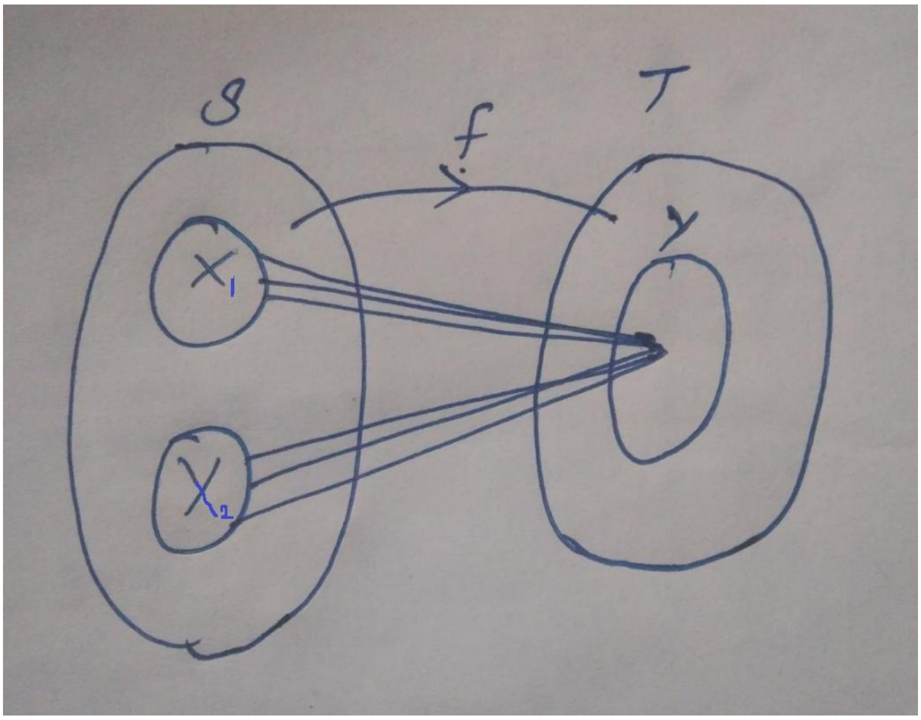
Unit I

Continuity and inverse images of open and closed sets.

Inverse image

Let $f : S \to T$ be a function from a set S to a set T.If Y is a subset of T,the inverse image of Y under f, denoted by $f^{-1}(Y)$ is defined to be the largest subset S which f maps into Y.

$f^{-1}(Y) = \{x : x \in S,\ f(x) \in Y\}$



Result:

Let A and B be subsets of T. then

$A \subset B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$

Proof:

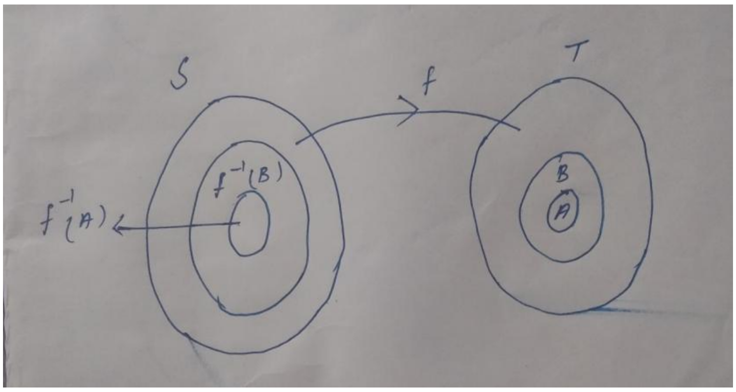
Let $x \in f^{-1}(A)$

$\Rightarrow f(x) \in A$ by defn of $f^{-1}(A)$

$\Rightarrow f(x) \in B$ as $A \subseteq B$

$\Rightarrow x \in f^{-1}(B)$

$\therefore f^{-1}(A) \subseteq f^{-1}(B)$



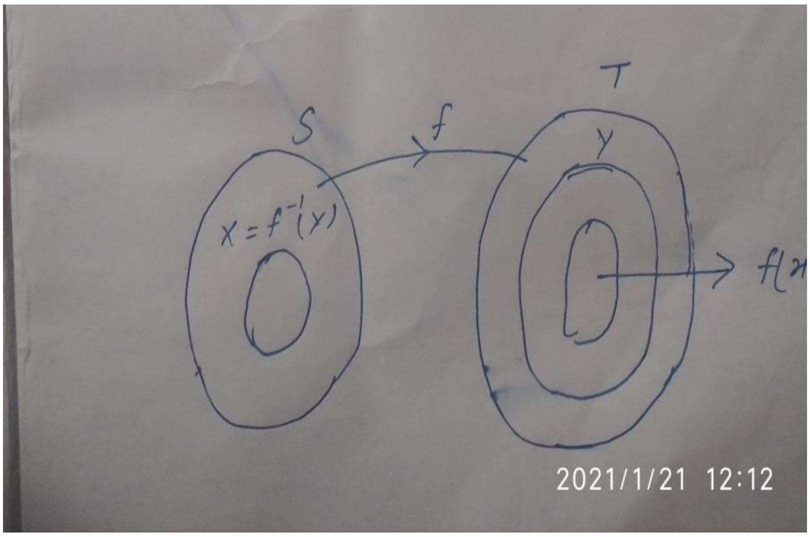
Theorem:

Let $f : S \to T$ be a function from S to T.

