Unit II

Definition : Uniform continuity

Let f : S→T be a function from on metric space (S, d_S) to another (T, d_T). Then ‘f’ is said to be uniformly continuous on a subset A of S if the following condition holds.

“For every ∈>0 there exists a δ >0 (depending only on ∈) such that if x ∈ A and p∈ A then, ds(f(x), f(p)) < ∈ whenever ds(x, p)< δ”.

Note :

Continuity is defined at a point where as uniform continuity is defined on a set.

Hence we say continuity of a function is a local property where as uniform continuity of a function is a global property.

Examples:

Consider f : (0,1] →R defined by

$$f(x) = 1/x$$

The function f is continuous on A = (0,1] but the function is not uniformly continuous

Let $\epsilon = 0$. Let $0< \delta <1$

Take $x= \delta$ and $p= \delta/11$

then $|x-p| = |\delta - \delta/11|$

$= \delta 10 /11 < \delta$

And $| f(x) – f(p)| = | 1/x – 1/p|$

$= | 1/\delta – 11/\delta|$

$= 1/\delta \, | -10 |$

$= 10/\delta > 10$

Hence for these two points we always have $| f(x) – f(p) | > 10$

Hence f is not uniformly continuous.

Consider $f : (0,1] \to R$ defined by

$$f(x) = x^2$$

f is uniformly continuous on A(0,1]

$| f(x) – f(p) | = | x^2 - p^2|$

$= | (x-p) \, (x+p) |$

$< | x+p | \, | x-p |$

$< 2 \, | x-p |$ as $x,p \in (0,1]$

If $x+p < \delta$ then $f(x) – f(p) < 2\delta$.

Hence if ε is given take $\delta = \varepsilon / 2$, so that $|x-p| < \delta$

$\Rightarrow | f(x) - f(p) | < \epsilon$

Thus f is uniformly continuous on A.

Note :

If the above function is defined on R, instead of (0,1] then F is not uniformly continuous on R.

Note:

Uniform continuity on A ⇒ continuity on A

Uniform continuity and compact sets

Theorem: (Heine)

Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Let A be a compact subset of S and assume that f is continuous on A. Then f is uniformly continuous on A.

Proof:

Let $\epsilon > 0$ be given. Then each point 'a' in A such that

$d_T(f(x), f(a)) < \varepsilon/2$ whenever $x \in B_S(a,r) \cap A$ --------------(1)

consider the collection of open balls $B_s(a, r/2)$ each with radius r/2. These balls cover A and since A is compact, a finite number of these also cover A.

$$\therefore A \subseteq \cup_{k=1}^{m} B_s(a_k, \frac{r_k}{2}) \text{ ------(2)}$$

In any ball of twice the radius $B(a_k, r_k)$ by (1) we have.

$d_T(f(x), f(a_k)) < \varepsilon/2$ whenever $x \in B_s(a_k, r_k) \cap A$ ----------(3)
Let $\delta = \min\{r_1/2, r_2/2, \ldots\ldots\ldots r_m/2\}$
Let $x, p \in A \ni ds(x,p) < \delta$ ------------------------(4)

Then $\exists$ a ball $B(a_k, r_k/2)$ containing x from (2)
using (3) we have,

$$d_T (f(x), f(a_k)) < \varepsilon/ 2 \text{ ---------------------------(5)}$$

by the triangle inequality we have,

$$\begin{aligned} d_S (p,a_k) &\leq d_S (p,x) + d_S (x, a_k) \\ &< \delta + r_k/2 \\ &= r_k/2 + r_k/2 \text{ (defn of } \delta) \\ &= r_k \end{aligned}$$

$\Rightarrow p \in B_S(a_k , r_k) \cap A$
$\Rightarrow d_T (f(p), f(a_k)) \leq \varepsilon/2$ by (3)
Once again by triangle inequality,

$$\begin{aligned} d_T (f(x), f(p)) &\leq d_T (f(x), f(a_k)) + d_T (f(a_k), f(p)) \\ &< \varepsilon /2 + \varepsilon /2 \\ &= \varepsilon \qquad \text{[using (3) and (2)]} \end{aligned}$$

Hence the proof.

Fixed point theorem for contractions:

Definition: Fixed point
Let (s,d) be a metric space be a function into itself. A point p in S is called a fixed point of r, If F(p) = p.

Definition: Contraction mapping
The function f: S→S is called a contract of S. If there is a positive number α<1 [called a contraction constant]
Such that, $d[f(x), f(y)] \leq \alpha\, d(x,y)$ for all x,y in S

Result:
A contraction mapping of any metric space S is uniformly continuous on S.

Suppose f: $S \to S$ is contraction mapping then $\exists$ $0<\alpha<1$ such that $d[f(x), f(y)] \leq d(x,y)$

thus give $\varepsilon>0$, $\exists\ \delta = \varepsilon/\alpha$ such that

$d(x,y) < \delta \Rightarrow d(f(x), f(y)) \leq \alpha.\, d(x,y) \Rightarrow < \alpha \,.\, \varepsilon/\alpha$

$= \varepsilon$ for every (x,y) in S

$\Rightarrow$ f is uniformly continuous on S.

Theorem:

<u>Fixed point theorem:</u>

A contraction of a complete metric space S, is a unique fixed point p.

Proof:

First let us prove that a contraction f : $S \to S$ a complete metric space S has almost a fixed point.

Let p and p' be a two fixed points.

Then $d[f(p), f(p')] \leq \alpha\, d(p,p')$ since f is a contraction.

$\Rightarrow d(p,p') \leq \alpha\, d(p,p')$

$\Rightarrow d(p,p') = 0$

$p = p'$

Hence the contraction f can be atmost one fixed point.

A contraction has exactly one fixed point. Take any point x in S and consider sequence x, f(x), f(f(x)), f(f(f(x))),

Let $p_0 = x$

$p_1 = f(p_0)$

$p_2 = f(f(x)) = f(p_1)$

.

.

.

.

.

$p_{n+1} = f(p_n)$ n= 0, 1, 2, 3,………. ----(1)

Then $\{p_n\}$ is a sequence.

