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Replacement Problem and System Reliability

"Conduct is wise or foolish only in reference to its results"

18:1. INTRODUCTION

The study of *replacement* is concerned with situations that arise when some items such as machines, men, electric-light bulbs, etc., need replacement due to their deteriorating efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. For example, a machine becomes more and more expensive to maintain after a number of years, a railway time-table gradually becomes more and more out of date, an electric-light bulb fails all of a sudden, pipeline is blocked, or an employee loses his job, and so like. In all such situations, there is a need to formulate a most economic replacement policy for replacing faulty units or to take some remedial special action to restore the efficiency of deteriorating units.

Following are the situations when the replacement of certain items needs to be done :

(*i*) An old item has failed and does not work at all, or the old item is expected to fail shortly.

(*ii*) The old item has deteriorated and works badly or requires expensive maintenance.

(*iii*) A better design of equipment has been developed.

Replacement problems can be broadly classified into the following two categories :

(*a*) When the equipment/assets deteriorate with time and the value of money

(*i*) does not change with time.

(*ii*) changes with time.

(*b*) When the items/units fail completely all of a sudden.

18:2. REPLACEMENT OF EQUIPMENT/ASSET THAT DETERIORATES GRADUALLY

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

At this point, a replacement is justified.

18:2.1. Replacement Policy when Value of Money does not change with time

The aim here is to determine the optimum replacement age of an equipment/item whose running/maintenance cost increases with time and the value of money remains static during that period. Let

C : capital cost of equipment, S : scrap value of equipment,
n : number of years that equipment would be in use,
$f(t)$: maintenance cost function, and $A(n)$: Average total annual cost.

Case 1. ***When t is a continuous variable.*** If the equipment is used for 'n' years, then the total cost incurred during this period is as under :

$$TC = \text{Capital cost} - \text{Scrap value} + \text{Maintenance cost} = C - S + \int_0^n f(t)\,dt.$$

Average annual total cost, therefore is :

$$A(n) = \frac{1}{n}\,TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt.$$

For minimum cost, we must have $\frac{d}{dn}[A(n)] = 0$. This implies that

$$\frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_0^n f(t)\,dt + \frac{1}{n}f(n) = 0 \quad \text{or} \quad f(n) = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt \equiv A(n).$$

Clearly, $$\frac{d^2}{dn^2}[A(n)] > 0 \quad \text{at} \quad f(n) = A(n).$$

This suggests that the *equipment should be replaced when maintenance cost equals the average annual total cost.*

Case 2. ***When t is a discrete variable.*** Here, the resale value at the end of nth year is denoted by $S(n)$, where $n = 1, 2, 3, \ldots$. Let $d(n)$ be the change in the resale value in the nth year and $f(n)$ be the maintenance cost during the nth year. Then

$$d(1) = S(0) - S(1),\quad d(2) = S(1) - S(2),\quad d(3) = S(2) - S(3),$$
$$d(4) = S(3) - S(4),\quad \ldots,\quad d(n) = S(n-1) - S(n).$$

So, $$\sum_{t=1}^{n} d(t) = d(1) + d(2) + d(3) + \ldots + d(n)$$
$$= S(0) - S(n) = C - S(n), \quad \text{where } S(0) = C.$$

Total cost during n years is given as

$$TC = C - S(n) + \sum_{t=1}^{n} f(t)$$
$$= \sum_{t=1}^{n} d(t) + \sum_{t=1}^{n} f(t) = \sum_{t=1}^{n} [d(t) + f(t)].$$

Let $e(t) = d(t) + f(t)$ be the effective maintenance cost in the nth year. Then

$$TC = \sum_{t=1}^{n} e(t)$$

and the average cost during the nth year is given as under :

$$A(n) = \frac{TC}{n} = \frac{1}{n}\sum_{t=1}^{n} e(t).$$

Now, $A(n)$ will be minimum for that value of n, for which

$$A(n+1) \ge A(n) \quad \text{and} \quad A(n-1) \ge A(n).$$

For this, we write

$$A(n+1) = \frac{1}{n+1}\sum_{t=1}^{n+1} e(t) = \frac{1}{n+1}\sum_{t=1}^{n} e(t) + \frac{1}{n+1}e(n+1) = \frac{nA(n)}{n+1} + \frac{e(n+1)}{n+1}.$$

$$\therefore \quad A(n+1) - A(n) = \frac{nA(n)}{n+1} + \frac{e(n+1)}{n+1} - A(n) = \frac{e(n+1) - A(n)}{n+1}.$$

So, $$A(n+1) - A(n) \ge 0 \Rightarrow e(n+1) \ge A(n).$$

Similarly, $$A(n) - A(n-1) \le 0 \Rightarrow e(n) \le A(n-1).$$

This suggests the optimal replacement policy :

Replace the equipment at the end of n years, if the effective maintenance cost in the (n + 1)th year is more than the average total cost in the nth year and the nth year's effective maintenance cost is less than the previous year's average total cost.

SAMPLE PROBLEMS

1801. *A firm is considering replacement of a machine, whose cost price is Rs. 12,200 and the scrap value, Rs. 200. The running (maintenance and operating) cost in rupees are found from experience to be as follows :*

Year :	1	2	3	4	5	6	7	8
Running cost :	200	500	800	1,200	1,800	2,500	3,200	4,000

When should the machine be replaced? **[Meerut M.Sc. (Math.) 2001; Madras M.B.A. 2004]**

Solution. We are given the running cost, $f(n)$, the scrap value S = Rs. 200 and the cost of machine, C = Rs. 12,200. In order to determine the optimal time 'n' when the machine should be replaced, we calculate an average total cost per year during the life of the machine as shown in the table given below :

Year of service n (1)	*Running cost (Rs.)* $f(n)$ (2)	*Resale value (Rs.)* $s(n)$ (3)	*Change in resale value (Rs.)* $d(n)$ (4)	*Effective running cost (Rs.)* $e(n)$ (5)	*Total cost (Rs.)* TC $\Sigma e(n)$ (6)	*Average cost (Rs.)* $A(n)$ (6)/(1) (7)
1	200	200	12,200 – 200 = 12,000	12,200	12,200	12,200
2	500	200	0	500	12,700	6,350
3	800	200	0	800	13,500	4,500
4	1,200	200	0	1,200	14,700	3,675
5	1,800	200	0	1,800	16,500	3,300
6	2,500	200	0	2,500	19,000	3,167
7	3,200	200	0	3,200	22,200	3,171
8	4,000	200	0	4,000	26,200	3,275

From the table, we observe that the average total cost per year, $A(n)$ is minimum in the 6th year (Rs. 3,167). Also, the average total cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence, the machine should be replaced after every 6 years.

