THE RIEMANN-STIELTJES INTEGRAL

7.1 INTRODUCTION

Calculus deals principally with two geometric problems: finding the tangent line to a curve, and finding the area of a region under a curve. The first is studied by a limit process known as *differentiation*; the second by another limit process—*integration*—to which we turn now.

The reader will recall from elementary calculus that to find the area of the region under the graph of a positive function f defined on $[a, b]$, we subdivide the interval $[a, b]$ into a finite number of subintervals, say n, the kth subinterval having length Δx_k, and we consider sums of the form $\sum_{k=1}^{n} f(t_k)\,\Delta x_k$, where t_k is some point in the kth subinterval. Such a sum is an approximation to the area by means of rectangles. If f is sufficiently well behaved in $[a, b]$—continuous, for example—then there is some hope that these sums will tend to a limit as we let $n \to \infty$, making the successive subdivisions finer and finer. This, roughly speaking, is what is involved in Riemann's definition of the definite integral $\int_a^b f(x)\,dx$. (A precise definition is given below.)

The two concepts, derivative and integral, arise in entirely different ways and it is a remarkable fact indeed that the two are intimately connected. If we consider the definite integral of a continuous function f as a function of its upper limit, say we write

$$F(x) = \int_a^x f(t)\,dt,$$

then F has a derivative and $F'(x) = f(x)$. This important result shows that differentiation and integration are, in a sense, inverse operations.

In this chapter we study the process of integration in some detail. Actually we consider a more general concept than that of Riemann: this is the *Riemann–Stieltjes integral*, which involves two functions f and α. The symbol for such an integral is $\int_a^b f(x)\,d\alpha(x)$, or something similar, and the usual Riemann integral occurs as the special case in which $\alpha(x) = x$. When α has a continuous derivative, the definition is such that the Stieltjes integral $\int_a^b f(x)\,d\alpha(x)$ becomes the Riemann integral $\int_a^b f(x)\,\alpha'(x)\,dx$. However, the Stieltjes integral still makes sense when α is not differentiable or even when α is discontinuous. In fact, it is in dealing with *discontinuous* α that the importance of the Stieltjes integral becomes apparent. By a suitable choice of a discontinuous α, any finite or infinite sum can be expressed as a Stieltjes integral, and summation and ordinary Riemann integration then become special cases of this more general process. Problems in physics which involve mass distributions that are partly discrete and partly continuous can also be treated by using Stieltjes integrals. In the mathematical theory of probability this integral is a very useful tool that makes possible the simultaneous treatment of continuous and discrete random variables.

In Chapter 10 we discuss another generalization of the Riemann integral known as the *Lebesgue integral.*

7.2 NOTATION

For brevity we make certain stipulations concerning notation and terminology to be used in this chapter. We shall be working with a compact interval $[a, b]$ and, unless otherwise stated, all functions denoted by f, g, α, β, etc., will be assumed to be real-valued functions defined and *bounded* on $[a, b]$. Complex-valued functions are dealt with in Section 7.27, and extensions to unbounded functions and infinite intervals will be discussed in Chapter 10.

As in Chapter 6, a partition P of $[a, b]$ is a finite set of points, say

$$P = \{x_0, x_1, \ldots, x_n\},$$

such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. A partition P' of $[a, b]$ is said to be *finer* than P (or a *refinement* of P) if $P \subseteq P'$, which we also write $P' \supseteq P$. The symbol $\Delta\alpha_k$ denotes the difference $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$, so that

$$\sum_{k=1}^{n} \Delta\alpha_k = \alpha(b) - \alpha(a).$$

The set of all possible partitions of $[a, b]$ is denoted by $\mathscr{P}[a, b]$.

The norm of a partition P is the length of the largest subinterval of P and is denoted by $\|P\|$. Note that

$$P' \supseteq P \quad \text{implies} \quad \|P'\| \le \|P\|.$$

That is, refinement of a partition decreases its norm, but the converse does not necessarily hold.