To prove $\{p_n\}$ converges to fixed point of f.

First let us prove $\{p_n\}$ is a Cauchy sequence.

$d(p_{n+1}, p_n) = d\{f(p_n), f(p_{n-1})\}$

$\leq \alpha\, d(p_n, p_{n-1})$ ------------------------------(2)

Now $d(p_n, p_{n+1}) \leq \alpha\, d(p_{n-1}, p_{n-2})$ using (1)

$d(p_{n-1}, p_{n-2}) \leq \alpha\, d(p_{n-2}, p_{n-3})$

by repeatedly using (2) we have,

$d(p_n, p_{n+1}) \leq \alpha^n d(p_0, p_1)$

$= c\, \alpha^n$ (where $c = d(p_0, p_1)$)

Now, let m>n

$d(p_m, p_n) \leq d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2}) + \ldots + d(p_{m-1}, p_m)$

by triangle inequality,

$= \sum_{k=n}^{m-1} d(p_k, p_{k+1})$

$\leq \sum_{k=n}^{m-1} \alpha^k$

$= c \sum_{k=n}^{m-1} \alpha^k$

$= c\ (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \ldots + \alpha^{n-1})$

$= c\ \alpha^n (1+ \alpha + \alpha^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \alpha^{m-n})$

$= c\ \alpha^n\ (1- \alpha^{m+n}) / 1-\alpha$

$< c\ \alpha^n / 1-\alpha$

$\to 0$ as $\alpha^n \to 0$ (when $n\to\infty$)

$\Longrightarrow \{p_n\}$ is a Cauchy sequence.

But S is a complete metric space. So that sequence $\{p_n\}$ converges in S.

Let $\{p_n\} \to p$

Now by continuity of f

$$f(p) = f(\lim_{n\to\infty} p_n)$$

$$= \lim_{n\to\infty} f(p_n)$$

$$= \lim_{n\to\infty} p_{n+1} = p$$

Thus $f(p) = p$ and so p is a fixed point of f.

Hence the proof.

<u>Discontinuous of real valued functions:</u>

<u>Right hand limit of F:</u>

Let f be defined on an interval (a,b). Assume $c \in [a,b]$. If $f(x) \to A$, as $x\to c$ through values greater than c then, A is the right hand limit of f at c.

Notation:

$$\lim_{x\to c} f(x) = A$$

The right hand limit A is also denoted by f (c+). In the ε, δ definition of right hand limit.

For over, $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - f(c+)| < \varepsilon$ whenever $c < x < c + \delta < b$

Note:

For the above case, f need not be defined at point c itself.

<u>Continuous from the right:</u>

If f is defined at c and if f(c+) = f(c), we say that f is continuous from the right at c.

<u>Left hand limit of f:</u>

Let f be defined on an interval (a,b).

Assume $c \in (a, b]$

If $f(x) \to A$ as $x \to c$ through the values less than c, then A is the left hand limit of f at c and $\lim_{x\to c} f(x) = A$

The left hand limit A is denoted by f (c-)

the ε, δ definition of left hand limit of f.

For $\varepsilon>0$, $\exists\ \delta>0$, $\ni |f(x) - f(c-)| < \varepsilon$ whenever $a<c-\delta<x<c$

<u>Continuous from the left:</u>

If f is defined at c and if f(c-) = f(c), we say f is continuous from the left at c.

Note:

If a<c<b then f is continuous at c if and only if

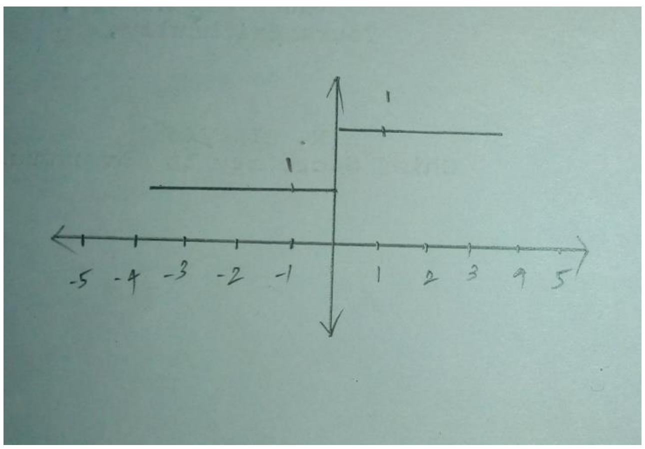
$$f(c) = f(c+)=f(c-)$$

Example:

Let f : R→R be defined by

$$f(x) = \begin{cases} 1 & if\ x < 0 \\ 2 & if\ x \geq 0 \end{cases}$$

Then f is not continuous



f is not continuous at 0 because

f(0) = 2, f (0+) = 2, f(0-) = 1

but f is continuous from the right

since f(0) = f(0+)

Discontinuity of f:

If f is not continuous at c, then c is discontinuous at c. In this case one of the following conditions is satisfied.

(a) Either f(c+) or f(c-) does not exist

(b) Both $f(c+)$ and $f(c-)$ exist but have different values [irremovable discontinuity]

(c) Both $f(c+)$ and $f(c-)$ exist and $f(c+) = f(c-) \neq f(c)$ [Removable discontinuity]

In case (c),the point c is a removable discontinuity, since the discontinuity could be removed by redefining f at c to have $f(c+) = f(c-)$.

In case (a) and (b), we call c is a irremovable discontinuity because the discontinuity cannot be removed by redefining at c.

Definition:

Let f be defined on a closed interval [a, b]. $f(c+)$ and $f(c-)$ both exist at some interior point c, then

(a) $f(c) – f(c-)$ is called the left hand jump of f at c.

(b) $f(c+) – f(c)$ is called the right hand jump of f at c.

(c) $f(c+) – f(c-)$ is called the jump of f at c.

If any one of these three numbers is non-zero, then c is called jump discontinuity of f.

