + \frac{2}{3}\right)s_2.$$

Let G_1 be the first Gomorian slack. Then, the corresponding cut or Gomorian constraint is :

$$-\frac{2}{3} = G_1 - \frac{5}{6}x_1 - \frac{5}{6}s_1 - \frac{2}{3}s_2.$$

On the addition of this constraint in the optimum simplex table (given above), we get

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	G_1
400	y_2	20/3	1/3	1	0	1/3	–1/3	0	0
300	y_3	50/3	5/6	0	1	–1/6	2/3	0	0
0	y_6	80/3	–5/3	0	0	–2/3	–1/3	1	0
0	G_1	–2/3	–5/6	0	0	–5/6	–2/3	0	1
		z (= 2000/3)	550/3	0	0	250/3	200/3	0	0

Since, $x_{B4}\left(=-\frac{2}{3}\right)$ is only negative, G_1 leaves the current basis. Further, maximum $\left\{\frac{550/3}{-5/6}, \frac{250/3}{-5/6}, \frac{200/3}{-2/3}\right\} = -100$. This indicate that y_5 enter the basis and $y_{45}\left(=-\frac{2}{3}\right)$ becomes the leading element. The next iterative simplex table is :

Final Iteration. Optimum Solution in Integers.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	G_1
400	y_2	7	3/4	1	0	3/4	0	0	−1/2
300	y_3	16	0	0	1	−1	0	0	1
0	y_6	27	−5/12	0	0	−1/4	0	1	−1/2
0	y_5	1	5/4	0	0	5/4	1	0	−3/2
		z (= 7600)	100	0	0	0	0	0	100

Hence, the all integer optimum solution is $x_1 = 0, x_2 = 7, x_3 = 16$ with maximum $z = 7{,}600$, *i.e.*, the company should manufacture 7 units of product P_2 and 16 units of product P_3 only to get the maximum profit of Rs. 7,600.

PROBLEMS

704. Find the optimum solution to the I.P.P. :

Maximize $z = x_1 - 2x_2$ subject to the constraints :

$4x_1 + 2x_2 \le 15$, $x_1, x_2 \ge 0$ and are integers.

705. Solve the following I.P.P. :

Maximize $z = 3x_2$ subject to the constraints :

$3x_1 + 2x_2 \le 7$, $-x_1 + x_2 \le 2$; $x_1 \ge 0$, $x_2 \ge 0$ and are integers. *[Annamalai M.B.A. (Nov.) 2009]*

706. Find the optimal solution to the following integer programming problem :

Maximize $z = x_1 - x_2$ subject to the constraints :

$x_1 + 2x_2 \le 4$, $6x_1 + 2x_2 \le 9$, $x_1, x_2 \ge 0$; x_1 and x_2 are integers.

707. Solve the following I.P.P. :

Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$-3x_1 + 7x_2 \le 14$, $7x_1 - 3x_2 \le 14$; $x_1, x_2 \ge 0$ and are integers.

[Meerut M.Sc. (Math.) 1996]

708. Describe any method of solving an integer programming problem. Use it to solve the problem :

Maximize $z = 2x_1 + 2x_2$ subject to the constraints :

$5x_1 + 3x_2 \le 8$, $x_1 + 2x_2 \le 4$; x_1, x_2 non-negative integers.

709. Solve the following I.P.P. :

Minimize $z = 9x_1 + 10x_2$ subject to the constraints :

$x_1 \le 9$, $x_2 \le 8$, $4x_1 + 3x_2 \ge 40$; $x_1, x_2 \ge 0$ and are integers.

710. Solve the following I.P.P. :

Maximize $z = 11x_1 + 4x_2$ subject to the constraints :

$-x_1 + 2x_2 \le 4$, $5x_1 + 2x_2 \le 16$, $2x_1 - x_2 \le 4$, x_1 and x_2 are non-negative integers.

711. Find the optimum integer solution to the following all-I.P.P. :

Maximize $z = x_1 + 2x_2$ subject to the constraints :

$x_1 + x_2 \le 7$, $2x_1 \le 11$, $2x_2 \le 7$, $x_1, x_2 \ge 0$ and are integers.