1802. (*a*) *Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?* **[Madras B.E. (Comp. Sc.) 2002]**

(*b*) *Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one-year old, should you replace it with B; if so when?* **[Delhi M.Com. 2006]**

Solution. (*a*) Let the machine have no resale value when replaced. Then, for machine A, the average total annual cost $ATC(n)$ is computed as follows :

Year (n)	$f(n)$	$s(n)$	$d(n)$	$e(n)$	TC	$A(n)$
1	200	0	9,000 – 0 = 9,000	9,200	9,200	9,200
2	2,200	0	0	2,200	11,400	5,700
3	4,200	0	0	4,200	15,600	5,200
4	6,200	0	0	6,200	21,800	5,450
5	8,200	0	0	8,200	30,000	6,000

This table shows that the best age for the replacement of machine A is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(b) For machine B, the average cost per year can similarly be computed as given in the following table :

Year (n)	f (n)	s (n)	d (n)	e (n)	TC	A (n)
1	400	0	10,000	10,400	10,400	10,400
2	1,200	0	0	1,200	11,600	5,800
3	2,000	0	0	2,000	13,600	4,533
4	2,800	0	0	2,800	16,400	4,100
5	3,600	0	0	3,600	20,000	4,000
6	4,400	0	0	4,400	24,400	4,066

Since, the minimum average cost for machine B is lower than that for machine A, machine A should be replaced.

To decide the time of replacement, we should compare the minimum average cost for B (Rs. 4,000) with yearly cost of maintaining and using the machine A. Since, there is no salvage value of the machine, we shall consider only the maintenance cost. We would keep the machine A so long as the yearly maintenance cost is lower than Rs. 4,000 and replace when it exceeds Rs. 4,000.

On the one-year old machine A, Rs. 2,200 would be required to be spent in the next year; while Rs. 4,200 would be needed in the year following. Thus, we should keep machine A for one year and replace it thereafter.

1803. *The data collected in running a machine, the cost of which is Rs. 60,000, are given below :*

Year		*1*	2	3	4	5
Resale value (Rs.)	:	42,000	30,000	20,400	14,400	9,650
Cost of spares (Rs.)	:	4,000	4,270	4,880	5,700	6,800
Cost of labour (Rs.)	:	14,000	16,000	18,000	21,000	25,000

Determine the optimum period for replacement of the machine. ***[Lucknow B.M.S. 2008]***

Solution. The operating or maintenance cost of machine in successive years is as follows :

Year	:	*1*	2	3	4	5
Operating cost (Rs.)	:	18,000	20,270	22,880	26,700	31,800

(The cost of spares and labour together determine the operating or running or maintenance cost.)

The average total annual cost is computed below :

Year of service n	*Operating cost (Rs.)* $f(n)$	*Resale value (Rs.)* $s(n)$	*Change in resale value (Rs.)* $d(n)$	*Effective op. cost (Rs.)* $e(n)$	*Total cost (Rs.)* TC	*Average cost (Rs.)* $A(n)$
1	18,000	42,000	18,000	36,000	36,000	36,000.00
2	20,270	30,000	12,000	32,270	68,270	34,135.00
3	22,880	20,400	9,600	32,480	1,00,750	33,583.30
4	26,700	14,400	6,000	32,700	1,33,450	**33,362.50**
5	31,800	9,650	4,750	36,550	1,70,000	34,000.00

The calculations in the above table shows that the average cost is *lowest* during the *fourth year*. Hence, the machine should be replaced after every fourth year.

PROBLEMS

1804. The cost of a machine is Rs. 6,100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows :

Year	:	*1*	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	100	250	400	600	900	1,200	1,600	2,000

When should the machine be replaced? ***[Kerala M.Com. 1996; Panjab M.B.A. (June) 2003]***

1812. A new tempo costs Rs. 80,000 and may be sold at the end of any year at the following prices :

Year (end)	1	2	3	4	5	6
Selling price (in Rs.) (at present value)	50,000	33,000	20,000	11,000	6,000	1,000

The corresponding annual operating costs are :

Year (end)	1	2	3	4	5	6
Cost/year (in Rs.) (at present value)	10,000	12,000	15,000	20,000	30,000	50,000

It is not only possible to sell the tempo after use but also to buy a second hand tempo.

It may be cheaper to do so than to replace by a new tempo.

Age of tempo	0	1	2	3	4	5
Purchase price (in Rs.) (at present value)	80,000	58,000	40,000	26,000	16,000	10,000

What is the age to buy and to sell tempo so as to minimize average annual cost?

1813. (*a*) A transport manager finds from his past records that the costs per year of running a truck whose purchase price is Rs. 6,000 are as given below :

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	1,000	1,200	1,400	1,800	2,300	2,800	3,400	4,000
Resale value (Rs.)	3,000	1,500	750	375	200	200	200	200

Determine at what age is replacement due? *[IAS 1993; Kerala M.Com. 1991; Jodhpur M.Sc. (Math.) 1992]*

(*b*) Let the owner of a fleet have three trucks, two of which are two years old and the third one year old. The cost price, running cost and resale vale of these trucks are same as given in (*a*). Now he is considering a new type of truck with 50% more capacity than one of the old ones at a unit price of Rs. 8,000. He estimates that the running costs and resale price for the truck will be as follows :

Year	1	2	3	4	5	6	7	8
Running costs (Rs.)	1,200	1,500	1,800	2,400	3,100	4,000	5,000	6,100
Resale price (Rs.)	4,000	2,000	1,000	500	300	300	300	300

Assuming that the loss of flexibility due to fewer trucks is of no importance, and that he will continue to have sufficient work for three of the old trucks, what should his policy be? *[Poona M.B.A. 1992]*

1814. Machine *A* costs Rs. 3,600. Annual operating costs are Rs. 40 for the first year and then increase by Rs. 360 every year. Assuming that machine *A* has no resale value, determine the best replacement age.

Another machine *B*, which is similar to machine *A*, costs Rs. 4,000. Annual running costs are Rs. 200 for the first year and then increase by Rs. 200 every year. It has resale value of Rs. 1,500, Rs. 1,000 and Rs. 500, if replaced at the end of first, second and third years respectively. It has no resale value during fourth year and onwards.

Which machine would you prefer to purchase? Future costs are not to be discounted. *[Guwahati M.C.A. 1992]*

1815. Machine *A* costs Rs. 45,000 and the operating costs are estimated at Rs. 1,000 for the first year, increasing by Rs. 10,000 per year in the second and subsequent years. Machine *B* costs Rs. 50,000 and operating costs are Rs. 2,000 for the first year, increasing by Rs. 4,000 in the second and subsequent years. If we now have a machine of type *A*, should we replace it with *B* ? If so when? Assume that both machines have no resale value and future costs are not discounted. *[Madras M.B.A. (Nov.) 2006; Lucknow B.M.S. 2008]*

18:2.2 Replacement Policy when Value of Money changes with time

When the time value of money is taken into consideration, we shall assume that (*i*) the equipment in question has no salvage value, and (*ii*) the maintenance costs are incurred in the beginning of the different time periods.

Since, it is assumed that the maintenance cost increases with time and each cost is to be paid just in the start of the period, let the money carry a rate of interest r per year. Thus, a rupee invested now will be worth $(1 + r)$ after a year, $(1 + r)^2$ after two years, and so on. In this way a rupee invested today will be worth $(1+r)^n$, n years hence, or, in other words, if we have to make a payment of one rupee in n years time, it is equivalent to making a payment of $(1+r)^{-n}$ rupees today. The quantity $(1+r)^{-n}$ is called the *present worth* factor (*Pwf*) of one rupee spent in n years time from now onwards. The expression $(1+r)^n$ is known as the *payment compound amount factor (Caf)*. of one rupee spent in n years time.