In the previous example

Left hand jump of f at 0 is,

$f(0) – f(0-) = 2-1 =1$

Right hand jump of $f(0+) – f(0) = 2-2 = 0$

Jump $f(0+) – f(0-) = 2-1 =1$

Jump at end-points of an interval:

Let f be defined on (a, b). Then for the end points only one sided jumps are considered.

At a, the right hand jump at a, f (a+) – f (a)

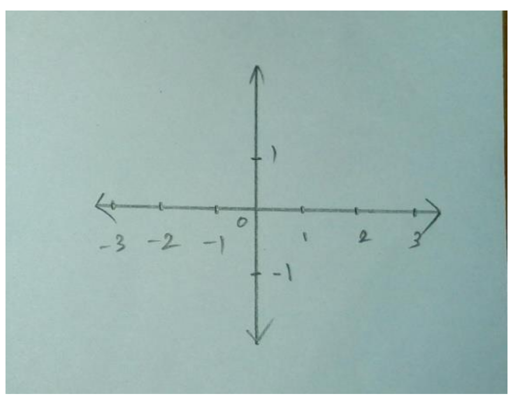
At b the left hand jump at b, f(b) – f(b-)

Example:

(i) Let f : $\mathbb{R} \to \mathbb{R}$ by
f (x) = x/ |x| , if x≠0
f (0) = A

(i.e) f(x) = $\begin{cases} 1 \; if \; x > 0 \\ -1 \; if \; x < 0 \end{cases}$

f (0) = A



Then f (0) = A; f(0+) = 1 and f(0-) = -1

Jump of f at 0 = f (0+) – f (0-)

= 1 – (-1)

= 2 ≠ 0

∴ f has jump discontinuity at 0

Left hand jump of f at 0= f(0) – f(0-)

= A – (-1)

= A+1

Right hand jump of f at 0 = f (0+) – f (0)

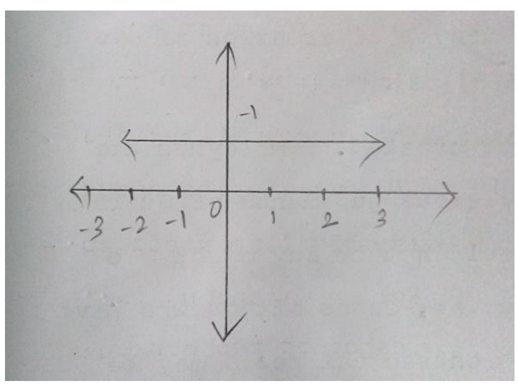
= 1-A

Example:

<u>Removable discontinuity</u>

Let f : R→R be defined by

f (x) = $\begin{cases} 1 \; if \; x \neq 0 \\ 0 \; if \; x = 0 \end{cases}$



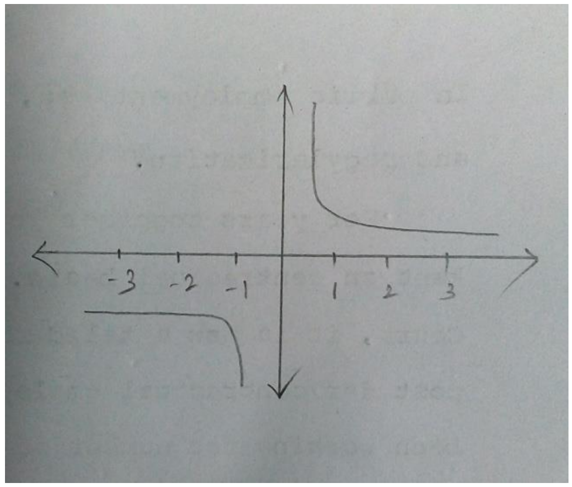
Here, f(0) = 0, f(0+) = 1 , f(0-) = 1

f has removable jump discontinuity at 0,

f can be made continuous by redefining f at 0 as f(0) = 1 (i.e) f (x) =1 , for all x.

(iii) Irremovable discontinuity :

Example:



f : R→R be defined by

f(x) = 1/x , x≠0

f(0) = A

f (0+) = +∞

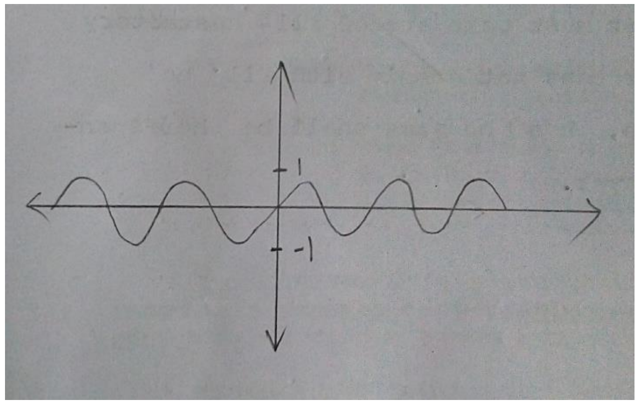
f (0-) = -∞

(i.e) f (0+) and f (0-) do not exist.

∴ f has irremovable discontinuity at 0.

(iv) f : R→R

f (x) = $\begin{cases} \sin\frac{1}{x} & if\ x \neq 0 \\ A & if\ x = 0 \end{cases}$

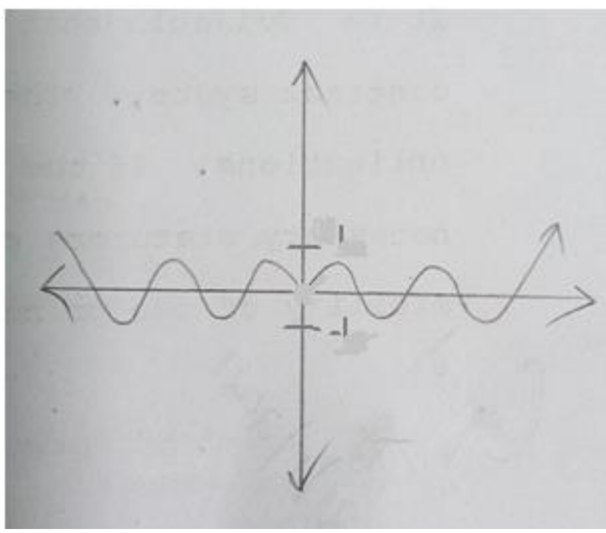


f has irremovable discontinuity at 0.

neither f(0+) nor f(0-) exists.