7.3 THE DEFINITION OF THE RIEMANN–STIELTJES INTEGRAL

Definition 7.1. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the subinterval $[x_{k-1}, x_k]$. A sum of the form*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k$$

is called a Riemann–Stieltjes sum of f with respect to α. We say f is Riemann-integrable with respect to α on $[a, b]$, and we write "$f \in R(\alpha)$ on $[a, b]$" if there exists a number A having the following property: For every $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that for every partition P finer than P_ε and for every choice of the points t_k in $[x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \varepsilon$.

When such a number A exists, it is uniquely determined and is denoted by $\int_a^b f\,d\alpha$ or by $\int_a^b f(x)\,d\alpha(x)$. We also say that the Riemann–Stieltjes integral $\int_a^b f\,d\alpha$ *exists*. The functions f and α are referred to as the *integrand* and the *integrator*, respectively. In the special case when $\alpha(x) = x$, we write $S(P, f)$ instead of $S(P, f, \alpha)$, and $f \in R$ instead of $f \in R(\alpha)$. The integral is then called a Riemann integral and is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$. The numerical value of $\int_a^b f(x)\,d\alpha(x)$ depends only on f, α, a, and b, and does not depend on the symbol x. The letter x is a "dummy variable" and may be replaced by any other convenient symbol.

NOTE. This is one of several accepted definitions of the Riemann–Stieltjes integral. An alternative (but not equivalent) definition is stated in Exercise 7.3.

7.4 LINEAR PROPERTIES

It is an easy matter to prove that the integral operates in a linear fashion on both the integrand and the integrator. This is the context of the next two theorems.

Theorem 7.2. *If $f \in R(\alpha)$ and if $g \in R(\alpha)$ on $[a, b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b (c_1 f + c_2 g)\,d\alpha = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b g\,d\alpha.$$

Proof. Let $h = c_1 f + c_2 g$. Given a partition P of $[a, b]$, we can write

$$S(P, h, \alpha) = \sum_{k=1}^{n} h(t_k)\,\Delta\alpha_k = c_1 \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k$$
$$= c_1 S(P, f, \alpha) + c_2 S(P, g, \alpha).$$

Given $\varepsilon > 0$, choose P'_ε so that $P \supseteq P'_\varepsilon$ implies $|S(P, f, \alpha) - \int_a^b f\,d\alpha| < \varepsilon$, and choose P''_ε so that $P \supseteq P''_\varepsilon$ implies $|S(P, g, \alpha) - \int_a^b g\,d\alpha| < \varepsilon$. If we take $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$, then, for P finer than P_ε, we have

$$\left| S(P, h, \alpha) - c_1 \int_a^b f\,d\alpha - c_2 \int_a^b g\,d\alpha \right| \le |c_1|\varepsilon + |c_2|\varepsilon,$$

and this proves the theorem.

Theorem 7.3. *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b f\,d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b f\,d\beta.$$

The proof is similar to that of Theorem 7.2 and is left as an exercise.

A result somewhat analogous to the previous two theorems tells us that the integral is also additive with respect to the interval of integration.

Theorem 7.4. Assume that $c \in (a, b)$. If two of the three integrals in (1) *exist, then the third also exists and we have*

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha. \tag{1}$$

Proof. If P is a partition of $[a, b]$ such that $c \in P$, let

$$P' = P \cap [a, c] \qquad \text{and} \qquad P'' = P \cap [c, b],$$

denote the corresponding partitions of $[a, c]$ and $[c, b]$, respectively. The Riemann–Stieltjes sums for these partitions are connected by the equation

$$S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha).$$

Assume that $\int_a^c f\,d\alpha$ and $\int_c^b f\,d\alpha$ exist. Then, given $\varepsilon > 0$, there is a partition P'_ε of $[a, c]$ such that

$$\left|S(P', f, \alpha) - \int_a^c f\,d\alpha\right| < \frac{\varepsilon}{2} \qquad \text{whenever } P' \text{ is finer than } P'_\varepsilon,$$

and a partition P''_ε of $[c, b]$ such that

$$\left|S(P'', f, \alpha) - \int_c^b f\,d\alpha\right| < \frac{\varepsilon}{2} \qquad \text{whenever } P'' \text{ is finer than } P''_\varepsilon.$$

Then $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ is a partition of $[a, b]$ such that P finer than P_ε implies $P' \supseteq P'_\varepsilon$ and $P'' \supseteq P''_\varepsilon$. Hence, if P is finer than P_ε, we can combine the foregoing results to obtain the inequality

$$\left|S(P, f, \alpha) - \int_a^c f\,d\alpha - \int_c^b f\,d\alpha\right| < \varepsilon.$$

This proves that $\int_a^b f\,d\alpha$ exists and equals $\int_a^c f\,d\alpha + \int_c^b f\,d\alpha$. The reader can easily verify that a similar argument proves the theorem in the remaining cases.