If $X \subseteq S$ and $Y \subseteq T$ then we have

(a) $X = f^{-1}(Y) \Rightarrow f(X) \subseteq Y$

(b) $Y = f(X) \Rightarrow X \subseteq f^{-1}(Y)$



(a)

Given $X = f^{-1}(Y)$

Let $y \in f(X)$

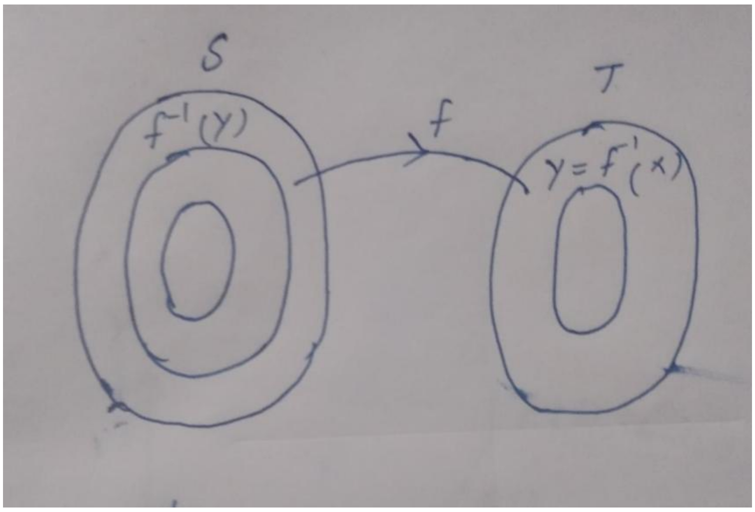
$\Rightarrow y = f(x)$ for some $x \in X$

$\Rightarrow y = f(x)$ for some $x \in f^{-1}(Y)$

$\Rightarrow y \in Y$ by definition of $f^{-1}(Y)$

$\therefore f(X) \subseteq Y$

$X = f^{-1}(Y) \Rightarrow f(X) \subseteq Y$



Given Y= f(X)
Let $x \in X$
$\Rightarrow f(x) \in f(X)$
$\Rightarrow f(x) \in Y$ since Y=f(X)
$\Rightarrow x \in f^{-1}(Y)$
Therefore $X \subseteq f^{-1}(Y)$
Hence Y=f(X) $\Rightarrow X \subseteq f^{-1}(y)$
Thus
$f(f^{-1}(Y)) \subseteq Y$ and $X \subseteq f^{-1}(f(X))$

Theorem:

Let f : S $\rightarrow$T be a function from one metric space (S,d_S) to another metric space (T,d_T) then f is continuous on S iff for every open set Y in T, the inverse image $f^{-1}(Y)$ is open is S.

Proof:

f is continuous $\Leftrightarrow$ inverse image of every open set is open under f

$\Rightarrow$ let f be continuous on S.
Let Y be open in T.

Claim:
$f^{-1}(Y)$ is open in S
Let $p \in f^{-1}(Y)$

To prove

p is an interior point of $f^{-1}(Y)$

Let f(p)=y

As $f(p)\in Y$ and Y is open, f(p) is an interior point of Y.

There exists an $\varepsilon>0$ such that $B_T(y,\varepsilon)\subseteq Y$ …(1)

Since f is continuous at p,there is an $\delta>0$ such that

$$f(B_S(p,\delta)\subseteq B_T(y,\varepsilon) \quad ..(2)$$

Now,

$$B_s(p,\delta)\subseteq f^{-1}(f(B_s(p,\delta))$$
$$\subseteq f^{-1}(B_T(y,\varepsilon) \text{ using (2)}$$
$$\subseteq f^{-1}(Y) \quad \text{using (1)}$$

$\Rightarrow$ p is an interior point of $f^{-1}(Y)$

$\Rightarrow$ $f^{-1}(Y)$ is open in (as p is arbitrary)

$\Leftarrow$

Let inverse image of every open set be open

To prove f is continuous

Let $p\in S$

Then $f(p)\in T$

Let f(p) =y

For every $\varepsilon>0$, the ball $B_T(y,\varepsilon)\subseteq T$

$\therefore$ By hypothesis

$f^{-1}(B_T(y,\varepsilon)$ is open in S.

and $p\in f^{-1}(B_T(y,\varepsilon))$

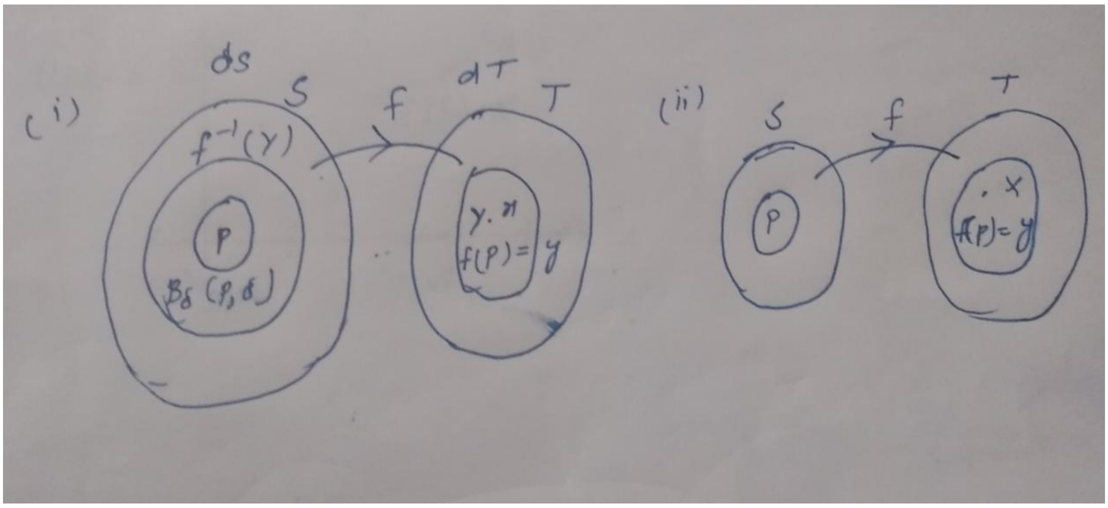
$\therefore$ there exist $\delta>0$ such that

$$B_s(p,\delta)\subseteq f^{-1}(B_T(y,\varepsilon))$$

$$\therefore f(B_S(P,\delta)\subseteq f(f^{-1}(B_T(y,\varepsilon))$$
$$\subseteq B_T(y,\varepsilon)$$

$\Rightarrow$ f is continuous at p. (since p is arbitrary)

$\Rightarrow$ f is continuous on S



Hence the proof.