(v) f: R→R defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$$



f has a removable jump discontinuity at 0.

f(0) = 1, f (0+) = 0, f (0-) = 0

<u>MONOTONIC FUNCTIONS</u>

Definition: **Monotonic increasing (or non decreasing)**

Let f be a real valued function defined on a subset S of $\mathbb{R}$. Then f is said to be increasing on S if for every pair of points x and y in S

$$x < y \Rightarrow f(x) \leq f(y)$$

If $x < y \Rightarrow f(x) < f(y)$, then f is said to be strictly increasing on S.

Definition: **Monotonic decreasing (or non increasing)**

Let f be a real valued function defined on a subset S of $\mathbb{R}$. Then f is said to be decreasing (or non increasing) on S,if for every pair of points x and y in S

$$x < y \Rightarrow f(x) \geq f(y)$$

If $x<y \Rightarrow f(x) > f(y)$, then f is said to be strictly decreasing on S.

Definition: <u>**Monotonic functions**</u>

A function is called monotonic on S if it is increasing on S or decreasing on S.

Result:

If f is an increasing function then -f is an decreasing function.

Proof:

Let f be increasing on S.

Then $x<y \Rightarrow f(x) \leq f(y) \qquad \forall\, x, y \in S$

$$\Rightarrow -f(x) \geq -f(y) \qquad \forall\, x, y \in S$$

Thus $x<y \Rightarrow (-f)(x) \geq (-f)(y) \qquad \forall\, x, y \in S$

(i.e) -f is a decreasing function on S.

Examples:

Let f : ℝ → ℝ be defined by

f (x) = 3x

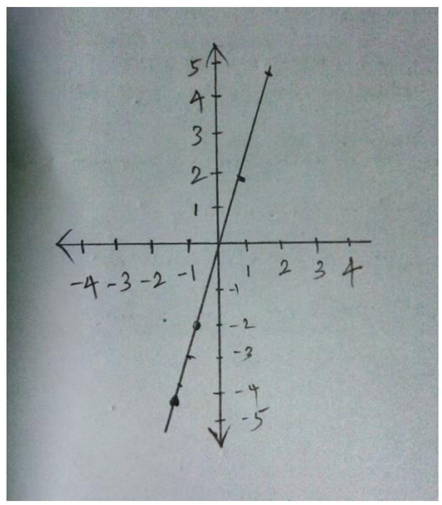
then f is increasing on ℝ.

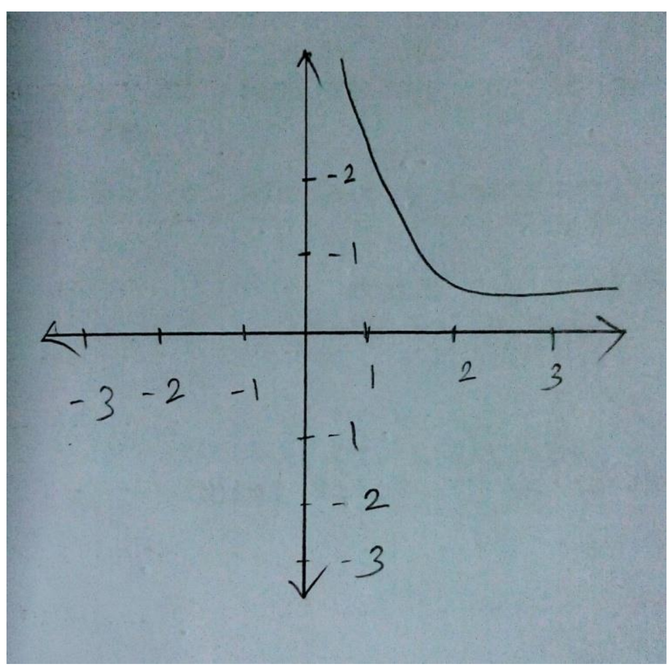
-2 < -1

f (-2) = -6

f (-1) = -3

-6 < -3





Consider,

g : ℝ → ℝ defined by

g(x) = 1/x

g is a decreasing function on R.

Theorem:

If f is increasing on [a, b], then f (c+) and f (c-) both exist for each c in (a, b) and we have

$$f(c-) \leq f(c) \leq f(c+)$$

At the end points we have

$$f(a) \leq f(a+) \text{ and } f(b-) \leq f(b)$$

Proof :

Let A = { f(x): a<x<c}

Since f is increasing A is bounded above by f(c).

∴ A has a supremum.

Let $\propto$ = sup A.

Then $\propto \leq f(c)$ (1)

To prove f(c) exists and f(c-) = $\propto$

To prove for every ε>0, there is a δ>0, such that

$$c - \delta < x < c \Rightarrow |f(x) - \propto| < \varepsilon$$

Since α =supA, by "Approximation Property" of supremum which satisfy

" If S is a non-empty set of real numbers with $b = \sup S$, then for every $a < b$ there is some x in S such that

$$a < x \leq b$$

We have $f(x_1)$ in A such that

$$\alpha - \varepsilon < f(x_1) \leq \alpha \quad \text{where } x_1 < c$$

Since f is increasing for every x in (x_1, c) we have,

$$x_1 < x, \ so \ f(x_1) \leq f(x)$$

$$\Rightarrow \alpha - \varepsilon < f(x_1) \leq \alpha \quad on \ x \in A$$

$$\Rightarrow |f(x) - \alpha| < \varepsilon$$

$$x_1 < x < c \Rightarrow |f(x) - \alpha| < \varepsilon$$

Thus $c - \delta < x < c \Rightarrow |f(x) - \alpha| < \varepsilon$

Hence $f(c_-)$ exists and $f(c_-)= \alpha$ where $\delta = c$

Similarly, there exists $B = \{f(x): c < x < b\}$ and for every $\varepsilon > 0$, there is a δ >0 such that

$$c < x < c_+ \delta \Rightarrow |f(x) - \beta| < \varepsilon$$

(i.e) $f(c_+)$ exists and $f(c_+)= \beta$

At the end points,

If $c = a \ \ then \ f(a-)$ does not exists and

$$f(a) \leq f(a+)$$

If $c = b \ \ then \ f(b+)$does not exist for $f \ on \ [a, b]$ and

$$f(b-) \leq f(b)$$

Hence the proof

Note:

Monotonic function on compact intervals always have right and left hand limits.