712. An exporter of ready-made garments makes two types of shirts : X and Y. He makes a profit of Rs. 10 and Rs. 40 per shirt on X and Y shirts respectively. He has two tailors, A and B, at his disposal to stitch these shirts. Tailors A and B can devote at the most 7 hours and 15 hours per day respectively. Both these shirts are to be stitched by both the tailors. Tailor A and tailor B spend two hours and five hours respectively in stitching a X shirt, and four hours and three hours respectively in stitching a Y shirt. How many shirts of both the types should be stitched in order to maximise daily profits? (A non-integer solution for this problem will not be accepted.)
[Delhi M.B.A. (Nov.) 1995, (PT) 2008]

713. A manufacturer of toys makes two types of toys, say A and B. Processing of these toys is done on two machines X and Y. Toy A requires two hours on machine X and six hours on machine Y. Toy B requires four hours on machine X and five hours on machine Y. There are 16 hours of time per day available on machine X and 30 hours on machine Y. The profit obtained on both the toys is same, that is Rs. 5 per toy. What should be the daily production of each type of the two toys? (A non-integer solution for this problem will not be accepted.)
[Delhi M.B.A. (Nov.) 2008]

714. A company manufactures two products P_1 and P_2, which require time on three different machines M_1, M_2 and M_3. Product P_1 requires 2 hours on machine M_1 and 1 hour on machine M_2. Product P_2 requires 4 hours on machine M_2 and 2 hours on machine M_3. There are 50 hours available on machine M_1, 16 hours on machine M_2, and 20 hours on machine M_3 per week. The per unit profit on products P_1 and P_2 are Rs. 30 and Rs. 50 respectively. Determine the output mix that will maximize the total profit per week, when only integer solution is acceptable.

715. The *ABC* company requires an output of at least 200 units of a product per day and to accomplish this target, it can buy machine A and B or both. Machine A costs Rs. 20,000 while machine B costs Rs. 15,000 and the company has a budget of Rs. 2,00,000 for the same. Machines A and B will produce 24 and 20 units respectively of this product per day. However, machine A will require a floor space of 12 square feet while machine B will require 18 square feet and the company has a total floor space of 180 square feet only. Determine the minimum number of machines that should be purchased. (A non-integer solution for this problem will not be accepted.)
[Delhi M.B.A. (Nov.) 2005]

716. Anna furniture firm manufactures tables (A) and chairs (B). The processing times are 3 hours and 4 hours per unit for A on operations one and two respectively, while 4 hours and 5 hours per unit for B on operations one and two respectively. The available time is 18 hours and 21 hours for operations one and two respectively. The product A can be sold at Rs. 3 profit per unit and B at Rs. 8 profit per unit. Solve the LPP. (A non-integer solution for this problem will not be accepted.)
[Madras M.B.A. 2000]

717. The *ABC* Electric Appliances Company produces two products : refrigerators and ranges. Production takes place in two separate departments. Refrigerators are produced in department I and ranges in department II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators in department I and 35 ranges in department II, because of the limited available facilities in these two departments. The company regularly employs a total of 50 workers in the departments. A refrigerator requires two man-weeks of labour, while a range requires one man-week of labour. A refrigerator contributes a profit of Rs. 300 and a range gives profit of Rs. 200. How many units of refrigerators and ranges should the company produce to realize a maximum profit? (A non-integer solution for this problem will not be accepted.)
[Delhi M.B.A. (PT) Nov. 2003]

718. A company produces two products, each of which requires stamping, assembly and painting operations. Total productive capacity by operation, if it were solely devoted to one product on the other is shown below:

Operation	Productive capacity (units/week)	
	Product A	Product B
Stamping	50	75
Assembly	40	80
Painting	90	45

Pro-rata allocation of productive capacity is permissible so that combination of production of the two products. Demand for the two products is unlimited. The profit on product A and B is Rs. 150 and Rs. 120 respectively. Determine the optimal product-mix. (A non-integer solution will not be accepted). *[Delhi M.B.A. (Nov.) 2009]*

719. The dietician at the local hospital is planning the breakfast menu for the maternity ward patients. She is primarily concerned with Vitamin E and iron requirements in planning the breakfast. According to the State Medical Association (SMA), new mothers must get at least 12 milligrams of Vitamin E and 24 milligrams of iron from breakfast. The SMA handbook reports that a scoop of scrambled egg contains 2 milligrams of Vitamin E and 8 milligrams of iron. The SMA handbook recommends that new mothers should eat at least two scoops of cottage cheese for their breakfast. The dietician considers this as one of the model constraints. The hospital accounting department estimate that a scoop of cottage cheese costs Rs. 2 and scoop of scrambled egg also costs Rs. 2. The dietician is attempting to determine the optimum breakfast menu that satisfies all the requirements and minimizes total cost. The cook insists that he can serve foods by only full scoop, thus necessitating an integer solution. Determine the optimum integer solution to the problem. *[Delhi M.B.A. (HCA) 2000]*

7:6. FRACTIONAL CUT METHOD—MIXED INTEGER LPP

In mixed integer linear programming problem, like the pure integer linear programming problem, all the coefficients and constants should be integers. Only some of the decision variables are restricted to integers, while the others may take integer or continuous values.