Theorem:

Let f : S→T be a function from one metric space (S,d_S) to another metric space (T,d_T). Then f is continuous on S iff for every closed set Y in T, the inverse image $f^{-1}(Y)$ is closed in S.

Proof:

Note that, $f^{-1}(T-Y) = S - f^{-1}(Y)$

f is continuous $\Leftrightarrow$ inverse image of every closed set is closed.

$\Rightarrow$ Let f be continuous

To prove inverse image of every closed set is closed.

Let Y be closed in T

Claim: $f^{-1}(Y)$ be closed in S.

Y is closed in T $\Rightarrow$ T-Y is open in T

$\Rightarrow f^{-1}(T-Y)$ is open in S

$\Rightarrow S - f^{-1}(Y)$ is open in S

$\Rightarrow f^{-1}(Y)$ is closed in S.

$\Rightarrow$ inverse image of every closed set is closed.

$\Leftarrow$ Let the inverse image of every closed set closed.

Claim: f is continuous

It is enough to prove that inverse image of every open set is open.

Let Y be open in T

Then T-Y is closed in T

$\Rightarrow f^{-1}(T-Y)$ is closed in S (by hypothesis)

$\Rightarrow S - f^{-1}(Y)$ is open in S

$\Rightarrow$ f is continuous

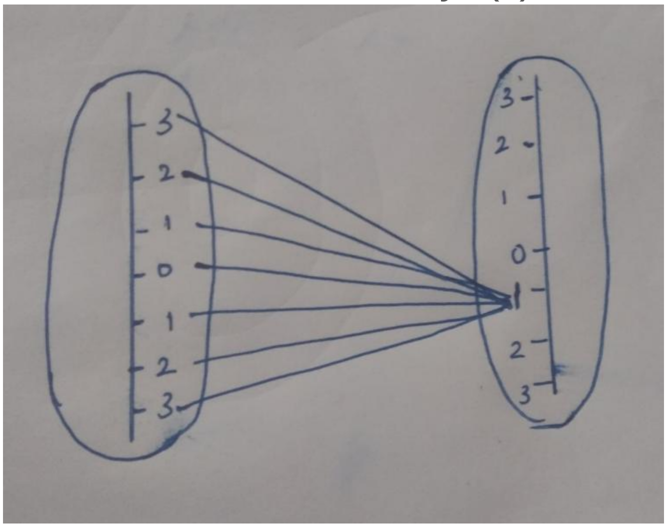
Remark:

The image of an open set under a continuous mapping is not necessarily open.

Counter example:

Consider the function $f : R \rightarrow R$

Define by $f(x) = 1 \ \forall x \in R$



R is open in R.

Its image set f(R)={1}

{1} is not open in R because 1 is not an interior point of R.

(ii) The image of a closed set under a continuous mapping need not be closed.

Consider the example

$f : R \rightarrow (-\Pi/2, \Pi/2)$ defined by $f(x) = \tan^{-1}x$

Then the image of a closed set is not closed in $(-\Pi/2, \Pi/2)$

Continuous functions on compact sets:

Definition of covering:-

A collection F of sets is said to be **covering** of a given set S if

$S \subseteq \cup_{A \in F} A$

The collection F is said to cover S.

If F is a collection of open sets then F is called an **open covering** of S.

Definition: Compact Set

A set S in R^n is said to be **compact** iff every open covering of S contains a finite subcover.

Theorem:

Let $f : S \to T$ be a function from one metric space (S,d_S) to another metric space (T,d_T). If f is continuous on a compact subset X of S, then the image f(X) is a compact subset of T. In particular, f(X) is closed and bounded in T.

Proof:

Given $X \subseteq S$ is compact and f is continuous

Claim: f(X) is compact in T

Let F be an open covering of f(X)

(ie) $f(X) \subseteq \cup_{A \in F} A$

Since each A is open in T and f is continuous on S, each $f^{-1}(A)$ is open in S. These sets $f^{-1}(A)$ form an open covering of X.

For,

$f(X) \subseteq \cup_{A \in F} A$ implies

$f^{-1}(f(X)) \subseteq f^{-1}(\cup_{A \in F} A) = \cup_{A \in F} f^{-1}(A)$

This implies $X \subseteq \cup_{A \in F} f^{-1}(A)$ as $X \subseteq f^{-1}(f(X))$

Since X is compact, a finite number of open sets of $\{f^{-1}(A)/A \subset F\}$ will cover X

(ie) $X \subseteq f^{-1}(A_1) \cup f^{-1}(A_2) \cup \ldots \cup f^{-1}(A_P)$

$\Rightarrow$ $f(X) \subseteq f(f^{-1}(A_1) \cup f^{-1}(A_2) \cup \ldots \cup f^{-1}(A_P))$

$= f(f^{-1}(A_1) \cup f(f^{-1}(A_2) \cup)\ldots\ldots \cup f(f^{-1}(A_P))$

$\Rightarrow$ f(X) is closed and bounded, by theorem "If S is subset of R^n, the following statements are equivalent.

(1) S is compact.
(2) S is closed and bounded
(3) Every infinite subset of S has an accumulation point in S".

Hence the proof.

Bounded set :

A set $X \subseteq R^n$ is a bounded set if there exists $a \in R^n$ and r > 0, such that $X \subseteq B(a,r)$.