Let the initial cost of the equipment be C and let R_n be the operating cost in year n. Let v be the rate of interest in such a way that $v = (1+r)^{-1}$ is the discount rate (present worth factor). Then the present value of all future discounted costs V_n associated with a policy of replacing the equipment at the end of each n years is given by

$$V_n = \{(C+R_0) + vR_1 + v^2R_2 + \ldots + v^{n-1}R_{n-1}\} + \{(C+R_0)v_n + v^{n+1}R_1 + v^{n+2}R_2 + \ldots + v^{2n-1}R_{n-1}\} + \ldots$$

$$= \left[C + \sum_{k=0}^{n-1} v^k R_k\right] \times \sum_{k=0}^{\infty} (v^n)^k = \left[C + \sum_{k=0}^{n-1} v^k R_k\right] \Big/ (1-v^n)^{-1}$$

Now, V_n will be a minimum for that value of n, for which

$$V_{n+1} - V_n > 0 \quad \text{and} \quad V_{n-1} - V_n > 0.$$

For this, we write

$$V_{n+1} - V_n = \left[C + \sum_{k=0}^{n} v^k R_k\right](1 - v^{n+1})^{-1} - V_n$$

$$= v^n\,[R_n - (1-v)\,V_n] / (1 - v^{n+1})$$

and similarly $\qquad V_n - V_{n-1} = V^{n-1}\,[R_{n-1} - (1-v)\,V_n] / (1 - v^{n-1})$

Since v is the depreciation value of money, it will always be less than 1, and therefore $1-v$ will always be positive. This implies that $v^n/(1-v^{n+1})$ will always be positive.

Hence, $\quad V_{n+1} - V_n > 0 \Rightarrow R_n > (1-v)\,V_n$ and $V_n - V_{n-1} < 0 \Rightarrow R_{n-1} < (1-v)\,V_n$

Thus,

$$R_{n-1} < (1-v)\,V_n < R_n$$

or

$$R_{n-1} < \frac{C + R_0 + vR_1 + v^2R_2 + \ldots + v^{n-1}R_{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} < R_n,$$

since

$$(1 - v^n)(1-v)^{-1} = \sum_{k=0}^{n-1} v^k.$$

The expression which lies between R_{n-1} and R_n is called the "*weighted average cost*" of all the previous n years with weights $1, v, v^2, \ldots, v^{n-1}$ respectively.

Hence, the optimal replacement policy of the equipment after n periods is :

(*a*) Do not replace the equipments if the next period's operating cost is less than the weighted average of previous costs.

(*b*) Replace the equipments if the next period's operating cost is greater than the weighted average of previous costs.

Remark. Procedure for determining the weighted average of costs (annualized cost) may be summarized in the following steps :

Step 1. Find the present value of the maintenance cost for each of the years,

i.e., $\qquad \Sigma R_{n-1} v^{n-1}$ $(n = 1, 2, \ldots)$; where $v = (1+r)^{-1}$.

Step 2. Calculate cost plus the accumulated present values obtained in *step 1*, *i.e.*, $C + \Sigma R v^{n-1}$.

Step 3. Find the cumulative present value factor up to each of the years 1, 2, 3, ..., *i.e.*, Σv^{n-1}.

Step 4. Determine the annualized cost $W(n)$, by dividing the entries obtained in *step 2* by the corresponding entries obtained in *step 3*, *i.e.*, $[C + \Sigma R_{n-1} v^{n-1}] / \Sigma v^{n-1}$.

Corollary. When the time value of money is not taken into consideration, the rate of interest becomes zero and hence v approaches unity. Therefore, as $v \to 1$, we get

$$R_{n-1} < \frac{C + R_0 + R_1 + \ldots + R_{n-1}}{1 + 1 + \ldots + n \text{ times}} < R_n$$

or

$$R_{n-1} < W(n) < R_n$$

Note. It may be noted that the above result is in complete agreement with the result that was obtained in 18 : 2.1.

Selection of the Best Equipment Amongst Two

Following is the procedure for determining a policy for the selection of an economically best item amongst the available equipments :

Step 1. Considering the case of two equipments, say A and B, we first find the best replacement age for both the equipments by making use of

$$R_{n-1} < (1 - v)\, V_n < R_n.$$

Let the optimum replacement age for A and B comes out to be n_1 and n_2 respectively.

Step 2. Next, compute the fixed annual payment (or weighted average cost) for each equipment by using the formula

$$W(n) = \frac{C + R_0 + vR_1 + v^2R_2 + \ldots + v^{n-1}R_{n-1}}{1 + v + \ldots + v^{n-1}}$$

and substitute $n = n_1$ for equipment A and $n = n_2$ for equipment B in it.

Step 3. (*i*) If $W(n_1) < W(n_2)$, choose equipment A.

(*ii*) If $W(n_1) > W(n_2)$, choose equipment B.

(*iii*) If $W(n_1) = W(n_2)$, both equipments are equally good.

SAMPLE PROBLEMS

1816. *Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given below :*

Year	:	*1*	2	3	4	5	6
Machine A	:	1,000	200	400	1,000	200	400
Machine B	:	1,700	100	200	300	400	500

Determine which machine should be purchased. **[Bharathidasan B.Com. 1999]**

Solution. Since the money carries the rate of interest, the present worth of the money to be spent over in a period of one year is

$$v = \frac{100}{100 + 10} = \frac{10}{11} = 0.9091$$

∴ The total discounted cost (present worth) of A for 3 years is

$$1000 + 200 \times (0.9091) + 400 \times (0.9091)^2 = \text{Rs. } 1512 \text{ approx.}$$

Again, the total discounted cost of B for six years is

$$1{,}700 + 100 \times (0.9091) + 200 \times (0.9091)^2 + 300 \times (0.9091)^3 + 400 \times (0.9091)^4 + 500 \times (0.9091)^5 = \text{Rs. } 2{,}765.$$

Average yearly cost of machine A = Rs. 1,512/3 = Rs. 504.

Average yearly cost of machine B = Rs. 2,765/6 = Rs. 461.

This shows that the apparent advantage is with machine B. But, the comparison is unfair since the periods for which the costs are considered are different. So, if we consider 6 years period for machine A also, then the total discounted cost of A will be

$$1{,}000 + 200 \times (0.9091) + 400 \times (0.9091)^2 + 1{,}000 \times (0.909)^3 + 200 \times (0.9091)^4 + 400 \times (0.9091)^5.$$

After simplification this comes out to be Rs. 2,647 which is Rs. 118 less costlier than machine B over the same period.

Hence, machine A should be purchased.

1817. *A pipeline is due for repairs. It will cost Rs. 10,000 and last for 3 years. Alternatively, a new pipeline can be laid at a cost of Rs. 30,000 and lasts for 10 years. Assuming cost of capital to be 10% and ignoring salvage value, which alternative should be chosen?*

Solution. Consider the two types of pipeline for infinite replacement cycles of 10 years for the new pipeline, and 3 years for the existing pipeline.

