FUNCTIONS OF BOUNDED VARIATION AND RECTIFIABLE CURVES

6.1 INTRODUCTION

Some of the basic properties of monotonic functions were derived in Chapter 4. This brief chapter discusses functions of bounded variation, a class of functions closely related to monotonic functions. We shall find that these functions are intimately connected with curves having finite arc length (rectifiable curves). They also play a role in the theory of Riemann–Stieltjes integration which is developed in the next chapter.

6.2 PROPERTIES OF MONOTONIC FUNCTIONS

Theorem 6.1. *Let f be an increasing function defined on $[a, b]$ and let $x_0, x_1, \ldots, x_n$ be $n + 1$ points such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then we have the inequality

$$\sum_{k=1}^{n-1} [f(x_k+) - f(x_k-)] \le f(b) - f(a).$$

Proof. Assume that $y_k \in (x_k, x_{k+1})$. For $1 \le k \le n - 1$, we have $f(x_k+) \le f(y_k)$ and $f(y_{k-1}) \le f(x_k-)$, so that $f(x_k+) - f(x_k-) \le f(y_k) - f(y_{k-1})$. If we add these inequalities, the sum on the right telescopes to $f(y_{n-1}) - f(y_0)$. Since $f(y_{n-1}) - f(y_0) \le f(b) - f(a)$, this completes the proof.

The difference $f(x_k+) - f(x_k-)$ is, of course, the jump of f at x_k. The foregoing theorem tells us that for every finite collection of points x_k in (a, b), the sum of the jumps at these points is always bounded by $f(b) - f(a)$. This result can be used to prove the following theorem.

Theorem 6.2. *If f is monotonic on $[a, b]$, then the set of discontinuities of f is countable.*

Proof. Assume that f is increasing and let S_m be the set of points in (a, b) at which the jump of f exceeds $1/m$, $m > 0$. If $x_1 < x_2 < \cdots < x_{n-1}$ are in S_m, Theorem 6.1 tells us that

$$\frac{n-1}{m} \le f(b) - f(a).$$

This means that S_m must be a finite set. But the set of discontinuities of f in (a, b) is a subset of the union $\bigcup_{m=1}^{\infty} S_m$ and hence is countable. (If f is decreasing, the argument can be applied to $-f$.)

6.3 FUNCTIONS OF BOUNDED VARIATION

Definition 6.3. *If $[a, b]$ is a compact interval, a set of points*

$$P = \{x_0, x_1, \ldots, x_n\},$$

satisfying the inequalities

$$a = x_0 < x_1 \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a, b]$. The interval $[x_{k-1}, x_k]$ is called the kth subinterval of P and we write $\Delta x_k = x_k - x_{k-1}$, so that $\sum_{k=1}^{n} \Delta x_k = b - a$. The collection of all possible partitions of $[a, b]$ will be denoted by $\mathscr{P}[a, b]$.

Definition 6.4. *Let f be defined on $[a, b]$. If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, write $\Delta f_k = f(x_k) - f(x_{k-1})$, for $k = 1, 2, \ldots, n$. If there exists a positive number M such that*

$$\sum_{k=1}^{n} |\Delta f_k| \le M$$

for all partitions of $[a, b]$, then f is said to be of bounded variation on $[a, b]$.

Examples of functions of bounded variation are provided by the next two theorems.

Theorem 6.5. *If f is monotonic on $[a, b]$, then f is of bounded variation on $[a, b]$.*

Proof. Let f be increasing. Then for every partition of $[a, b]$ we have $\Delta f_k \ge 0$ and hence

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} \Delta f_k = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = f(b) - f(a).$$

Theorem 6.6. *If f is continuous on $[a, b]$ and if f' exists and is bounded in the interior, say $|f'(x)| \le A$ for all x in (a, b), then f is of bounded variation on $[a, b]$.*

Proof. Applying the Mean-Value Theorem, we have

$$\Delta f_k = f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}), \qquad \text{where } t_k \in (x_{k-1}, x_k).$$

This implies

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} |f'(t_k)| \, \Delta x_k \le A \sum_{k=1}^{n} \Delta x_k = A(b - a).$$

Theorem 6.7. *If f is of bounded variation on $[a, b]$, say $\sum |\Delta f_k| \le M$ for all partitions of $[a, b]$, then f is bounded on $[a, b]$. In fact,*

$$|f(x)| \le |f(a)| + M \qquad \text{for all } x \text{ in } [a, b].$$

Proof. Assume that $x \in (a, b)$. Using the special partition $P = \{a, x, b\}$, we find

$$|f(x) - f(a)| + |f(b) - f(x)| \le M.$$

This implies $|f(x) - f(a)| \le M$, $|f(x)| \le |f(a)| + M$. The same inequality holds if $x = a$ or $x = b$.