Bounded function :

A function $f : S \to R^n$ is called bounded on S if there is a positive number M such that

$$||f(x)|| \leq M \ \forall \ x \in S.$$

Result:

Let $f : S \to R^n$, then f is bounded on S iff f(S) is a bounded set of R^n.

Proof:

$\Rightarrow$

Let $f : S \to R^n$ be bounded on S.
Then $\exists$ M > 0 such that

$$||f(x)|| \leq M \ \forall \ x \in S$$

Claim: f(S) is a bounded subset in R^n.

Since $||f(x)|| \leq M$, $\forall \ x \in S$

$f(S) \subset B(0,M)$ where B(0,M) is an open ball with centre at origin and radius M.

i.e f(S) is a bounded subset in R^n

$\Leftarrow$

Let f(S) be a bounded subset of R^n
Then $\exists$ $a \in R^n$ and r > 0 such that

$$f(S) \subseteq B(a,r)$$

Let $x \in S$

$$\begin{aligned} \|f(x)\| &= \|f(x)-0\| \\ &= \|f(x)-a+a-0\| \\ &\leq \|f(x)-a\|+\|a\| \\ &< (r+\|a\|) \\ &= M_1 \end{aligned}$$

This is true for every $x \in S$.
$\Rightarrow$ f is bounded on S.

Theorem:

Let f : S $\rightarrow$ R^n be a function from a metric space S to the euclidean space R^n. If f is continuous on a compact set X of S then f is bounded on X.

Proof:

Let f be a continuous function on the compact subset X of S.
Then by theorem

"Let f : S $\rightarrow$ T be a function from one metric space (S,d_S) to another metric space (T,d_T). If f is continuous on a compact subset X of S, then the image f(X) is a compact subset of T. In particular, f(X) is closed and bounded in T"

f(X) is compact in R^n and f(X) is bounded in R^n
$\Rightarrow$ f is bounded on X, by the previous result

Result:

If f is a real valued function bounded on X, then f(X) is a bounded subset of R and so it has int f(X) and sup f(X) and

$$\text{Int } f(X) \leq f(X) \leq \sup f(X), \; x \in X$$

Theorem:

Let f : S → R be a real valued function from a metric space S to the euclidean space R. Assume f is continuous on a compact subset X of S. Then there exists a point (p,q) in X such that f(p) = int f(X) ,and f(q) = sup f(X)

Proof:

Since f is a continuous function on the compact subset S, f(X) is compact in R.

Also f(X) is closed and bounded in R

Since f(X) is bounded, $m \leq f(x) \leq M$ with $x \in X$ where m=inf f(X) and M=sup f(X).

$\Rightarrow$ every open ball with m as centre intersects f(X)

$\Rightarrow$ m is an adherent point of f(x)

$\Rightarrow$ m=f(p) for some $p \in X$

Similarly,

$\Rightarrow$ every open ball with M a centre will also intersect f(X)

$\therefore$ M is also an adherent point of f(X)

$M \in f(X)$ as f(X) is closed.

Let M=f(q) for some $q \in X$

Thus f(p)= inf f(X) and f(q) = sup f(X).

Theorem : Let f: S→T be a function from one metric space(S, ds) to another metric space(T,d_T). Assume that f is one-to-one on S,so that the inverse function "f^{-1} " exists. If S is compact and if f is continuous on S, then f^{-1} is continuous on f(S).

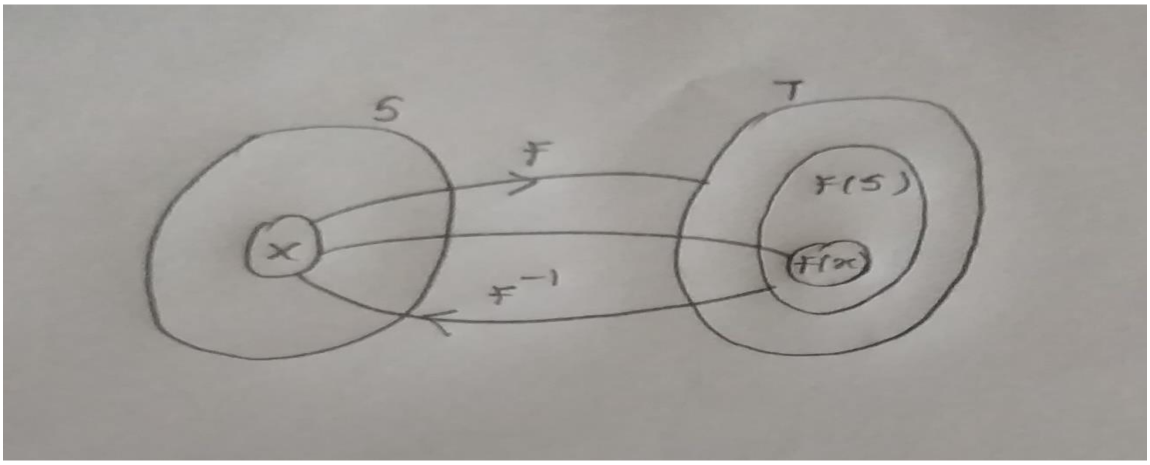
Proof :

Given f is a continuous function on the compact

space S. To prove, f^{-1}: f(S)→S is continuous

We have, to prove that inverse image of every closed set in S is closed in f(S), it is enough is prove that for every closed set X in S, the image f(X)is closed in f(S).

Since X is closed and S is compact.



By theorem,

"Every closed subset of a compact space is compact."

X is compact.

⇒ f(X) is compact. By theorem,

"continuous image of a compact set is compact."

⇒ f(X) is closed by theorem,"

compact subset of a metric space is closed and bounded".