Since, the discount rate of money per year is 10%, theretore the present worth of money to be spent over in a period of one year is

$$v = \frac{100}{100 + 10} = 0.9091$$

Let k_n denote the discounted value of all future costs associated with a policy of replacing the equipment after n years. Then, if we designate the initial outlay by C,

$$k_n = C + Cv^n + Cv^{2n} + \ldots + \infty = C(1 + v^n + v^{2n} + \ldots + \infty) = C/(1 - v^n)$$

Making use of values of C, v and n for two types of pipelines, the discounted value, therefore, yields

$$k_3 = \frac{10{,}000}{1 - (0.9091)^3} = \text{Rs. } 4{,}021 \text{ for the existing pipeline,}$$

and

$$k_{10} = \frac{30{,}000}{1 - (0.9091)^{10}} = \frac{30{,}000}{1 - 0.3855} = \text{Rs. } 48{,}820 \text{ for the new pipeline.}$$

Since $k_3 < k_{10}$, the existing pipeline should be continued. Alternatively, the comparison may be made over $3 \times 10 = 30$ years.

1818. *A person is considering to purchase a machine for his own factory. Relevant data about alternative machines are as follows :*

	Machine A	*Machine B*	*Machine C*
Present investment (Rs.)	10,000	12,000	15,000
Total annual cost (Rs.)	2,000	1,500	1,200
Life (years)	10	10	10
Salvage value (Rs.)	500	1,000	1,200

As an adviser to the buyer, you have been asked to select the best machine, considering 12% normal rate of return.

You are given that :

(a) Single payment present worth factor (pwf) at 12% interest for 10 years (= 0.322).

(b) Annual series present worth factor (Pwf) at 12% interest for 10 years (= 5.650).

Solution. The present value of total cost of each of the three machines for a period of ten years is computed in the following table :

Machine	*Present investment*	*Present value of total annual cost*	*Present value of salvage value*	*Net cost (Rs.)*
(1)	(2)	(3)	(4)	(5) = (2) + (3) − (4)
A	10,000	$2000 \times 5.65 = 11{,}300$	$500 \times 0.322 = 161.00$	21,139.00
B	12,000	$1500 \times 5.65 = 8{,}475$	$1000 \times 0.322 = 322.00$	**20,153.00**
C	15,000	$1200 \times 5.65 = 6{,}780$	$1200 \times 0.322 = 386.40$	21,393.60

From the information in the table, we observe that the present value of total cost for machine B is the least. Hence, machine B should be purchased.

1819. *The cost of a new machine is Rs. 5,000. The maintenance cost of nth year is given by $C_n = 500(n-1)$; $n = 1, 2, \ldots$ Suppose that the discount rate per year is 0.5. After how many years it will be economical to replace the machine by a new one?*

[Madras M.B.A. 1996; Meerut M.Sc. (Math.) 1996]

Solution. Since the discount rate of money per year is 0.05, the present worth of the money to be spent over a period of one year is

$$v = (1 + 0.05)^{-1} = 0.9523.$$

The optimum replacement time is determined in the following table :

Year (n)	R_{n-1}	v^{n-1}	$R_{n-1}v^{n-1}$	$C + \sum_k R_{k-1}v^{k-1}$	$\sum_k v^{k-1}$	$W(n)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0	1.0000	0	5,000	1.0000	5,000
2	500	0.9523	476	6,476	1.9523	2,805
3	1,000	0.9070	907	6,383	2.8593	2,232
4	1,500	0.8638	1,296	7,679	3.7231	2,063
5	2,000	0.8227	1,645	9,324	4.5458	2,051*
6	2,500	0.7835	1,959	11,283	5.3293	2,117

Since, $W(n)$ is minimum for $n = 5$ and R_4 (= 1,500) < $W(5)$ as well as $W(5) > R_6$ (= 2,500); it is economical to replace the machine by a new one at the end of five years.

1820. *A manufacturer is offered two machines A and B. A is priced at Rs. 5,000, and running costs are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.*

If money is worth 10% per year, which machine should be purchased? (Assume that the machine will eventually be sold for scrap at a negligible price.)

[Annamalai M.B.A. 2009]

Solution. Since the money is worth 10% per year, the discount rate for both the machines is given by

$$v = \frac{1}{1 + 0.10} = 0.9091$$

For the solution of this problem, we compute the following tables for machines *A* and *B* separately, by using *Pwf* table given at the end of the book.

For Machine A

Year (n)	R_{n-1}	v^{n-1}	$v^{n-1}R_{n-1}$	$C + \sum_k v^{k-1}R_{k-1}$	$\sum_k v^{k-1}$	$W(n)$
(1)	(2)	(3)	(4)	(5)	(6)	(7) = (5)/(6)
1	800	1.0000	800	5,800	1.0000	5,800.00
2	800	0.9091	727	6,527	1.9091	3,418.88
3	800	0.8264	661	7,188	2.7355	2,627.67
4	800	0.7513	601	7,789	3.4868	2,233.85
5	800	0.6830	546	8,335	4.1698	1,998.89
6	1,000	0.6209	621	8,956	4.7907	1,869.45
7	1,200	0.5645	677	9,633	5.3552	1,798.81
8	1,400	0.5132	718	10,351	5.8684	1,763.85
9	1,600	0.4665	746	11,097	6.3349	1,751.72
10	1,800	0.4241	763	11,860	6.7590	1,754.70

For Machine B

Year (n)	R_{n-1}	v^{n-1}	$v^{n-1} R_{n-1}$	$C + \sum_k v^{k-1} R_{k-1}$	$\sum_k v^{k-1}$	$W(n)$
(1)	(2)	(3)	(4)	(5)	(6)	(7) = (5)/(6)
1	1,200	1.0000	1,200.00	3,700.00	1.0000	3,700.00
2	1,200	0.9091	1,090.91	4,790.91	1.9091	2,509.51
3	1,200	0.8264	991.98	5,782.59	2.7353	2,113.91
4	1,200	0.7513	901.56	6,684.15	3.4868	1,916.99
5	1,200	0.6830	819.60	7,503.75	4.1698	1,799.55
6	1,200	0.6209	745.08	8,248.83	4.7907	1,721.84
7	1,400	0.5645	790.30	9,039.13	5.3552	1,687.92
8	1,600	0.5132	821.12	9,860.25	5.8684	1,680.23
9	1,800	0.4665	839.70	10,699.95	6.3349	1,689.05
10	2,000	0.4241	848.20	11,548.15	6.7590	1,708.56

From the above tables we observe that for machine A, $1,600 < 1,751.72 < 1,800$.

Now, since the running cost of 9th year is Rs. 1,600 and that of 10th year is Rs. 1,800 and since $1,800 > 1,751.72$, it is better to replace the machine A after 9th year.

Similarly, for machine B since $1,800 > 1,680.23$, it is better to replace the machine B after 8th year.

Further since the weighted average cost in 9 years of machine A is Rs. 1751.72 and the weighted average cost in 8 years of machine B is Rs. 1,680.23, it is advisable to purchase machine B.

PROBLEMS

1821. Let $v = 0.9$ and initial price is Rs. 5,000. Running cost varies as follows :

Year :	1	2	3	4	5	6	7
Running cost (in Rs.) :	400	500	700	1,000	1,300	1,700	2,100

What would be the optimum replacement interval?