The iterative procedure for the solution of such problems is as follows :

Step 1. Determine an optimum solution to the given linear programming problem by using simplex method ignoring the restriction of integer value.

Step 2. Test the integrality of the optimum solution.

(*a*) If all $x_{Bi} \geq 0$ $(i = 1, 2, \ldots, m)$ and are integers, the current solution is an optimum one.

(*b*) If all $x_{Bi} \geq 0$ $(i = 1, 2, \ldots, m)$ but the integer restricted variables are not integers, go to the next step.

Step 3. Choose the largest fraction value among the basic variables which are restricted to integers. Let it be x_{Bk} $(= f_{ko}$, say).

Step 4. Generate the Gomorian constraint in the form

$$\sum_{j \in R_+} f_{kj} x_j + \left(\frac{f_{ko}}{f_{ko} - 1}\right) \sum_{j \in R_-} f_{kj} x_j \geq f_{ko},$$

where, $0 < f_{ko} < 1$ and $R_+ = \{ j : f_{kj} > 0 \}$, $R_- = \{ j : f_{kj} < 0 \}$

i.e.,
$$G_1 = -f_{ko} + \sum_{j \in R_+} f_{kj} x_j + \left(\frac{f_{ko}}{f_{ko} - 1}\right) \sum_{j \in R_-} f_{kj} x_j,$$

where G_1 is known as the Gomorian slack variable and this constraint as the Gomory's cut.

Step 5. Add the Gomorian constraint generated in *step 4* at the bottom of the optimum simplex table. Use dual simplex method to find an improved optimum solution.

Step 6. Go to *step* 2 and repeat the procedure until all $x_{Bi} \geq 0$ $(i = 1, 2, \ldots, m)$ and all restricted variables are integers.

SAMPLE PROBLEM

~~720. Solve~~ *the following mixed integer programming problem :*

$$\text{Maximize } z = 4x_1 + 6x_2 + 2x_3 \text{ subject to the constraints :}$$

$$4x_1 - 4x_2 \leq 5, \quad -x_1 + 6x_2 \leq 5, \quad -x_1 + x_2 + x_3 \leq 5$$

$$x_1, x_2, x_3 \geq 0; \quad x_1 \text{ and } x_3 \text{ are integers.}$$

Solution. Introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$; an initial basic feasible solution is $s_1 = 5$, $s_2 = 5$ and $s_3 = 5$.

Ignoring the integer restrictions, the optimum solution of the given L.P.P. is displayed in the table below :

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
4	y_1	5/2	1	0	0	3/10	1/5	0
6	y_2	5/4	0	1	0	1/20	1/5	0
2	y_3	25/4	0	0	1	1/4	0	1
		$z\ (= 30)$	0	0	0	2	2	2

Since, x_1 and x_3 both are not integers and max. $\{f_1, f_3\}$ = max. $\left\{\frac{1}{2}, \frac{1}{4}\right\} = \frac{1}{2}$; from the first row, we have

$$\left(2 + \frac{1}{2}\right) = x_1 + \frac{3}{10}s_1 + \frac{1}{5}s_2$$

Here, both s_1 and s_2 have positive coefficients, therefore the mixed fractional cut is :

$$G_1 = \frac{-1}{2} + \frac{3}{10}s_1 + \frac{1}{5}s_2$$

First Iteration. Drop G_1 and enter $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
4	$\mathbf{y}_1$	5/2	1	0	0	3/10	1/5	0	0
6	$\mathbf{y}_2$	5/4	0	1	0	1/20	1/5	0	0
2	$\mathbf{y}_3$	25/4	0	0	1	1/4	0	1	0
0	$\mathbf{G}_1$	–1/2	0	0	0	–3/10	–1/5	0	1
		z (= 30)	0	0	0	2	2	2	0

Here, since $x_{B4} < 0$, $\mathbf{G}_1$ leaves the basis and since max. $\left\{\frac{2}{-3/10}, \frac{2}{-1/5}\right\}$ = max. $\left\{\frac{-20}{3}, -10\right\}$, *i.e.*, $-\frac{20}{3}$; $\mathbf{y}_4$ *enters the basis.*

Second Iteration. Improved Solution.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
4	$\mathbf{y}_1$	2	1	0	0	0	0	0	1
6	$\mathbf{y}_2$	7/6	0	1	0	0	1/6	0	1/6
2	$\mathbf{y}_3$	35/6	0	0	1	0	–1/6	1	5/6
0	$\mathbf{y}_4$	5/3	0	0	0	1	2/3	0	–10/3
		z (= 80/3)	0	0	0	0	2/3	2	20/3