Topological Mappings (Homeomorphisms)

Let f : S →T be a function from one metric space(S,ds) to another metric space (T,d_T). Assume that f is one-to-one on S, so that the inverse function f^{-1} exist. If f is continuous on S and if f^{-1} is continuous on f(S) then f is called a topological mapping or a homeomorphism.

In this case the metric spaces (S,ds) and (T,d_T) are said to be homeomorphic.

Note:

1. f is a homeomorphism then f^{-1} is also a homeomorphism.
2. A homeomorphism maps open subsets of S onto open subsets of f(S) and
3. It maps closed subsets of S onto closed subsets of f(S)

Topological property:

Definition:

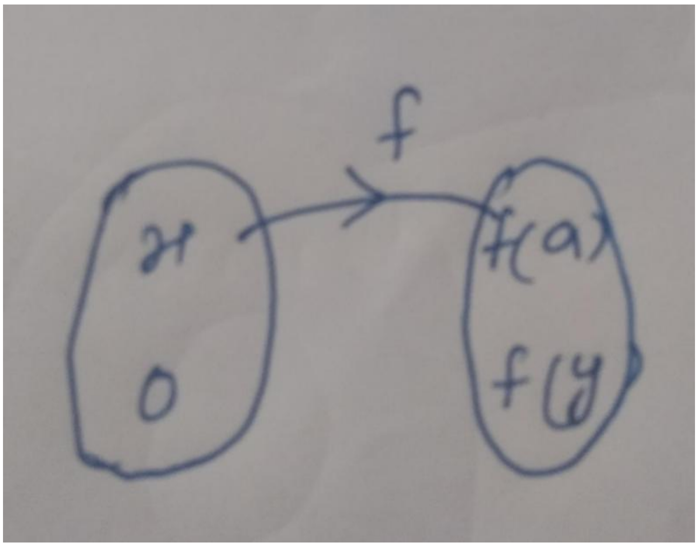
A property of a set that remains invariant under every topological mapping is called a **topological property**.

Example:

The properties of being open, closed and compact are topological properties.

Definition: **Isometry**

A function f : S→T which is one to one on S and which preserves the metric is called an isometry.



(ie) If S is a isometry then,

$$d_S(x,y) = d_T\ (f(x),f(y)) \quad \text{for every } x,y \in S$$

If there is an isometry from (S,ds)→(f(S),d_T) the two metric spaces are called isometric.

Sign preserving property of continuous functions:

Theorem:

Let f be defined on an interval S in R. Assume that f is continuous at a point c in S and that f(c)≠0. Then there exist a one ball B(c,δ) such that f(x) has the same sign as f(c) in B(c,δ)∩S.

Proof:

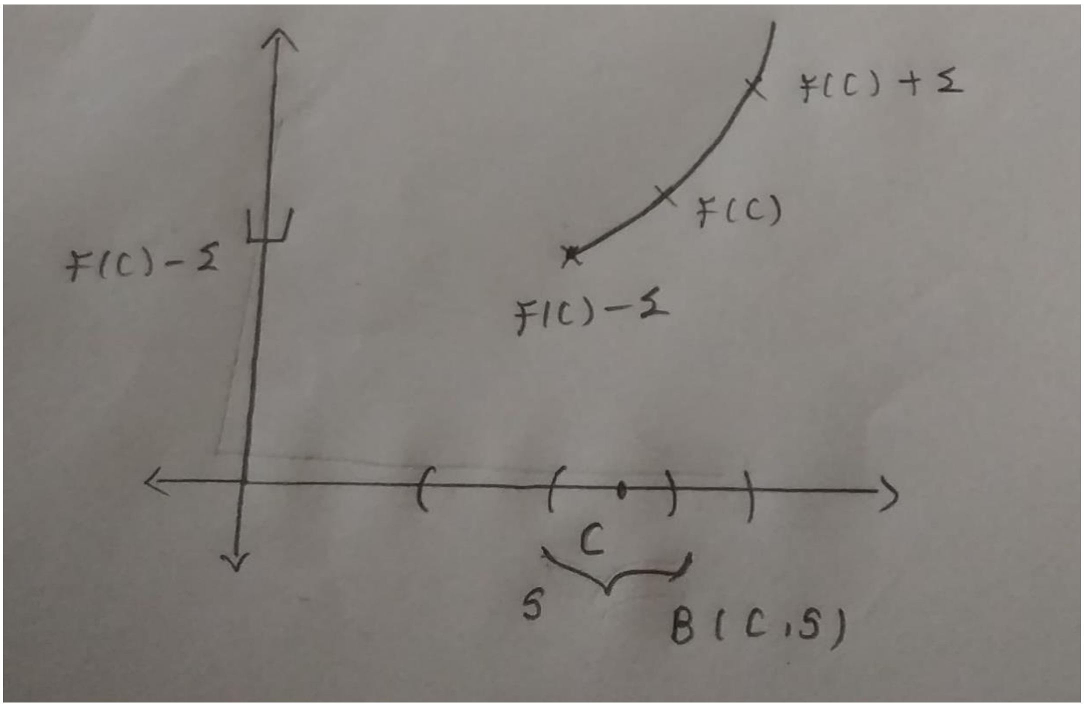
Since f is continuous at the point c there exist and ε >0 for the given δ>0 such that

$$f(c)-\varepsilon < f(x) < f(c)+\varepsilon \quad x \in B(c,\delta) \cap S. \tag{1}$$

It's given that $c \in S$,

$f(c) \neq 0$ Suppose $f(c) > 0$

Then take $\varepsilon = f(c)/2$ in (1)



$\Rightarrow f(c) - f(c)/2 < f(x) < f(c) + f(c)/2$

$\Rightarrow \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c) \quad x \in B(c,\delta) \cap S$

$\Rightarrow f(x) > 0$ for every

$x \in B(c,\delta) \cap S$

Suppose $f(c) < 0$

$\varepsilon = -f(c)/2$ in (1)

$\Rightarrow f(c) + f(c)/2 < f(c) < f(c) - f(c)/2 \quad$ □ $x \in B(c,\delta) \cap S$

$\Rightarrow \frac{3}{2} f(c) < f(x) < f(c)/2$

Let f(c) = -m where m is positive

Then $-\frac{3}{2}m < f(x) < -\frac{m}{2} \quad \forall\, x \in B(c,\delta) \cap S$

$\Rightarrow f(x) < 0 \quad \forall\, x \in B(c,\delta) \cap S$

Hence the proof.