Theorem:

Let f be strictly increasing on a set S in $\mathbb{R}$. Then f^{-1} exists and is strictly increasing on $f(S)$.

Proof:

Let f be strictly increasing on S. then

$$x < y \Rightarrow f(x) < f(y)$$

(i.e) different elements have different images

$\Rightarrow f$ is 1-1 on S

$\Rightarrow f^{-1}$ exists on $f(S)$

Claim: $f^{-1} : f(S) \to S$ is strictly increasing

Let $y_1 < y_2$, where $y_1 y_2 \in f(S)$

Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Since f is $1-1$, for $some$ $x^1, x_2 \in S$

$$x_1 = f^{-1}(y_1) \quad \& \quad x_2 = f^{-1}(y_2)$$

Suppose $f^{-1}(y_1) > f^{-1}(y_2)$

$\Rightarrow x_1 > x_2$

$\Rightarrow f(x_1) > f(x_2)$

$\Rightarrow y_1 > y_2$

which is contradiction as $y_1 < y_2$

$\therefore f^{-1}(y_1) < f^{-1}(y_2)$

$$y_1 < y_2 \quad \Rightarrow \quad f^{-1}(y_1) < f^{-1}(y_2) \quad \forall\, y^1, y_2 \in f(S)$$

Hence f^{-1} is strictly increasing on $f(S)$

Theorem:

Let f be strictly increasing and continuous on a compact interval $[a,b]$. Then f^{-1} is continuous and strictly increasing on the interval $[f(a), f(b)]$.

Proof:

By previous theorem, "f is strictly increasing on $[a,b]$. f^{-1} exists and is strictly increasing on $[f(a), f(b)]$.

By theorem,

Let $f : S \to T$ be a function from one metric space (S, d_S) is another (T, d_T). Let f be 1-1. If S is compact and f is continuous on S. Then f^{-1} is continuous on $f(S)$. We have

f is continuous on $[a,b] \Rightarrow f^{-1}$ is continuous on $[f(a), f(b)]$.

Hence the proof

DIFFERENTIATION

Definition: Difference of Quotient

Let f be defined on an open interval(a,b). Then for two distinct points x and c in (a,b) we can form the quotient

$$\frac{f(x)-f(c)}{x-c}$$

This is called Difference Quotient

Definition: Derivative of f (Differentiability of f)

Let f be defined on an open interval (a,b) and let $c \in (a,b)$ then f is said to be differentiable at c whenever the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \quad \text{exists}$$

The limit, denoted by $f'(c)$, is called the derivative of f and c

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Successive of Derivatives

f' is called the first derivative of f. The second derivative of f is f'' at c and is defined by

$$f''(c)=\lim_{x\to c}\frac{f'(x)-f'(c)}{x-c}$$

The successive derivatives of f are defined similarly and the n^{th} derivative is denoted by $f^{(n)}$

Notation:

$f'(c)=\left.\frac{dy}{dx}\right|_{x=c} = Df(c)=y'(c)$ all denote the first derivative of $y=f(x)$ at c

Theorem:

If f is defined on (a,b) and differentiable at a point c in (a,b), then there is a function f^* (depending on f and on c) which is continuous at c and which satisfies the equation

$$f(x)-f(c)=(x-c)f^*(x) \qquad \text{......(1)}$$

for all x is (a,b), with $f^*(c) = f'(c)$. Conversely, if there is a function f^*, continuous at c, which satisfies (1), then f is differentiable at c and $f'(c) = f^*(c)$

Proof:

Let f be differentiable at c . Define f^* on (a,b) by

$$f^*(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & if\ x \neq c \\ f'(c) & if\ \ x = c \end{cases}$$

This satisfies the equation

$$f(x) - f(c) = (x-c)f^*(x) \text{ and}$$

$$f^*(x) = f'(c)$$

Now to prove f^* is continuous at c

(i.e) To prove: As $x \to c$, $f^*(x) = f'(c)$

$$\text{As } x \to c\ ,\ f^*(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$$

$$= f^*(c)$$

Thus $f^*(x) \to f^*(c)$

$\Rightarrow f^*$ is continuous at c.

Conversely,

Let there exist a function f^* on (a,b) continuous at c and which satisfies

$$f(x) - f(c) = (x-c)f^*(x) \qquad \text{……………..(1)}$$

To prove, f is differentiable at c and

$$f'(c) = f^*(c)$$

Divide (1) throughout by $x - c$

$$\frac{f(x)-f(c)}{x-c} = f^*(c)$$

Taking limit on both sides

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f^*(c)$$

$$= f^*(c) \dots\dots\dots\dots\dots (2)$$

As f^* is continuous at c, $f'(c)$ exist. LHS of (2) exist. (i.e) f is differentiable at c. And

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$\Rightarrow \quad f'^{(c)} = f^*(c) \quad by(\,2)$$

Hence proved

Theorem:

If f is differentiable at c then f is continuous at c.

Proof:

Let f be differentiable at c. Then the by previous theorem,

there exist a function f^* continuous at c and it satisfies

$$f(x) - f(c) = (x - c)f^*(x) \dots\dots\dots\dots\dots. (1)$$

As $x \to c$, $(1) \Rightarrow f(x) - f(c) \to 0$

$$\Rightarrow \quad f(x) \to f(c)$$

Thus $\quad x \to c \Rightarrow f(x) \to f(c)$

That is, f is continuous at c.