Examples

1. It is easy to construct a continuous function which is not of bounded variation. For example, let $f(x) = x \cos \{\pi/(2x)\}$ if $x \ne 0$, $f(0) = 0$. Then f is continuous on $[0, 1]$, but if we consider the partition into $2n$ subintervals

$$P = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right\},$$

an easy calculation shows that we have

$$\sum_{k=1}^{2n} |\Delta f_k| = \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \cdots + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

This is not bounded for all n, since the series $\sum_{n=1}^{\infty} (1/n)$ diverges. In this example the derivative f' exists in $(0, 1)$ but f' is not bounded on $(0, 1)$. However, f' is bounded on any compact interval not containing the origin and hence f will be of bounded variation on such an interval.

2. An example similar to the first is given by $f(x) = x^2 \cos (1/x)$ if $x \ne 0$, $f(0) = 0$. This f is of bounded variation on $[0, 1]$, since f' is bounded on $[0, 1]$. In fact, $f'(0) = 0$ and, for $x \ne 0$, $f'(x) = \sin (1/x) + 2x \cos (1/x)$, so that $|f'(x)| \le 3$ for all x in $[0, 1]$.

3. Boundedness of f' is not necessary for f to be of bounded variation. For example, let $f(x) = x^{1/3}$. This function is monotonic (and hence of bounded variation) on every finite interval. However, $f'(x) \to +\infty$ as $x \to 0$.

6.4 TOTAL VARIATION

Definition 6.8. *Let f be of bounded variation on $[a, b]$, and let $\sum(P)$ denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. The number*

$$V_f(a, b) = \sup \{\textstyle\sum (P) : P \in \mathscr{P}[a, b]\},$$

is called the total variation of f on the interval $[a, b]$.

NOTE. When there is no danger of misunderstanding, we will write V_f instead of $V_f(a, b)$.

Since f is of bounded variation on $[a, b]$, the number V_f is finite. Also, $V_f \ge 0$, since each sum $\sum(P) \ge 0$. Moreover, $V_f(a, b) = 0$ if, and only if, f is constant on $[a, b]$.

Theorem 6.9. *Assume that f and g are each of bounded variation on $[a, b]$. Then so are their sum, difference, and product. Also, we have*

$$V_{f\pm g} \le V_f + V_g \qquad \text{and} \qquad V_{f\cdot g} \le AV_f + BV_g,$$

where

$$A = \sup\{|g(x)| : x \in [a, b]\}, \qquad B = \sup\{|f(x)| : x \in [a, b]\}.$$

Proof. Let $h(x) = f(x)g(x)$. For every partition P of $[a, b]$, we have

$$\begin{aligned} |\Delta h_k| &= |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\ &= |[f(x_k)g(x_k) - f(x_{k-1})g(x_k)] \\ &\quad + [f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})]| \le A|\Delta f_k| + B|\Delta g_k|. \end{aligned}$$

This implies that h is of bounded variation and that $V_h \le AV_f + BV_g$. The proofs for the sum and difference are simpler and will be omitted.

NOTE. Quotients were not included in the foregoing theorem because the reciprocal of a function of bounded variation need not be of bounded variation. For example, if $f(x) \to 0$ as $x \to x_0$, then $1/f$ will not be bounded on any interval containing x_0 and (by Theorem 6.7) $1/f$ cannot be of bounded variation on such an interval. To extend Theorem 6.9 to quotients, it suffices to exclude functions whose values become arbitrarily close to zero.