Bolzano's theorem:

Let f be a real valued and continuous function on an interval [a,b] in R and suppose that f(a)and f(b) have opposite signs (i,e) f(a)f(b)<0 then there is atleast one point c in open interval (a,b) such that f(c) =0

Proof:

Given that f(a) and f(b) have opposite signs.

Assume that f(a)>0 and f(b)<0

Let $A = \{x : x \in [a,b],\ f(x) \geq 0\}$

Then A is non-empty. Since f(a)>0 , $a \in A$

Also A is bounded

above by b. Therefore

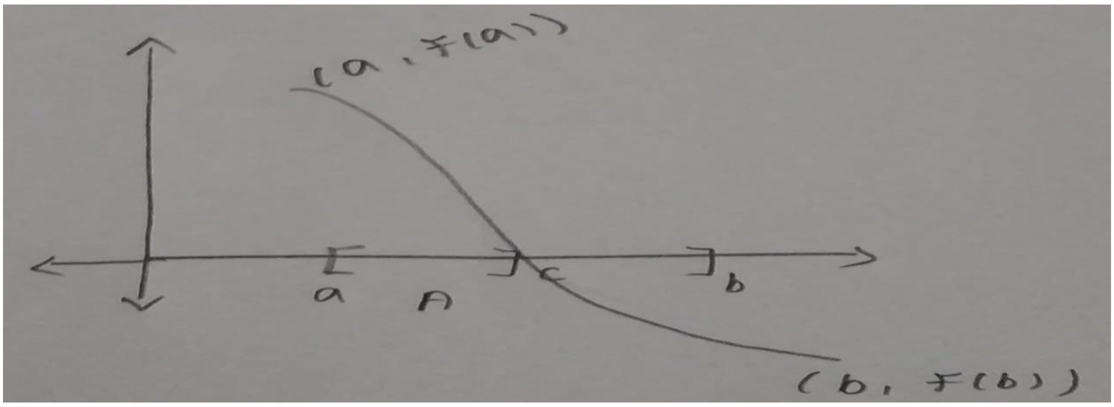
A has a supremum.

Let c = sup A

Now a < c < b

To prove, f(c) =0

If f(c) ≠ 0 then by the sign preserving property of real valued continuous function there is a 1-ball $B(c,\delta)$ in which f has the same sign as f(c). If f(c) >0 there are points x>c at which f(x) > 0, contradicting the definition of C.



If f(c)<0, then c - $\delta/2$ is an upper bound for A again contradicting the definition of c. Hence f(c) =0.

Theorem:

Assume f is real valued and continuous on a compact interval S in R. Suppose there are two points α < β in S such that f(α) ≠ f(β) then f takes every value between f(α) and f(β) in the interval (α,β).

Proof:

Let α and β be such that f(α) ≠ f(β).

Let k be a number between f(α) and f(β)

Define: g: α,β]→R

g(x) = f(x)-k

then

g(α) = f(α)-k

g(β) = f(β)-k

then $g(\alpha)$ and $g(\beta)$ have opposite sides

as $f(\alpha)<k<f(\beta)$ (or) $f(\beta)<k<f(\alpha)$

By Bolzano's theorem, there exists $c \in (\alpha,\beta)$ such that

$$g(c) = 0$$

(i,e) $f(c)-k = 0$

$\Rightarrow$ $f(c) = k$

$\Rightarrow$ for every k in between $f(\alpha)$ and $f(\beta)$ there exists a

$c \in (\alpha,\beta)$ э $f(c) = k$ (k is arbitrary)

$\Rightarrow$ f takes every value between $f(\alpha)$ and $f(\beta)$.

Remark:

The continuous image of a compact interval S under a real valued function is another compact interval [inf f(S),supf(S)].

Proof:

By intermediate value theorem, the function f: $S\rightarrow R$ defined on a compact interval takes every value between f(a) and f(b) if S = [a, b]. This together with the theorem "Let f: $S\rightarrow R$ be a real valued function from a metric space S to Euclidean space R.

Assume that f is continuous on a compact subset of S.

Then there exists points p and q in x such that, f(p) = inf f(X) and f(q) = sup f(X)" we have, f(S) = [inf f(S), sup f(S)].

Connectedness

Definition:

A metric space S is called disconnected, if $S = A\cup B$ where A and B are disjoint non-empty open sets in S. S is connected if it is not disconnected.

Note:

A subset X of a metric space S is called connected if when regarded as a metric subspace of S, it is a connected metric space.

Examples:

1. Consider the metric space S = R-{0} with usual Euclidean metric.

   $R-\{0\} = (-\infty,0)\cup(0,\infty)$

   Therefore R-{0} is disconnected.

2. Every open interval is connected.

   Consider, T = {0,1} with the discrete metric
   $d(x, y) = \{1 \text{ if } x\neq y, 0 \text{ if } x=y\}$
   then T becomes the discrete metric space. The possible subsets of T are ϕ,{0},{1},{0,1}. We know

   $$B(a, r) = \{x: d(x, a)<r\}$$

   $$B(0,1/2) = \{0\}, \text{ and } B(1,1/2) = \{1\}$$

   Consider {0}. This is an open set of T.