1822. The initial cost of an item is Rs. 15,000 and maintenance or running costs for different years are given below :

Year :	1	2	3	4	5	6	7
Running cost (in Rs.) :	2,500	3,000	4,000	5,000	6,500	8,000	10,000

What is the replacement policy to be adopted if the capital is worth 10% and there is no salvage value?

1823. The yearly cost of 2 machines A and B when the money value is neglected is as follows :

Year :	1	2	3	4	5
Machine A :	1,800	1,200	1,400	1,600	1,000
Machine B :	2,800	200	1,400	1,100	600

Find their cost patterns if money value is 10% per year and hence find which machine is most economical.

[Madras B.E. (Mech.) 1999]

1824. A manual stamper currently valued at Rs. 1,000 is expected to last 2 years and costs Rs. 4,000 per year to operate. An automatic stamper which can be purchased for Rs. 3,000 will last 4 years and can be operated at an annual cost of Rs. 3,000. If money carries the rate of interest 10% per annum, determine which stamper should be purchased.

1825. A manufacturer is offered two machines A and B. A is priced at Rs. 5,000 and running costs are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.

If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price.)

[Madras B.E. 1999]

1826. An engineering company is offered two types of material handling equipment A and B. A is priced at Rs. 60,000 including cost of installation, and the costs for operation and maintenance are estimated to be Rs. 10,000 for each of the first five years, increasing by Rs. 3,000 per year in the sixth and subsequent year. Equipment B with a rated capacity same as A, requires an initial investment of Rs. 30,000 but in terms of operation and maintenance costs more than A. These costs for B are estimated to be Rs. 13,000 per year for the first six years, increasing by Rs. 4,000 per year for each year from the 7th year onwards. The company expects a return of 10 per cent on all its investments. Neglecting the scrap value of the equipment at the end of its economic life, determine which equipment the company should buy.

1827. An individual is planning to purchase a car. A new car will cost Rs. 1,20,000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation costs during the first year are Rs. 20,000 and they increase by 15% every year. The minimum resale value of car can be Rs. 40,000.

(*i*) When should the car be replaced to minimise average annual cost (ignore interest)?

(*ii*) If interest of 12% is assumed, when should the car be replaced? *[Kerala M.Com. 1990]*

18:3. REPLACEMENT OF EQUIPMENT THAT FAILS SUDDENLY

It is difficult to predict that a particular equipment will fail at a particular time. This difficulty can be overcome by determining the probability distribution of failures. Here it is assumed that the failures occur only at the end of the period, say t. Thus the objective becomes to find the value of t which minimizes the total cost involved for the replacement.

We shall consider the following two types of replacement policies :

Individual Replacement Policy. Under this policy, an item is replaced immediately after its failure.

Group Replacement Policy. Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with a provision that if any item fails before the optimal time, it may be individually replaced.

Mortality Tables. These are used to derive the probability distribution of the life span of an equipment. Let

$$M(t) = \text{number of survivors at any time } t,$$
$$M(t-1) = \text{number of survivors at any time } t-1, \text{ and}$$
$$N = \text{initial number of equipments}$$

Then the probability of failure during time period t is given by

$$p(t) = [M(t-1) - M(t)]/N$$

The probability that an equipment survived till age $(t-1)$, will fail during the interval $(t-1)$ to t can be defined as the *conditional probability* of failure. It is given by

$$p_c(t) = [M(t-1) - M(t)]/M(t-1)$$

The probability of survival till age t is given by

$$p_s(t) = M(t)/N.$$

Theorem 18-1 **(Mortality).** *A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exists. Then the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant (which is equal to the size of the total population divided by the mean age at death).*

Proof. Let k be a constant such that no item of the system can survive upto and beyond time $k+1$, *i.e.*, the life span of any item lies between $t = 0$ and $t = k$. We define

$f(t)$: the number of births at time t, and

$p(x)$: the probability that an equipment will die (fail) just before achieving the age $x+1$, *i.e.*, at age x.

It is easy to note that $\sum_{x=0}^{k} p(x) = 1$.

Now $f(t-x)$ births take place at time $t-x$, $t = k, k+1, \ldots$ Such newly born items attain the age x at time t. Therefore, the expected number of deaths of such alive numbers at time t is $p(x) f(t-x)$. Thus the mathematical expectation of the number of deaths before time $t+1$ is $\Sigma f(t-x) p(x)$, $t = k, k+1, \ldots$ Moreover, since deaths are immediately replaced by births, we must have

$$f(t+1) = \sum_{x=0}^{k} f(t-x) p(x), \qquad t = k, k+1, \ldots$$

The solution to this difference equation in t can be obtained by making use of

$$f(t) = A\alpha^t, \text{ where } A \text{ is some constant and } |\alpha| < 1.$$

$$\therefore \qquad A\alpha^{t+1} = \sum_{x=0}^{k} A\alpha^{t-x} p(x) = A[\alpha^t p(0) + \alpha^{t-1} p(1) + \ldots + \alpha^{t-k} p(k)]$$

or

$$\alpha^{k+1} = \alpha^k [\sum_{x=0}^{k} \alpha^{-x} p(x)] = \alpha^k [p(0) + \alpha^{-1} p(1) + \ldots + \alpha^{-k} p(k)]$$

Thus

$$\alpha^{k+1} - [\alpha^k p(0) + \alpha^{k-1} p(1) + \ldots + p(k)] = 0.$$

This is a linear homogeneous difference equation of degree $k+1$ and thus has exactly $k+1$ roots. Let the roots be $\alpha_0, \alpha_1, \ldots, \alpha_k$.

For $\alpha = 1$, the equation yields

$$L.H.S. = 1 - \sum_{x=0}^{k} p(x) = 1 - 1 = 0 = R.H.S.$$

Thus, $\alpha = 1$ is a root of the above equation. Let us denote it by $\alpha_0 = 1$. The most general solution of the difference equation will be of the form

$$f(t) = A_0\alpha_0^t + A_1\alpha_1^t + \ldots + A_k\alpha_k^t$$
$$= A_0 + A_1\alpha_1^t + \ldots + A_k\alpha_k^t$$

where $A_0, A_1, \ldots, A_k$ are constants whose values are to be determined. We observe that, since $|\alpha_i| < 1$ as $t \to \infty$, $\lim_{t \to \infty} f(t) = A_0$. Thus, under our assumption for a long period t, the number of deaths per unit time is equal to A_0.

Now, the problem is to determine the value of the constant A_0.

Let $\qquad g(x)$ = Probability of surviving for more than x years

or $\qquad g(x) = 1 - P$ (survivor will die before attaining the age x)

$$= 1 - \{p(0) + p(1) + \ldots + p(x-1)\}$$

Obviously, it can be assumed that $g(0) = 1$.

Since, the number of births as well as deaths have become constant, each equal to A_0, therefore expected number of survivors of age x is given by $A_0 \cdot g(x)$.