Algebra of Derivatives

Theorem:

Assume $f \; and \; g$ are defined on (a, b) and differentiable at c. Then $f+g, f-g \; and \; f.g$ are also differentiable at c. This is also true of $f \,/\, g$ if $g(c) \neq 0$. the derivatives at c are given by the following formula.

(i) $\quad (f \pm g)'(c) = f'(c) \pm g'(c)$

(ii) $\quad (f.g)'^{(c)} = f(c)g'^{(c)} + g(c)f'(c)$

(iii) $\quad \left({}^{f}\!/_{g}\right)'(c) = \frac{g(c)f\prime(c)-f(c)g\prime(c)}{g(c)^2}$ provided $g(c) \neq 0$

Proof:

(i) $\quad (f+g)'(c) = \lim\limits_{x \to c} \frac{(f+g)(x)-(f+g)(c)}{x-c}$

$$= \lim_{x \to c} \frac{(f(x)+g(x))-(f(c)+g(c))}{x-c}$$

$$= \lim_{x \to c} \left[\frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}\right]$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} + \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$$

$$= f'(c) + g'(c)$$

Thus if $f \; and \; g$ are differentiable at c, then $(f+g)$ is differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$

Similarly, $(f - g)$ is differentiable at c and

$$(f-g)'(c) = f'(c) - g'(c)$$

(ii) Since f is differentiable at c, by theorem, we have

f^* is continuous (i.e) $\lim_{x\to c} f'(x) = f$*(c)

satisfying $f(x) - f(c) = (x-c)f^*(c) \quad \forall\, x \; in \; (a,b) \ldots\ldots(2)$

And $f^*(c) = f'(c) \ldots\ldots\ldots\ldots\ldots\ldots.(3)$

Since g is differentiable at c, there exists a function g^* continuous at c

(i.e) $\left[\lim_{x\to c} g^*(c) = g'(c)\right] \ldots\ldots\ldots\ldots\ldots.(4)$

satisfying $g(x) - g(c) = (x-c)g^*(x) \quad \forall\; x \in (a,b) \ldots\ldots\ldots\ldots(5)$
and

$$g^*(c) = g'(c) \ldots\ldots\ldots\ldots\ldots(6)$$

$$f(x)g(x) = (f(c) + (x-c)f^*(c))(g(c) + (x-c)g^*(x))$$

$$= f(c)g(c) + (x-c)f(c)g^*(x) + (x-c)g(c)f^*(c) + (x-c)^2 g^*(x)f^*(c)$$

$$f(x)g(x) - f(c)g(c) = (x-c)[f(c)g^*(x) + g(c)f^*(c)] + (x- \ c)^2 f^*(x)g^*(x)$$

$$\lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} = \lim_{x\to c} f(c)\, g^*(x) + \lim_{x\to c} g(c)f^*(x) + \lim_{x\to c} g^* \ (c)f^*(x)$$

$$= f(c)\lim_{x\to c} g^*(x) + g(c)\lim_{x\to c} g^*(x) + 0$$

$$= f(c)g^*(c) + g(c)f^*(c) \;\; by \; (1) \; and \; (4)$$

$$= f(c)g'(c) + g(c)f'(c) \;\; by(3)\; and\; (6)$$

$$\lim_{x \to c} \frac{(fg)(x)-(fg)(c)}{x-c} = f(c)g'^{(c)} + g(c)f'(c) \quad \ldots\ldots\ldots\ldots\ldots.(7)$$

Since the R.H.S of (7) exists, the limit on L.H.S also exists

$\therefore fg$ is differentiable and

$$(fg)'(c) = f(c)g'(c) + g(c)f'(c)$$

(iii) $$\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)} = \frac{g(c)f(x)-f(c)g(x)}{g(x)g(c)}$$

$$= \frac{g(c)[f(c)+(x-c)f^*(x)]-f(c)[g(c)+(x-c)g^*(x)]}{[g(c)+(x-c)g^*(x)]g(c)}$$

$$= \frac{g(c)f(c)+(x-c)g(c)f^*(x)-f(c)g(c)-(x-c)f(c)g^*(x)}{\left(g(c)\right)^2+(x-c)g(c)g^*(x)}$$

$$= \frac{(x-c)[g(c)f^*(x)-f(c)g^*(x)]}{\left(g(c)\right)^2+(x-c)g(c)g^*(x)}$$

$$\lim_{x \to c} \frac{\left({}^{f}/_{g}\right)x-\left({}^{f}/_{g}\right)c}{x-c} = \lim_{x \to c} \frac{g(c)f^*(x)-f(c)g^*(x)}{\left(g(c)\right)^2+(x-c)g(c)g^*(x)}$$

$$\lim_{x \to c} \frac{\left({}^{f}/_{g}\right)x-\left({}^{f}/_{g}\right)c}{x-c} = \lim_{x \to c} \frac{g(c)f^*(x)-f(c)g^*(x)}{\left(g(c)\right)^2+(x-c)g(c)g^*(x)}$$

$$= \frac{\lim_{x \to c} g(c)f^*(x)-f(c)g^*(x)}{\lim_{x \to c}\left[\left(g(c)\right)^2+(x-c)g(c)g^*(x)\right]}$$

$$= \frac{g(c)f^*(c)-f(c)g^*(c)}{g(c)^2+0}$$

$$= \frac{g(c)f'(c)-f(c)g'(c)}{g(c)^2} \quad \ldots\ldots\ldots\ldots\ldots..(8)$$

$$\Rightarrow \left({}^{f}/_{g}\right)'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{g(c)^2}$$

R.H.S of (8) exist

$\therefore$ the limit on the L.H.S exists

(i.e) the function $\left({}^{f}/_{g}\right)$ is differentiable and

$$\left({}^{f}/_{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2} \qquad \text{if } g(c) \neq 0$$

Results:

(i) Derivative of a constant function is zero.