   Similarly, ϕ,{1},{0,1} are also open

   Thus every subset of T is open.

   $\Rightarrow$Every subset of T is closed.

3. The set Q of rational numbers regarded as a metric subspace of Euclidean space is disconnected for $Q = A\cup B$ where,

A consists of all rationals < $\sqrt{2}$ and B consists of all rationals > $\sqrt{2}$. Also every ball in Q is disconnected.

4. Every metric space S contains non-empty connected subsets. In fact that for each p in S, the set {p} is connected.

Definition:

A real valued function f which is continuous on a metric space S is said to be two valued on f if $f(S) \subseteq \{0,1\}$

In other words a two valued function is a continuous function whose only possible values are 0 and 1.

Note:

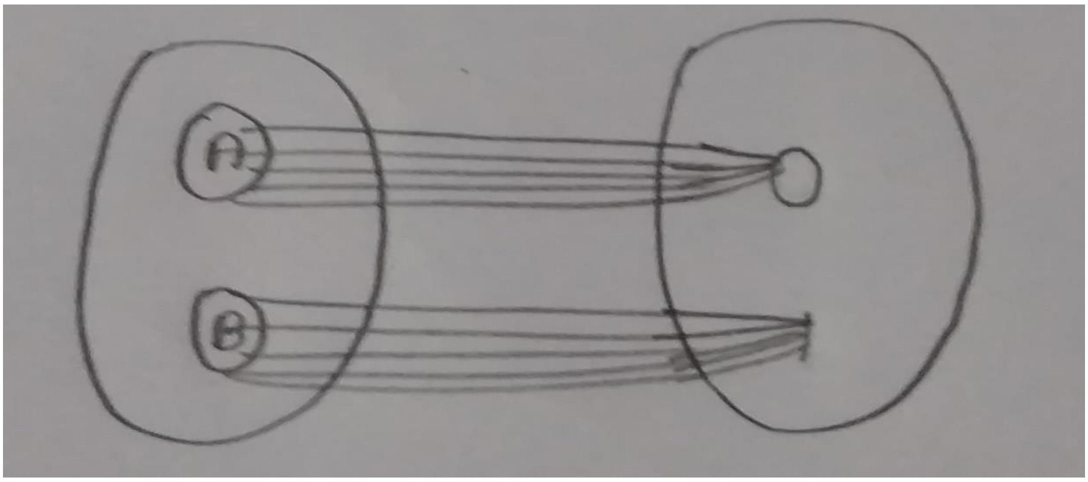
We usually consider the set T = {0, 1} with discrete metric space T, where every subset is both open and closed in T.

Theorem:

A metric space S is connected if and only if every two valued function on S is constant.

Proof:

Assume S is connected . Let f be two valued on S.



Claim: f is constant

Let $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ be the inverse of the subsets {0} and {1}. {0} and {1} are the open subsets of the discrete metric space {0,1}. Since f is continuous, both A and B are open in S.

Also $A \cap B = \emptyset$

Hence $S = A \cup B$ where A and B are disjoint and open.

$\Longrightarrow$ $A = S$ and $B = \emptyset$ (or) $B = S$ and $A = \emptyset$ as (S is not disconnected)

$\Longrightarrow$ f is constant on S.

conversely,

$\Longleftarrow$ To prove, if every two valued function on S is constant, then S is connected.

Suppose S is disconnected.

Then $S = A \cup B$, where A and B are disjoint non-empty open sets of S.

To prove there exists a two valued function on S, which is not constant.

Let f(x)= $\begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \in B \end{cases}$

Since A and B are non-empty,f takes both values 0 and 1.

So f is not a constant. f is continuous on S because the inverse image of every open subset of {0,1} is open in S.

Thus f is two valued but not a constant.

$\Longrightarrow \Leftarrow$ to the hypotheses

Hence S is connected

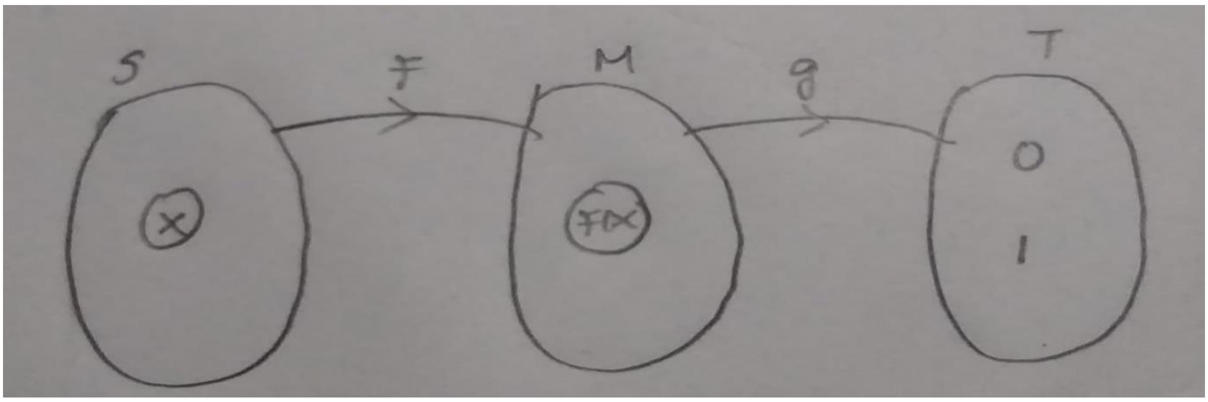
Hence the proof.

Continuous image of a connected set is connected

Theorem :

Let f: S→M be a function from a metric space S to another metric space M. Let X be a connected subset of S. If f is continuous on X, then f(X) is a connected subset of M.

Proof :



Let g be a two valued function on f(X).

To prove : g is a constant.