As deaths are immediately replaced by births and therefore size N of the population remains constant. Thus, we must have

$$N = A_0 \sum_{x=0}^{k} g(x) \quad \text{or} \quad A_0 = N \Big/ \sum_{x=0}^{k} g(x).$$

The denominator represents the average age at death. This can also be proved as follows :

From finite differences, we know that

$$\Delta(x) = (x+1) - x = 1$$

and $$\sum_{x=a}^{b} f(x)\,\Delta h(x) = f(b+1)\,h(b+1) - f(a)\,h(a) - \sum_{x=a}^{b} h(x+1)\,\Delta f(x)$$

Therefore, we can write,

$$\sum_{x=0}^{k} g(x) = \sum_{x=0}^{k} g(x)\,\Delta(x) = \Big[g(x)\cdot x \Big]_0^{k+1} = (k+1)\,g(k+1) - 0 \times g(0) - \sum_{x=0}^{k} (x+1)\,\Delta g(x).$$

$$= (k+1)\,g(k+1) - \sum_{x=0}^{k} (x+1)\,\Delta g(x).$$

But $$g(k+1) = 1 - \{p(0) + p(1) + \ldots + p(k)\} = 0,$$ since no one can survive for more than $(k+1)$ years of age.

and
$$\begin{aligned} \Delta g(x) &= g(x+1) - g(x) \\ &= [1 - p(0) - p(1) - \ldots - p(x)] - [1 - p(0) - \ldots - p(x-1)] \\ &= -p(x). \end{aligned}$$

$\therefore$ $$\sum_{x=0}^{k} g(x) = (k+1)\,g(k+1) - \sum_{x=0}^{k} (x+1)\,[-p(x)]$$

$$= \sum_{x=0}^{k} (x+1)\,p(x)\,; \text{ since } g(k+1) = 0.$$

This happens to be the mean (expected age at death).

Hence, $$A_0 = N\,/\text{Mean age at death}.$$

***Theorem 18-2* (Group Replacement).** *Let all the items in a system be replaced after a time interval 't' with provisions that individual replacements can be made if and when any item fails during this time period. Then*

(a) Group replacement must be made at the end of t^{th} period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of t periods.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacements at the end of period t – 1 is less than the average cost per unit period through the end of period t.

Proof. Let,

N = total number of items in the system,

C_2 = cost of replacing an individual item,

C_1 = cost of replacing an item in group,

$C(t)$ = total cost of group replacement after time period t,

$f(t)$ = number of failures during time period t.

Then, clearly

$$C(t) = NC_1 + C_2 \sum_{x=1}^{t-1} f(x)$$

The average cost of group replacement per unit period of time during a period t, is thus given by

$$A(t) = \frac{C(t)}{t} = \left[NC_1 + C_2 \sum_{x=1}^{t-1} f(x) \right] \Big/ t.$$

We shall determine the optimum t so as to minimize $C(t)/t$.

Note that whenever $\dfrac{C(t-1)}{t-1} > \dfrac{C(t)}{t}$ and $\dfrac{C(t+1)}{t+1} > \dfrac{C(t)}{t}$, it is better to replace all the items after time period t.

Now,

$$\frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t)/t\,;$$

and $$\frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t)/t$$

$\therefore$
$$t \cdot C_2 f(t-1) < C(t) < t \cdot C_2 f(t).$$
or
$$t f(t-1) - \sum_{x=1}^{t-1} f(x) < \frac{NC_1}{C_2} < t f(t) - \sum_{x=1}^{t-1} f(x)$$

SAMPLE PROBLEMS

1828. *The following failure rates have been observed for a certain type of transistors in a digital computer :*

End of the week	:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Probability of failure to date	:	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.00

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is made to replace all these transistors simultaneously at fixed intervals, and to replace the individual transistors as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? At what group replacement price per transistor would a policy of strictly individual replacement become preferable to the adopted policy? **[Guwahati MCA 1992]**

Solution. Suppose there are 1,000 transistors in use. Let p_i be the probability that a transistor, which was new when placed in position for use, fails during the ith week of its life. Thus, we have

$p_1 \equiv 0.05,$ $\qquad p_2 \equiv 0.13 - 0.5 = 0.08,$
$p_3 \equiv 0.25 - 0.13 = 0.12,$ $\qquad p_4 \equiv 0.43 - 0.25 = 0.18,$
$p_5 \equiv 0.68 - 0.43 = 0.25,$ $\qquad p_6 \equiv 0.88 - 0.68 = 0.20,$
$p_7 \equiv 0.96 - 0.88 = 0.08,$ $\qquad p_8 \equiv 1.00 - 0.96 = 0.04.$

Let N_i denote the number of replacements made at the end of the ith week. Then, we have

N_o = number of transistors in the beginning $= 1{,}000$
$N_1 = N_o p_1 = 1{,}000 \times 0.05 = 50$
$N_2 = N_o p_2 + N_1 p_1 = 1{,}000 \times 0.08 + 50 \times 0.05 = 82$
$N_3 = N_o p_3 + N_1 p_2 + N_2 p_1 = 1{,}000 \times 0.12 + 50 \times 0.08 + 82 \times 0.05 = 128$
$N_4 = N_o p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1 = 199$
$N_5 = N_o p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1 = 289$
$N_6 = N_o p_6 + N_1 p_5 + N_2 p_4 + N_3 p_3 + N_4 p_2 + N_5 p_1 = 272$
$N_7 = N_o p_7 + N_1 p_6 + N_2 p_5 + N_3 p_4 + N_4 p_3 + N_5 p_2 + N_6 p_1 = 194$
$N_8 = N_o p_8 + N_1 p_7 + N_2 p_6 + N_3 p_5 + N_4 p_4 + N_5 p_3 + N_6 p_2 + N_7 p_1 = 195$

From the above calculations, we observe that the expected number of transistors failing each week increases till 5th week and then starts decreasing and later again increasing from 8th week.

Thus, N_i will oscillate till the system acquires a steady state. The expected life of each transistor is
$$1 \times 0.5 + 2 \times .08 + 3 \times 0.12 + 4 \times 0.18 + 5 \times 0.25 + 6 \times 0.2 + 7 \times 0.08 + 8 \times 0.04$$
$$= 4.62 \text{ weeks.}$$

Average number of failures per week
$$= 1{,}000/4.62 = 216 \text{ approximately.}$$

Therefore, the cost of individual replacement
$$= 216 \times 1.25 = \text{Rs. } 270.00 \text{ per week.}$$

Now, since the replacement of all the 1,000 transistors simultaneously cost 30 paise per transistors and the replacement of an individual transistor on failure cost Rs. 1.25, the average cost for different group replacement policies is given as under :

End of week	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
1	50	$50 \times 1.25 + 1{,}000 \times 0.30 = 363$	363
2	132	$132 \times 1.25 + 1{,}000 \times 0.30 = 465$	232.50
3	260	$260 \times 1.25 + 1{,}000 \times 0.30 = 625$	208.30
4	459	$459 \times 1.25 + 1{,}000 \times 0.30 = 874$	218.50

Since, the average cost is lowest against week 3, the optimum interval between group replacements is 3 weeks. Further, since the average cost is less than Rs. 270 (for individual replacement), the policy of group replacement is better.

1829. *A computer has a large number of electronic tubes. They are subject to mortality as given below :*

Period	Age of failure (hours)	Probability of failure
1	0—200	0.10
2	201—400	0.26
3	401—600	0.35
4	601—800	0.22
5	801—1000	0.07

If the tubes are group replaced, the cost of replacement is Rs. 15 per tube. Group replacement can be done at fixed intervals in the night shift when the computer is not normally used. Replacement of individual tubes which fail in service costs Rs. 60 per tube. How frequently should the tubes be replaced?