Theorem 6.10. *Let f be of bounded variation on $[a, b]$ and assume that f is bounded away from zero; that is, suppose that there exists a positive number m such that $0 < m \le |f(x)|$ for all x in $[a, b]$. Then $g = 1/f$ is also of bounded variation on $[a, b]$, and $V_g \le V_f/m^2$.*

Proof.

$$|\Delta g_k| = \left|\frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})}\right| = \left|\frac{\Delta f_k}{f(x_k)f(x_{k-1})}\right| \le \frac{|\Delta f_k|}{m^2}.$$

6.5 ADDITIVE PROPERTY OF TOTAL VARIATION

In the last two theorems the interval $[a, b]$ was kept fixed and $V_f(a, b)$ was considered as a function of f. If we keep f fixed and study the total variation as a function of the interval $[a, b]$, we can prove the following *additive* property.

Theorem 6.11. *Let f be of bounded variation on $[a, b]$, and assume that $c \in (a, b)$. Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have*

$$V_f(a, b) = V_f(a, c) + V_f(c, b).$$

Proof. We first prove that f is of bounded variation on $[a, c]$ and on $[c, b]$. Let P_1 be a partition of $[a, c]$ and let P_2 be a partition of $[c, b]$. Then $P_0 = P_1 \cup P_2$ is a partition of $[a, b]$. If $\sum(P)$ denotes the sum $\sum |\Delta f_k|$ corresponding to the partition P (of the appropriate interval), we can write

$$\sum(P_1) + \sum(P_2) = \sum(P_0) \le V_f(a, b). \tag{1}$$

This shows that each sum $\sum(P_1)$ and $\sum(P_2)$ is bounded by $V_f(a, b)$ and this means that f is of bounded variation on $[a, c]$ and on $[c, b]$. From (1) we also obtain the inequality

$$V_f(a, c) + V_f(c, b) \leq V_f(a, b),$$

because of Theorem 1.15.

To obtain the reverse inequality, let $P = \{x_0, x_1, \ldots, x_n\} \in \mathscr{P}[a, b]$ and let $P_0 = P \cup \{c\}$ be the (possibly new) partition obtained by adjoining the point c. If $c \in [x_{k-1}, x_k]$, then we have

$$|f(x_k) - f(x_{k-1})| \leq |f(x_k) - f(c)| + |f(c) - f(x_{k-1})|,$$

and hence $\sum(P) \leq \sum(P_0)$. Now the points of P_0 in $[a, c]$ determine a partition P_1 of $[a, c]$ and those in $[c, b]$ determine a partition P_2 of $[c, b]$. The corresponding sums for all these partitions are connected by the relation

$$\sum(P) \leq \sum(P_0) = \sum(P_1) + \sum(P_2) \leq V_f(a, c) + V_f(c, b).$$

Therefore, $V_f(a, c) + V_f(c, b)$ is an upper bound for every sum $\sum(P)$. Since this cannot be smaller than the least upper bound, we must have

$$V_f(a, b) \leq V_f(a, c) + V_f(c, b),$$

and this completes the proof.

6.6 TOTAL VARIATION ON $[a, x]$ AS A FUNCTION OF x

Now we keep the function f and the left endpoint of the interval fixed and study the total variation as a function of the right endpoint. The additive property implies important consequences for this function.

Theorem 6.12. *Let f be of bounded variation on $[a, b]$. Let V be defined on $[a, b]$ as follows: $V(x) = V_f(a, x)$ if $a < x \leq b$, $V(a) = 0$. Then:*

i) *V is an increasing function on $[a, b]$.*

ii) *$V - f$ is an increasing function on $[a, b]$.*

Proof. If $a < x < y \leq b$, we can write $V_f(a, y) = V_f(a, x) + V_f(x, y)$. This implies $V(y) - V(x) = V_f(x, y) \geq 0$. Hence $V(x) \leq V(y)$, and (i) holds.

To prove (ii), let $D(x) = V(x) - f(x)$ if $x \in [a, b]$. Then, if $a \leq x < y \leq b$, we have

$$D(y) - D(x) = V(y) - V(x) - [f(y) - f(x)] = V_f(x, y) - [f(y) - f(x)].$$

But from the definition of $V_f(x, y)$ it follows that we have

$$f(y) - f(x) \leq V_f(x, y).$$

This means that $D(y) - D(x) \geq 0$, and (ii) holds.