Consider the composite function h: X→T defined on X by h(x) = g(f(x)). Since, composition of two continuous functions is continuous, h is continuous. As h takes only the values 0 and 1 it is two valued, As X is connected and h is two valued on X, by previous theorem, h is a constant.

⇒ g is constant on f(X).

⇒ f(X) is connected.

Hence the proof.

Result :

Every curve in R^n is connected :

Since an interval X⊆R is connected its continuous image f(X) is connected. If f is real valued the image f(X) is another interval. If f has values in R^n the image, f(X) is called a curve in R^n and it is connected.

Theorem : Intermediate value theorem :

Let f be real valued and continuous on a connected subset of R^n. If f takes two different values in S, say a and b, then for each real c between a and b there exists a point x in S, such that f(x) = c.

Proof :

f(S) being the continuous image of a connected set is connected.

As f(S) ⊆ R and it is connected and it is an interval. Since f takes the values a and b in S, f(S) is an interval containing a and b.

⇒All values in between a and b are taken by f

⇒for $a<c<b$ there exists $x \in S$ such that f(x) = c

Hence the proof.

Components of a metric space

Theorem :

Let F be a collection of connected subsets of a metric space S such that the intersection $T = \bigcap_{A \in F} A$ is non-empty. Then their union $U = \bigcup_{A \in F} A$ is connected.

Proof :

Given $T = \bigcap_{A \in F} A$ where F is a collection of connected subsets of S. Since T is non-empty, let t∈T

⇒$t \in A$ for every $A \in F$ →(1)

To prove, U is connected

Let f be a two valued function on $U = \bigcup_{A \in F} A$. It is enough if we show that f is a constant, for S is connected iff every two valued function on S is connected.

Let $x \in U$ be arbitrary.

⇒ $x \in A$ for some A in F

⇒ $f(x) = f(t)$ $\forall x \in A$

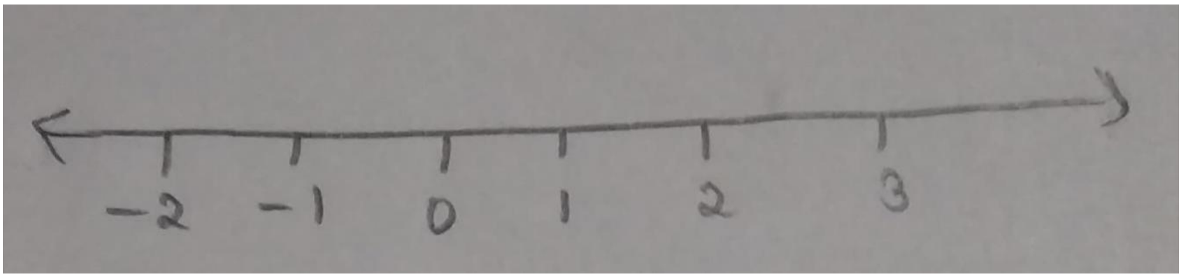
Since A is connected and f is constant on A , by(1)$\forall x \in U$,

$$f(x) = f(t)$$

$\Rightarrow$f is constant on U.

Hence the proof.

Example :



Let A = (0, 1)

B = (1⁄2, 2)

C = (-2, 1)

$A \cap B \cap C \neq \phi$

$A \cup B \cup C$ = (-2, 2) which is connected.

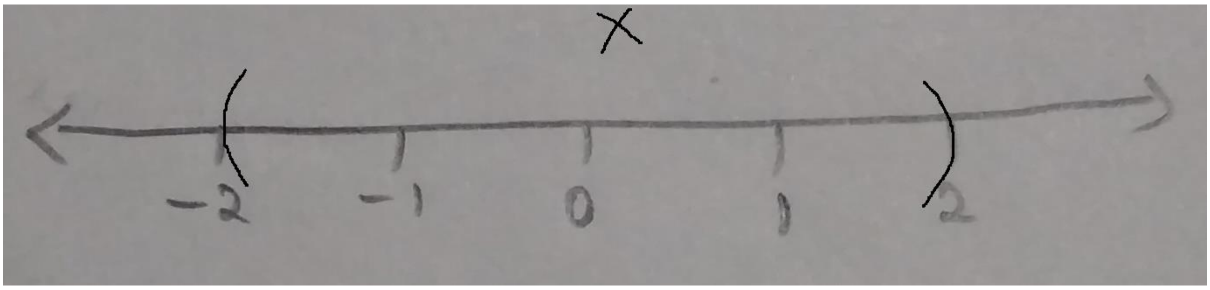
Component of X :

Every point x in a metric space S belongs to atleast one connected subset of S, namely {x}. The union of all the connected subsets which contain x is connected and is called the component of S and is denoted by U(x).

Note :

U(x) is the maximal connected subset of S which contains x.

Example :



In the above example

Let A = (0, 1), B = (1⁄2, 2), C = (-2, 1)

Each of A,B and C is connected consider the element 3⁄4.

3⁄4 ∈ A, 3⁄4 ∈ B, and 3⁄4 ∈ C

The largest connected set containing 3⁄4 is A∪B∪C = (-2, 2) which is connected by above.

then,

U(¾) = (-2, 2)

Theorem :

Every point of a metric space S belongs to a uniquely determined component of S. in other words the components of S form a collection of disjoint sets whose union is S.

Proof :

Since every point x∈S belongs to atleast one connected subset called {x}, we can say that x belongs to atleast one component of S.

Union of components of S is S.

Components are disjoint.

Suppose $x \in U_1 \cap U_2$ where U_1 and U_2 are two components of S.

$\Rightarrow U_1 \cup U_2$ is a maximal connected set containing X condratracting the fact that U_1 and U_2 are maximal connected sets containing X.

$\Rightarrow U_1 \cap U_2 = \phi$

Hence the proof.