Solution. Consider each block of 200 hours as one period and assume that there are 1000 tubes initially.

Let N_i be the number of replacements made at the end of the ith period, if all the 1000 tubes are new initially. Then the expected number of failures at different weeks can be calculated as shown below :

$$N_1 = N_o p_1 = 1000 \times 0.10 = 100$$
$$N_2 = N_o p_2 + N_1 p_1 = 1000 \times 0.26 + 100 \times 0.10 = 270$$
$$N_3 = N_o p_3 + N_1 p_2 + N_2 p_1 = 1000 \times 0.35 + 100 \times 0.26 + 270 \times 0.10 = 403$$
$$N_4 = N_o p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1$$
$$= 1000 \times 0.22 + 100 \times 0.35 + 270 \times 0.26 + 403 \times 0.10 = 365$$
$$N_5 = N_o p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1$$
$$= 1000 \times 0.07 + 100 \times 0.22 + 270 \times 0.35 + 403 \times 0.26 + 365 \times 0.10 = 328$$

From the above calculations, we observe that the number of tubes failing in each period increases till the third period, and then starts decreasing. Thus the value of N_i will oscillate till the system settles down to a steady state. In the steady state, the proportion of tubes failing during each period is the reciprocal of their average life.

Expected life of a tube $= 1 \times 0.10 + 2 \times 0.26 + 3 \times 0.35 + 4 \times 0.22 + 5 \times 0.07 = 2.90$ periods.

Expected number of failures per period $= \dfrac{1000}{2.90} = 345$

$\therefore$ Cost of individual replacements per period $= 345 \times 60 =$ Rs. 20,700.

Now, since the replacement of all the 1,000 tubes simultaneously cost Rs. 15 per tube, and the replacement of an individual tube on failure cost Rs. 60; the average cost for different group replacement policies is given as follows :

End of period (hours)	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
0—200	100	$100 \times 60 + 1000 \times 15 = 21{,}000$	21,000
201—400	370	$370 \times 60 + 1000 \times 15 = 37{,}200$	18,600
401—600	773	$773 \times 60 + 1000 \times 15 = 61{,}380$	20,460
601—800	1138	$1138 \times 60 + 1000 \times 15 = 83{,}280$	20,820

Since, the average cost is lowest against period 2, the optimum interval between group replacements is two periods, *i.e.*, after 400 hours. Further, since the average cost of group replacement (which is Rs. 18,600 is less Rs. 20,700 (cost of individual replacement), the policy of group replacement is better.

1830. *At time zero all items in a system are new. Each item has a probability p of failing immediately before the end of the first month of life, and a probability $q = (1 - p)$ of failing immediately before the end of the second month (i.e., all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures $f(x)$ at the end of month x is given by*

$$f(x) = \frac{N}{1+q}\,[1 - (-q)^{x+1}].$$

where N is the number of items in the system.

If the cost per item of individual replacement is C_1, and the cost per item of group replacement is C_2, find the condition under which

(*a*) *A group replacement policy at the end of each month is the most profitable.*

(*b*) *No group replacement policy is better than a policy of pure individual replacement.*

[Delhi B.Sc. (Stat.) 1994]

Solution. Let

N = number of items in the system in the beginning

N_1 = number of items expected to fail at the end of 1st month

$= N_0 p = N(1-q)$, since $p = 1-q$.

N_2 = number of items expected to fail at the end of 2nd month

$= N_0 q + N_1 p = Nq + N(1-q)^2 = N(1-q+q^2)$,

N_3 = number of items expected to fail at the end of 3rd month

$= N_1 q + N_2 p = N(1-q)\,q + N(1-q+q^2)(1-q) = N(1-q+q^2+q^3)$,

and so on. In general,

$$N_k = N[1 - q + q^2 - q^3 + \ldots + (-q)^k].$$

$$\begin{aligned}\therefore \quad N_{k+1} &= N_{k-1}\,q + N_k p \\ &= N[1 - q + q^2 + \ldots + (-q)^{k-1}]\,q + N[1 - q + q^2 + \ldots + (-q)^k](1-q) \\ &= N[1 - q + q^2 + \ldots + (-q)^{k+1}]\end{aligned}$$

Hence, by mathematical induction, the expected number of failures at the end of month x will be given by

$$f(x) = N[1 - q + q^2 + \ldots + (-q)^x] = N[1 - (-q)^{x+1}]/(1+q).$$

The value of $f(x)$ at the end of month x will vary for different values of $(-q)^{x+1}$ and it will reach the steady-state as $x \to \infty$.

Hence, in the steady state case, the expected number of failures will be

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} N[1 - (-q)^{x+1}]/(1+q) \\ &= N/(1+q)\,;\ \text{since } q < 1 \text{ and } (-q)^{x+1} \to 0 \text{ as } x \to \infty. \\ &= \text{Total number of items in the system / Mean age,}\end{aligned}$$

Now, since C_1 is the cost of replacement per item individually and C_2 is the cost of an item in group, therefore

(*i*) If we have a group replacement at the end of each month, then the cost of replacement is NC_2.

(*ii*) If we have a group replacement policy at the end of every other month, then the cost is $NC_2 + NpC_1$.

The average cost per month, therefore, is $(NC_2 + NpC_1)/2$, and

Average life of an item $= 1 \times p + 2 \times q = 1 \times (1-q) + 2q = 1 + q$.

Therefore, the average number of failures is $N/(1 + q)$ and hence the cost of individual replacement is $NC_1/(1+q)$.

(*a*) A group replacement at the end of first month will be better than individual replacement, if total cost of group replacement is less than the average monthly cost of individual replacement.

Thus, $N(1-q)C_1 + NC_2 < NC_1/(1+q)$, *i.e.*, $C_2 < c_1q^2/(1+q)$.

For a group replacement at the end of every second month, the total cost of replacement will be

$$(N_1 + N_2)C_1 + NC_2 = N(2 - 2q + q^2)C_1 + NC_2.$$

$\therefore$ Average monthly cost of group replacement at the end of second month is

$$[N(2 - 2q + q^2)C_1 + NC_2]/2$$

In this case, the group replacement policy will be better than the individual replacement policy, if

$$\begin{matrix}\text{Average monthly cost of}\\ \text{group replacement}\end{matrix} < \begin{matrix}\text{Average monthly cost of}\\ \text{individual replacement}\end{matrix}$$

or $$[N(2 - 2q + q^2)C_1 + NC_2]/2 < NC_1/(1+q),$$

or $$C_2 < q^2(1-q)C_1/(1+q).$$

(*b*) For the individual replacement policy to be better than any of the group replacement policies discussed above, we must have

$$C_2 > C_1q^2/(q+1) \quad \text{and} \quad C_2 > C_1q^2(1-q)/(q+1)$$

or $$C_1 < C_2(1+q)/q^2 \quad \text{and} \quad C_1 < C_2(1+q)/[q^2(1-q)]$$

But $q < 1$, therefore $(1+q)/q^2 < (1+q)/q^2(1-q)$

Hence, $C_1 < (1+q)C_2/q^2$.