Using mathematical induction, we can prove a similar result for a decomposition of $[a, b]$ into a finite number of subintervals.

NOTE. The preceding type of argument cannot be used to prove that the integral $\int_a^c f\,d\alpha$ exists whenever $\int_a^b f\,d\alpha$ exists. The conclusion is correct, however. For integrators α of bounded variation, this fact will later be proved in Theorem 7.25.

Definition 7.5. If $a < b$, we define $\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha$ whenever $\int_a^b f\,d\alpha$ exists. We also define $\int_a^a f\,d\alpha = 0$.

The equation in Theorem 7.4 can now be written as follows:

$$\int_a^b f\,d\alpha + \int_b^c f\,d\alpha + \int_c^a f\,d\alpha = 0.$$

7.5 INTEGRATION BY PARTS

A remarkable connection exists between the integrand and the integrator in a Riemann–Stieltjes integral. The existence of $\int_a^b f\,d\alpha$ implies the existence of $\int_a^b \alpha\,df$, and the converse is also true. Moreover, a very simple relation holds between the two integrals.

Theorem 7.6. *If $f \in R(\alpha)$ on $[a, b]$, then $\alpha \in R(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

NOTE. This equation, which provides a kind of reciprocity law for the integral, is known as the *formula for integration by parts.*

Proof. Let $\varepsilon > 0$ be given. Since $\int_a^b f\,d\alpha$ exists, there is a partition P_ε of $[a, b]$ such that for every P' finer than P_ε, we have

$$\left|S(P', f, \alpha) - \int_a^b f\,d\alpha\right| < \varepsilon. \tag{2}$$

Consider an arbitrary Riemann–Stieltjes sum for the integral $\int_a^b \alpha\,df$, say

$$S(P, \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)\,\Delta f_k = \sum_{k=1}^{n} \alpha(t_k)f(x_k) - \sum_{k=1}^{n} \alpha(t_k)f(x_{k-1}),$$

where P is finer than P_ε. Writing $A = f(b)\alpha(b) - f(a)\alpha(a)$, we have the identity

$$A = \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1}).$$

Subtracting the last two displayed equations, we find

$$A - S(P, \alpha, f) = \sum_{k=1}^{n} f(x_k)[\alpha(x_k) - \alpha(t_k)] + \sum_{k=1}^{n} f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})].$$

The two sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where P' is that partition of $[a, b]$ obtained by taking the points x_k and t_k together. Then P' is finer than P and hence finer than P_ε. Therefore the inequality (2) is valid and this means that we have

$$\left|A - S(P, \alpha, f) - \int_a^b f\,d\alpha\right| < \varepsilon,$$

whenever P is finer than P_ε. But this is exactly the statement that $\int_a^b \alpha\,df$ exists and equals $A - \int_a^b f\,d\alpha$.

7.6 CHANGE OF VARIABLE IN A RIEMANN–STIELTJES INTEGRAL

Theorem 7.7. *Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that $a = g(c)$,*

$b = g(d)$. *Let h and β be the composite functions defined as follows:*

$$h(x) = f[g(x)], \qquad \beta(x) = \alpha[g(x)], \qquad \text{if } x \in S.$$

Then $h \in R(\beta)$ on S and we have $\int_a^b f\, d\alpha = \int_c^d h\, d\beta$. That is,

$$\int_{g(c)}^{g(d)} f(t)\, d\alpha(t) = \int_c^d f[g(x)]\, d\{\alpha[g(x)]\}.$$

Proof. For definiteness, assume that g is strictly increasing on S. (This implies $c < d$.) Then g is one-to-one and has a strictly increasing, continuous inverse g^{-1} defined on $[a, b]$. Therefore, for every partition $P = \{y_0, \ldots, y_n\}$ of $[c, d]$, there corresponds one and only one partition $P' = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $x_k = g(y_k)$. In fact, we can write

$$P' = g(P) \qquad \text{and} \qquad P = g^{-1}(P').$$

Furthermore, a refinement of P produces a corresponding refinement of P', and the converse also holds.