Since x_3 is still not an integer, we write from the third row of second iteration

$$\left(5 + \frac{5}{6}\right) = x_3 - \frac{1}{6}s_2 + s_3 + \frac{5}{6}G_1$$

Here, since s_2 has negative coefficient, the mixed fractional cut is :

$$G_2 = -\frac{5}{6} + \left(\frac{5/6}{-1 + 5/6}\right)\left(-\frac{1}{6}\right) s_2 + \frac{5}{6}G_1$$

$$= -\frac{5}{6} + \frac{5}{6}s_2 + \frac{5}{6}G_1$$

Third Iteration. Drop G_2 and enter $\mathbf{y}_5$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$	$\mathbf{G}_2$
4	$\mathbf{y}_1$	2	1	0	0	0	0	0	1	0
6	$\mathbf{y}_2$	7/6	0	1	0	0	1/6	0	1/6	0
2	$\mathbf{y}_3$	35/6	0	0	1	0	–1/6	1	5/6	0
0	$\mathbf{y}_4$	5/3	0	0	0	1	2/3	0	–10/3	0
0	$\mathbf{G}_2$	–5/6	0	0	0	0	–5/6	0	–5/6	1
		z (= 80/3)	0	0	0	0	2/3	2	20/3	0

Here, $x_{B5} < 0$ implies $\mathbf{G}_2$ leaves the basis and since y_{55} is the only negative in G_2-row, $\mathbf{y}_5$ enters the basis.

Final Iteration. Optimum Integer Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	G_1	G_2
4	y_1	2	1	0	0	0	0	0	1/6	1/5
6	y_2	1	0	1	0	0	0	0	5/6	−1/5
2	y_3	6	0	0	1	0	0	1	−10/3	4/5
0	y_4	1	0	0	0	1	0	0	1	−6/5
0	y_5	1	0	0	0	0	1	0	20/3	4/5
		$z (= 26)$	0	0	0	0	0	2		

Solution obtained above is optimum and satisfies the condition that x_1 and x_3 are integers. Hence, the required solution is :

$$x_1 = 2, \ x_2 = 1, \ x_3 = 6, \text{ and maximum } z = 26.$$

PROBLEMS

Solve the following mixed-integer programming problems, using Gomory's cutting plane method :

721. Maximize $z = x_1 + x_2$ subject to the constraints :

$$3x_1 + 2x_2 \le 5, \ x_2 \le 2 \, ; \ x_1, x_2 \ge 0 \text{ and } x_1 \text{ is an integer.}$$

[Madras B.E. (Mech. & Prod.) 1990]

722. Minimize $z = x_1 - 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 5, \ -2x_1 + 4x_2 \le 11 \, ; \ x_1, x_2 \ge 0 \text{ and } x_2 \text{ is an integer.}$$

723. Maximize $z = 7x_1 + 9x_2$ subject to the constraints :

$$-x_1 + 3x_2 \le 6, \ 7x_1 + x_2 \le 35 \, ; \ x_1, x_2 \ge 0 \text{ and } x_1 \text{ is an integer.}$$

724. Maximize $z = 3x_1 + x_2 + 3x_3$ subject to the constraints :

$$-x_1 + 2x_2 + x_3 \le 4, \ 4x_2 - 3x_3 \le 2, \ x_1 - 3x_2 + 2x_3 \le 3 \, ;$$

$$x_j \ge 0 \ (j = 1, 2, 3) \text{ with } x_1 \text{ and } x_3 \text{ as integers.}$$

725. Maximize $z = 1.5x_1 + 3x_2 + 4x_3$ subject to the constraints :

$$2.5x_1 + 2x_2 + 4x_3 \le 12, \ 2x_1 + 4x_2 - x_3 \le 7 \, ; \ x_1, x_2, x_3 \ge 0 \text{ and } x_3 \text{ is an integer.}$$

7:7. BRANCH AND BOUND METHOD

The concept behind this method is to divide the entire feasible solution space of LP problem into smaller parts called sub-problems and then search each of them for an optimal solution. This approach is useful in those cases where there is large number of feasible solutions and enumeration of those becomes economically impractical or impossible.

The branch and bound method starts by imposing feasible and infeasible upper and/or lower bounds for the decision variables in each sub-problem. This helps in reducing the number of simplex method iterations to arrive at the optimal solution, because each sub-problem worse than the current feasible bound is discarded and only the remaining sub-problems are examined. At a point where no more sub-problems can be created, we will find an optimal solution.

The iterative procedure is summarized below :

Step 1. Obtain an optimum solution of the given L.P.P. ignoring the integer restriction.

Step 2. Test the integrability of the optimum solution obtained in *step 1*. There are two cases :

(*i*) If the solution is in integers, the current solution is optimum to the given integer programming problem.

(*ii*) If the solution is not in integers, go to next step.