PROBLEMS

1831. In a machine shop, a particular cutting tool costs Rs. 6 to replace. If a tool breaks on the job, the production disruption and associate costs amount to Rs. 30. The past life of a tool is given as follows :

Job No.	1	2	3	4	5	6	7
Proportion of broken tools on job	0.01	0.03	0.09	0.13	0.25	0.55	0.95

After how many jobs, should the shop replace a tool before it breaks down? *[IAS 1989]*

1832. A group of process plants in an oil refinery are fitted with valves. Over a period of time, the failure pattern of these 400 valves has been observed and it is as follows :

Month	1	2	3	4	5	6	7	8	*Total*
Number of valves failing	8	20	48	104	120	56	32	12	400

It costs Rs. 100 to replace each valve individually. If all the valves are replaced at a time, it costs Rs. 50 per valve.

The maintenance department is considering following replacement policies :

(*a*) Replace all valves simultaneously at fixed intervals, in addition to replacing valves as and when they fail.

(*b*) Replace valves as and when they fail.

Suggest the optimal replacement policy. *[Bangalore M.B.A. (June) 1998]* from the production line before replacing a sprinkler at the end of the shift or on weekends. Due to the high setup cost for this task, management is considering a policy of group replacement. What replacement policy should the management adopt when the information regarding sprinkler breakdowns and cost is as given below :

Run Time (months)	*1*	*2*	*3*	*4*	*5*	*6*
Probability of failure	0.05	0.05	0.10	0.10	0.30	0.40

REPLACEMENT COST

	Purchase	*Installation*	*Per unit*
Individual	Rs. 50	Rs. 175	Rs. 225
Group	Rs. 50	Rs. 25	Rs. 75

[Delhi M.B.A. (Nov.) 1998]

18:4. RECRUITMENT AND PROMOTION PROBLEM

Like industrial replacement problems principles of replacement can also be applied to the problems of recruitment and promotion of staff. For this we assume that life distribution of the service of staff in the organization is already known.

SAMPLE PROBLEM

1840. *An airline requires 200 assistant hostesses, 300 hostesses, and 50 supervisors. Girls are recruited at age 21 and if still in service, retire at the age of 60. Given the following Life Table, determine (i) How many girls should be recruited each year? (ii) At what age promotion should take place?*

Life Table for Airline Hostesses

Age	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*
No. in service	1,000	600	480	384	307	261	228	206
Age	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*
No. in Service	190	181	173	167	161	155	150	146
Age	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*
No. in service	141	136	131	125	119	113	106	99
Age	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*
No. in service	93	87	80	73	66	59	53	46
Age	*53*	*54*	*55*	*56*	*57*	*58*	*59*	—
No. in service	39	33	27	22	18	14	11	—

Solution. The total number of girls recruited at age 21 and those serving up to the age of 59 will be equal to 6,480. We require 200 + 300 + 50 = 550 girls in all in the airline.

The recruitment every year is 1,000 when total number of girls is 6,480 after 59 years. Therefore, in order to maintain a strength of 550 hostesses we should recruit

$$550 \times 1{,}000/6{,}480 = 85 \text{ (nearly) every year}$$

If we want to promote the assistant hostesses at the age x, then up to age $x-1$ we need 200 assistant hostesses. Among 550, there are 200 assistant hostesses. Therefore, out of a strength of 1,000 there will be

$$200 \times 1{,}000/550 = 364 \text{ assistant hostesses,}$$

and from the life Table this number is available up to the age of 24 years. Hence the promotion of assistant hostesses will be due in 25th year.

Also, out of 550 recruitments, we need only 300 hostesses.

Therefore, if we recruit 1,000 girls, then we shall require

$$1{,}000 \times 300/550 = 545 \text{ hostesses.}$$

Hence, the number of hostesses and assistant hostesses in a recruitment of 1,000 will be

$$364 + 545 = 909.$$

So, we shall need only 1,000 – 909 = 91 supervisors, whereas at the age of 46 only 86 will survive. Hence promotion of hostesses to supervisors will be due in 46th year.

PROBLEMS

1841. Calculate the probability of a staff resignation in each year from the following survival table :

Year	0	1	2	3	4	5	6	7	8	9	10
Staff at year end	1,000	940	820	580	400	280	190	130	70	30	0

1842. A research team is planned to raise to a strength of 50 chemists and then to remain at that level. The number of recruits depends on their length of service which is as follows :

Year	1	2	3	4	5	6	7	8	9	10
Percentage left at year ends	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain the required strength? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant expects his promotion to one of these posts?

[Bharthidasan B.Com. 1999]

1843. A public sector bank requires 400 clerks, 250 officers, and 50 managers for a certain state. The employees are recruited at the age of 21 years as clerks. The promotions take place only from within on the basis of the length of service. You are given the following life table :

Age (years)	21	22	23	24	25	26	27	28	29	30
No. of employees in service	1,000	640	560	450	400	355	300	205	190	182
Age (years)	31	32	33	34	35	36	37	38	39	40
No. of employees in service	175	170	162	155	150	148	145	142	140	132
Age (years)	41	42	43	44	45	46	47	48	49	50
No. of employees in service	125	118	112	105	101	100	95	90	88	85
Age (years)	51	52	53	54	55	56	57	58	59	60
No. of employees in service	80	72	70	63	55	40	32	25	20	0

Determine :

(*a*) how many employees should be recruited each year?

(*b*) at what age should the promotions take place?

18:5. EQUIPMENT RENEWAL PROBLEM

Knowledge of probability distribution of failure time of certain equipments enables us to solve many practical problems of replacement. In statistical terminology, by the term renewal we mean either to insert a new equipment in place of an old one or to repair the old equipment, so that the *p.d.f.* of its future life-time is that of a new equipment. Thus the problem of replacement can be dealt with what is called the *Renewal Theory*, where the life span of an equipment is considered to be a random variable.

The probability that a renewal occurs during the small time interval $(t, t + dt)$ is called the renewal rate at time t, where t is measured from the instant the first equipment was started. It is generally denoted by $h(t)\, dt$ and $h(t)$ is called the *renewal density function.*

Theorem 18-3. *The renewal rate of a machine is asymptotically reciprocal of the mean life of the machine.*

Proof. Let the *p.d.f.* of failure-time of a machine be $f(x)$. Also let the life-time of all the items in the system follow the same probability distribution, say $f(x)$. that is, if X_i be the life-span for ith machine $(i = 1, 2, \ldots)$ then $X_1, X_2, \ldots$ each has $f(x)$ as if its *p.d.f.*

Let the machine fail $(n-1)$ times during the period $(0, t)$, and at each failure an immediate replacement being made with a similar machine. Then, if at the end of this period, n the machine is in service, we have

$$X_1 + X_2 + \ldots + X_{n-1} < t \quad \text{and} \quad X_1 + X_2 + \ldots + X_n > t.$$

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[18 : 18.1 to 18.4]

(i) Introduction

(ii) Replacement of equipment/asset that deteriorates gradually

(iii) Replacement of equipment that fails suddenly

(iv) Recruitment and Promotion problem.