Let $f : (a,b) \to \mathbb{R}$. Defined by $f(x) = m\ (a\ constant)$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{m - m}{x - c} = 0$$

$$f'(x) = 0$$

Since $c \in (a,b)$ is arbitrary, $f'(x) = 0 \quad \forall\, x \in (a,b)$

(ii) Let $f : (a,b) \to \mathbb{R}$ by defined by $f(x) = x$

Let $c \in (a,b)$ be arbitrary

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{x - c}{x - c} = 1$$

$\Rightarrow f'(x) = 1$

Thus $f(x) = x \Rightarrow f'(x) = 1$

(iii) Let $f : (a,b) \to \mathbb{R}$ by defined by $f(x) = x^2$

$$f(x) = x^2$$

$$= x.x$$

$$= f_1(x) + f_2(x)$$

$$f'(x) = (f_1 f_2)'x$$

$$= f_1(x) f_2{}'(x) + f_1{}'(x) f_2(x)$$

$$= x.1 + x.1$$

$$f'(x) = 2x$$

Thus $f(x) = x^2 \Rightarrow f'(x) = 2x$

(i) Let $f : (a, b) \to \mathbb{R}$ by defined by $f(x) = x^3$

$$f(x) = x^3$$

$$= x^2.x$$

$$= f_1(x) + f_2(x)$$

$$f'(x) = (f_1 f_2)'x$$

$$= f_1(x) f_2{}'(x) + f_1{}'(x) f_2(x)$$

$$= x^2.1 + 2x.x = 3x^2$$

Thus $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

Thus, we see that every polynomial has a derivative everywhere in $\mathbb{R}$ and every rational function has derivative wherever it is defined.

Chain Rule

Theorem:

Let f be defined on an open interval S, let g be defined on $f(S)$ and consider the composite function $g_{\circ}f$ defined on S by the equation

$$(g_{\circ}f)(x) = g[f(x)]$$

Assume there is a point c in S such that $f(c)$ is an interior point of $f(S)$. If f is differentiable at c and if g is differentiable at $f(c)$, then $g_{\circ}f$ is differentiable at c and we have

$$(g_{\circ}f)'(c) = g'[f(c)]f'(c)$$

Proof:

Since f is differentiable at c, there exists a function f continuous at c ………………(1)

satisfying $f(x) - f(c) = (x - c)(x) \quad \forall\, x \in S \quad \ldots\ldots\ldots\ldots.(2)$

And $f^*(c) = f'(x)$ …………………………....(3)

Since g is differentiable at $f(c)$, there exists a function g* continuous at $f(c)$.

Satisfying $g(y) - g\big(f(c)\big) = \big(y - f(c)\big)g^*(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots..(4)$

And $g^* f(c) = g'\left(f(c)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$

Choose x in S such that $f(x) = y$

Then (5) becomes

$$g(f(x)) - g\big(f(c)\big) = \big(f(x) - f(c)\big)g^* f(x) \quad (since\ y = f(x))$$

$$= (x-c)f^*(x)g^*(f(x))$$

$$\frac{g(f(x))-g\big(f(c)\big)}{x-c} = f^*(x)g^*\big(f(x)\big) \quad \ldots\ldots\ldots\ldots.(6)$$

Since f and g is continuous at c. $g^*{}_{\circ}f$ is continuous at c

As $\quad x \to c, \quad g^*(f(x)) \to g^*(f(c) = g'\big(f(c)\big) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(7)$

Taking limit on both sides on eqn 6 we have

$$\lim_{x\to c}\frac{g(f(x))-g\big(f(c)\big)}{x-c} = \lim_{x\to c} f^*(x)g^*\big(f(x)\big)$$

$$= f^*(c)g^*\big(f(c)\big)$$

$$f'(c)g\big(f'(c)\big)$$

Since R.H.S exist, the limit on L.H.S exist

Thus $\quad (g_{\circ}f)$ is differentiable at c and

$$(g_{\circ}f)'(c) = g'[f(c)]f'(c)$$

One sided derivatives and infinite derivatives

Definition: Right hand derivative and Left hand derivative

Let f defined on a closed interval S and assume that f is continuous at the point c in S. Then f is said to have a right hand derivative at c. If the right hand limit

$$\lim_{x\to c+}\frac{f(x)-f(c)}{x-c}$$

exists as a finite value (or) if the limit is $+\infty$ $(or) - \infty$. This limit will be denoted by $f_+{}'(c)$

$$f_+{}'(c) = \lim_{x \to c+} \frac{f(x)-f(c)}{x-c}$$

f is said to have the left hand derivative and c is the left hand limit if

$$\lim_{x \to c_-} \frac{f(x)-f(c)}{x-c}$$

exists as a finite value (or) if the limit is $+\infty\ (or) - \infty$. This limit will be denoted by $f_-{}'(c)$

$$f_-{}'(c) = \lim_{x \to c_-} \frac{f(x)-f(c)}{x-c}$$

Infinite derivatives:

If c is an interior point of S. Then

$$f'(c) = \infty \; if \quad f_+{}'(c) = \infty = f_-{}'(c)$$

Similarly, $\quad f'(c) = -\infty \; if \; f_+{}'(c) = -\infty = f_-{}'(c)$

Note:

Thus f has a derivative (finite or infinite) at an interior point c. iff

$$f_+{}'(c) = f_-{}'(c) = f'(c)$$

Functions with non-zero derivative

Theorem:

Let f be defined on a open interval (a, b) and assume that for some c in (a, b) we have $f'(c) > 0 \;\; or \; f'(c) = +\infty$. Then there is a one-ball $B(c)$ subset of (a, b) in which

$$f(x) > f(c) \quad if \quad x > c \; and$$

$$f(x) < f(c) \quad if \quad x < c$$

Proof:

Let $f'(c)$ be finite and positive. Since f is differentiable at c, there exist a function f^* such that

f^* continuous at c

satisfying $$f(x) - f(c) = (x - c)f^*(x) \quad \forall \, x \; in \; S \;\; \to 1$$

$$f^*(c) = f'(c) > 0$$

$$f^*(c) > 0 \Rightarrow f^*(c) \neq 0$$

Thus f^* is continuous at c and $f^*(c) \neq 0$

By sign preserving property

" Let f be defined on a interval S in $\mathbb{R}$. Assume that f is continuous at point c in S and that $f^*(c) \neq 0$ then there exist a open ball $B(c, \delta)$ such that $f(x)$ has the same sign as $f(c)$ in $B(c, \delta) \cap S$"