If $\varepsilon > 0$ is given, there is a partition P'_ε of $[a, b]$ such that P' finer than P'_ε implies $|S(P', f, \alpha) - \int_a^b f\, d\alpha| < \varepsilon$. Let $P_\varepsilon = g^{-1}(P'_\varepsilon)$ be the corresponding partition of $[c, d]$, and let $P = \{y_0, \ldots, y_n\}$ be a partition of $[c, d]$ finer than P_ε. Form a Riemann–Stieltjes sum

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k)\, \Delta\beta_k,$$

where $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$. If we put $t_k = g(u_k)$ and $x_k = g(y_k)$, then $P' = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$ finer than P'_ε. Moreover, we then have

$$\begin{aligned} S(P, h, \beta) &= \sum_{k=1}^{n} f[g(u_k)]\{\alpha[g(y_k)] - \alpha[g(y_{k-1})]\} \\ &= \sum_{k=1}^{n} f(t_k)\{\alpha(x_k) - \alpha(x_{k-1})\} = S(P', f, \alpha), \end{aligned}$$

since $t_k \in [x_{k-1}, x_k]$. Therefore, $|S(P, h, \beta) - \int_a^b f\, d\alpha| < \varepsilon$ and the theorem is proved.

NOTE. This theorem applies, in particular, to Riemann integrals, that is, when $\alpha(x) = x$. Another theorem of this type, in which g is not required to be monotonic, will later be proved for Riemann integrals. (See Theorem 7.36.)

7.7 REDUCTION TO A RIEMANN INTEGRAL

The next theorem tells us that we are permitted to replace the symbol $d\alpha(x)$ by $\alpha'(x)\, dx$ in the integral $\int_a^b f(x)\, d\alpha(x)$ whenever α has a continuous derivative α'.

Theorem 7.8. *Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\,dx$ exists and we have*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

Proof. Let $g(x) = f(x)\alpha'(x)$ and consider a Riemann sum

$$S(P, g) = \sum_{k=1}^{n} g(t_k)\,\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\,\Delta x_k.$$

The same partition P and the same choice of the t_k can be used to form the Riemann–Stieltjes sum

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k.$$

Applying the Mean-Value Theorem, we can write

$$\Delta\alpha_k = \alpha'(v_k)\,\Delta x_k, \qquad \text{where } v_k \in (x_{k-1}, x_k),$$

and hence

$$S(P, f, \alpha) - S(P, g) = \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\,\Delta x_k.$$

Since f is bounded, we have $|f(x)| \le M$ for all x in $[a, b]$, where $M > 0$. Continuity of α' on $[a, b]$ implies uniform continuity on $[a, b]$. Hence, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ (depending only on ε) such that

$$0 \le |x - y| < \delta \quad \text{implies} \quad |\alpha'(x) - \alpha'(y)| < \frac{\varepsilon}{2M(b - a)}.$$

If we take a partition P'_ε with norm $\|P'_\varepsilon\| < \delta$, then for any finer partition P we will have $|\alpha'(v_k) - \alpha'(t_k)| < \varepsilon/[2M(b - a)]$ in the preceding equation. For such P we therefore have

$$|S(P, f, \alpha) - S(P, g)| < \frac{\varepsilon}{2}.$$

On the other hand, since $f \in R(\alpha)$ on $[a, b]$, there exists a partition P''_ε such that P finer than P''_ε implies

$$\left|S(P, f, \alpha) - \int_a^b f\,d\alpha\right| < \frac{\varepsilon}{2}.$$

Combining the last two inequalities, we see that when P is finer than $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$, we will have $|S(P, g) - \int_a^b f\,d\alpha| < \varepsilon$, and this proves the theorem.

NOTE. A stronger result not requiring continuity of α' is proved in Theorem 7.35.