NOTE. For some functions f, the total variation $V_f(a, x)$ can be expressed as an integral. (See Exercise 7.20.)

6.7 FUNCTIONS OF BOUNDED VARIATION EXPRESSED AS THE DIFFERENCE OF INCREASING FUNCTIONS

The following simple and elegant characterization of functions of bounded variation is a consequence of Theorem 6.12.

Theorem 6.13. *Let f be defined on $[a, b]$. Then f is of bounded variation on $[a, b]$ if, and only if, f can be expressed as the difference of two increasing functions.*

Proof. If f is of bounded variation on $[a, b]$, we can write $f = V - D$, where V is the function of Theorem 6.12 and $D = V - f$. Both V and D are increasing functions on $[a, b]$.

The converse follows at once from Theorems 6.5 and 6.9.

The representation of a function of bounded variation as a difference of two increasing functions is by no means unique. If $f = f_1 - f_2$, where f_1 and f_2 are increasing, we also have $f = (f_1 + g) - (f_2 + g)$, where g is an arbitrary increasing function, and we get a new representation of f. If g is *strictly* increasing, the same will be true of $f_1 + g$ and $f_2 + g$. Therefore, Theorem 6.13 also holds if "increasing" is replaced by "strictly increasing."

6.8 CONTINUOUS FUNCTIONS OF BOUNDED VARIATION

Theorem 6.14. *Let f be of bounded variation on $[a, b]$. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Then every point of continuity of f is also a point of continuity of V. The converse is also true.*

Proof. Since V is monotonic, the right- and lefthand limits $V(x+)$ and $V(x-)$ exist for each point x in (a, b). Because of Theorem 6.13, the same is true of $f(x+)$ and $f(x-)$.

If $a < x < y \le b$, then we have [by definition of $V_f(x, y)$]

$$0 \le |f(y) - f(x)| \le V(y) - V(x).$$

Letting $y \to x$, we find

$$0 \le |f(x+) - f(x)| \le V(x+) - V(x).$$

Similarly, $0 \le |f(x) - f(x-)| \le V(x) - V(x-)$. These inequalities imply that a point of continuity of V is also a point of continuity of f.

To prove the converse, let f be continuous at the point c in (a, b). Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon/2$. For this same ε, there also exists a partition P of $[c, b]$, say

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad x_0 = c, \quad x_n = b,$$

such that

$$V_f(c, b) - \frac{\varepsilon}{2} < \sum_{k=1}^{n} |\Delta f_k|.$$

Adding more points to P can only increase the sum $\sum |\Delta f_k|$ and hence we can assume that $0 < x_1 - x_0 < \delta$. This means that

$$|\Delta f_1| = |f(x_1) - f(c)| < \frac{\varepsilon}{2},$$

and the foregoing inequality now becomes

$$V_f(c, b) - \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \sum_{k=2}^{n} |\Delta f_k| \le \frac{\varepsilon}{2} + V_f(x_1, b),$$

since $\{x_1, x_2, \ldots, x_n\}$ is a partition of $[x_1, b]$. We therefore have

$$V_f(c, b) - V_f(x_1, b) < \varepsilon.$$

But

$$\begin{aligned} 0 \le V(x_1) - V(c) &= V_f(a, x_1) - V_f(a, c) \\ &= V_f(\overset{x_1,\, c}{c, x_1}) = V_f(c, b) - V_f(x_1, b) < \varepsilon. \end{aligned}$$

Hence we have shown that

$$0 < x_1 - c < \delta \quad \text{implies} \quad 0 \le V(x_1) - V(c) < \varepsilon.$$

This proves that $V(c+) = V(c)$. A similar argument yields $V(c-) = V(c)$. The theorem is therefore proved for all interior points of $[a, b]$. (Trivial modifications are needed for the endpoints.)

Combining Theorem 6.14 with 6.13, we can state

Theorem 6.15. *Let f be continuous on $[a, b]$. Then f is of bounded variation on $[a, b]$ if, and only if, f can be expressed as the difference of two increasing continuous functions.*

NOTE. The theorem also holds if "increasing" is replaced by "strictly increasing."

Of course, discontinuities (if any) of a function of bounded variation must be jump discontinuities because of Theorem 6.13. Moreover, Theorem 6.2 tells us that they form a countable set.