There exists an open ball $B(c) \subseteq (a, b)$

such that $f^*(x)$ has the same sign as $f^*(c)$ for every $x \; in \; B(c)$

Since $\qquad f^*(c) > 0 \Rightarrow f^*(c) > 0$

$$\frac{f(x)-f(c)}{x-c} > 0$$

$$\Rightarrow f(x) - f(c) \text{ and } x - c$$

$$\Rightarrow (x-c) > 0 \;\Rightarrow f(x) - f(c) > 0 \text{ and}$$

$$(x-c) < 0 \;\Rightarrow f(x) - f(c) < 0$$

$$x < c \Rightarrow f(x) > f(c) \text{ and}$$

$$x < c \Rightarrow f(x) < f(c)$$

Suppose, $\quad f'(c) = +\infty$

Then $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} \to \infty$

(i.e) there is an one-ball $B(c)$. Which

$$\frac{f(x)-f(c)}{x-c} > 1 > 0 \qquad x \neq c$$

By the same argument of this above

$$x < c \Rightarrow f(x) < f(c) \text{ and}$$

$$x > c \Rightarrow f(x) > f(c)$$

Theorem:

Let f be defined on an open interval (a,b) and assume that for some c in open interval (a,b),with $f'(c) < 0 \; or \; f(c) = -\infty$. Then there is an one-ball $B(c) \subseteq (a,b)$ in which

$$x < c \Rightarrow f(x) > f(c) \quad and \quad x > c \Rightarrow f(x) < f(c)$$

Proof:

Let $f'(c)$ be finite and $f'(c) < 0$. Since f is differentiable at c, there exist a function f^* such that

f^* is continuous at c

satisfying $f(x) - f(c) = (x - c)f^*(x) \quad \forall\, x\, in\, S$

$$f^*(c) = f'(c) < 0$$

$$f^*(c) < 0 implies f^*(c) \neq 0$$

Thus $f*$ is continuous at c and $f^*(c) \neq 0$. By the sign preserving property of continuous function there exist an open ball $B(c) \subseteq (a,b)$ such that $f^*(x)$ has the same sign as $f^*(c)$ for every $x\ in\ B(c)$.

Since $f^*(c) < 0 \Rightarrow f^*(x) < 0$

$$\frac{f(x)-f(c)}{x-c} < 0$$

$f(x) - f(c)\ and\ x - c$ have opposite signs

Thus $x - c > 0 \Rightarrow f(x) - f(c) < 0 \quad and$

$$x - c < 0 \Rightarrow f(x) - f(c) > 0$$

(i.e) $x > c \Rightarrow f(x) < f(c) \quad and$

$$x < c \Rightarrow f(x) > f(c)$$

$f'(c) = -\infty$ there is a one ball in which

$$\frac{f(x)-f(c)}{x-c} < 1 < 0 \ for \ x \neq c$$

By the same argument above

$$x > c \Rightarrow f(x) < f(c) \quad and \ x > c \Rightarrow f(x) > f(c \ )$$

Zero Derivatives and Local Extrema

Definition: Local maximum and Local minimum

Let f be a real valued function defined on a subset S of a metric space M. Assume $a \in S$. Then f is said to have a local maximum at a if there is a ball $B(a)$ such that

$$f(x) \leq f(a) \quad \forall \ x \ in \ B(a) \cap S$$

If $f(x) \geq f(a) \ \ \forall x \ in \ B(a) \cap S$,then f is said to have a local minimum at a.

Note:

A local maximum at a is the absolute maximum of f on the subset $B(a) \cap S$. If f has an absolute maximum at 'a', then 'a' is also a local maximum. However, f can have local maxima at several points in S without having an absolute maximum on the whole set S.

Example:

Consider $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^3$

The function f has neither absolute maximum nor absolute minimum in $\mathbb{R}$

Consider the interval $S = [1,2]\ \ in\ S$. The local minimum is $f(1) = 1$. The local maximum is $f(2) = 8$.

Theorem:

Let f be defined on an open interval (a, b) and assume that f has a local maximum or local minimum at an interior point c of (a, b). If f has a derivative (finite or infinite) at c, then $f'(c) = 0$

Proof:

Let $f'(c)$ be +ve or +∞.

Then by theorem, "Let f be defined on an open interval (a, b) and assume that for some c in open interval (a, b) we have

$f'(c) > 0\ \ or\ f'(c) = +\infty$ then there exist a one ball

$$B(c) \subseteq (a, b) \text{ in which}$$

$$x > c \Rightarrow f(x) > f(c)\ \ \ and\ \ x < c \Rightarrow f(x) < f(c)"$$

f cannot have a local maximum or local minimum at c . If $f(c) < 0\ or - \infty$ then by theorem ,

"Let f be defined on an open interval (a, b) and assume that for some c in open interval (a, b) we have $f'(c) < 0\ \ or\ f'(c) = -\infty$ then there exist a one ball $B(c) \subseteq (a, b)$ in which

$$x > c \Rightarrow f(x) < f(c)\ \ \ and\ \ x < c \Rightarrow f(x) > f(c)"$$

f cannot have a local maximum or local minimum at c. But, by hypothesis, the derivative of f at c exists

$$\therefore f'(c) = 0$$

Hence the proof

Note:

(i) Converse of the theorem is not true. Consider $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$f'(0) = 0$$

But f has neither local maximum nor local minimum at x=0

(ii) In the statement of previous theorem

Derivative of f exist at c is important. For, consider $f: \mathbb{R} \to \mathbb{R}$ defined $by f(x) = |x|$. This function attains it minima at zero. But the function is not differentiable at $x = 0$. Note that this f is continuous at $x = 0$ but not differentiable at $x = 0$

(iii) The fact that c is an interior point of $(a, b) in$ the statement of above theorem is important.