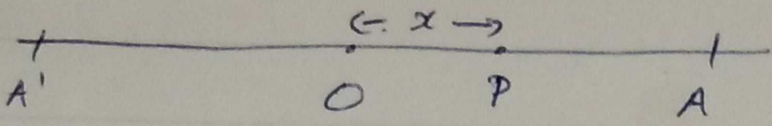
Simple Harmonic Motion

Defn

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called <u>Simple Harmonic Motion</u> (S.H.M.).

S.H.M. in a Straight Line



Let O be a fixed point on the st. line A'OA.

P be the position of the particle at time t.

Let $OP = x$.

The acceleration is d^2x/dt^2.

If the particle executes S.H.M. then

$$\frac{d^2x}{dt^2} \propto x.$$

$$\therefore \frac{d^2x}{dt^2} = -\mu x \quad \text{where } \mu \text{ is +ve constant} \qquad (1).$$

and as acceleration is in the direction opposite to x

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This is the fundamental differential eqn representing a S.H.M.

$$\frac{d^2x}{dt^2} = -\mu x.$$

(ie) $$\frac{dv}{dt} = -\mu x.$$

$$\frac{dv}{dx} \cdot \frac{dx}{dt} = -\mu x.$$

$$\frac{dv}{dx} \cdot v = -\mu x.$$

$$v \cdot dv = -\mu x \, dx.$$

Integrating $\int v\,dv = \int -\mu x\,dx.$

$$\frac{v^2}{2} = -\mu \frac{x^2}{2} + c.$$

$$v^2 = -\mu x^2 + c. \quad \text{——— (2)}$$

when x increases v decreases.

∴ The particle comes to rest at some point A.

Let $OA = a$.

Suppose the particle starts from A, so the initial conditions are $t=0$, $x=a$ & $v=0$.

sub. in ②, $0 = -\mu a^2 + c$.

$\Rightarrow c = \mu a^2$.

$\therefore v^2 = -\mu x^2 + \mu a^2 = \mu(a^2 - x^2)$.

and $v = \pm \sqrt{\mu(a^2-x^2)}$ ——— ③.

This eqn. gives the velocity v corresponding to any displacement x.

As t increases, x decreases, so $\frac{dx}{dt}$ is $-ve$.

(ie). velocity v is $-ve$.

$\therefore v = -\sqrt{\mu(a^2-x^2)}$.

The particle comes to rest when $x = \pm a$.

(ie) The particle oscillates between A and A'.

"The maximum displacement of the particle on either side of origin is 'a'. This is called the Amplitude of the moving particle".

Maximum and minimum values of the accelerations are $\mp \mu a$ and it occurs at A & A'

$v = -\sqrt{\mu(a^2-x^2)}$.

(ie) $\frac{dx}{dt} = -\sqrt{\mu(a^2-x^2)}$.

$$-\frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\mu}\, dt.$$

Integrating, $\cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}\cdot t + c_1$.

Initially when $t=0$, $x=a$.

$$\therefore \cos^{-1}\left(\frac{a}{a}\right) = \sqrt{\mu}\,(0) + c_1$$

$$\Rightarrow c_1 = 0.$$

$$\therefore \cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}\cdot t.$$

$$\frac{x}{a} = \cos\sqrt{\mu}\, t.$$

$$x = a\cos\sqrt{\mu}\, t. \quad \text{———} (4).$$

This eqn gives the displacement at time t.

We know that $\cos\sqrt{\mu}\,t = \cos(2\pi + \sqrt{\mu}\,t)$.

$$= \cos\sqrt{\mu}\left(\frac{2\pi}{\sqrt{\mu}} + t\right).$$

Thus x remains the same when t is increased by $\frac{2\pi}{\sqrt{\mu}}$.

$\frac{2\pi}{\sqrt{\mu}}$ is called the period of S.H.M.

This is independent of the displacement x

Note:

(1) Since $\frac{d^2x}{dt^2} = -\mu x$, maximum acceleration corresponds to the greatest value of x & so. $= \mu \cdot a = \mu$ (amplitude)

and the amplitude a.

This is the characteristic property of S.H.M.

When the particle moves from A to A', it is said to execute one vibration.

When it moves from A to A' and back to A it is said to execute one oscillation.

The number of oscillation per second is called the frequency and it is equal to the reciprocal of the period

Thus the frequency $= \frac{\sqrt{\mu}}{2\pi}$

Periodic time (or Period)

The period or periodic time of a simple harmonic motion is the interval of time that elapses from any instant till a subsequent instant when the particle is again moving through the same position with the same velocity in the same direction.

Note:

General soln. of SHM eqn.

The differential eqn. of SHM is

$$\frac{d^2x}{dt^2} = -\mu x. \quad \text{——— (1)}.$$

Note
(2) $\because v = \sqrt{\mu(a^2-x^2)}$, the greatest value of
v is got at $x=0$. (ie) $v = a\sqrt{\mu}$
$= \sqrt{\mu}$ (amplitude)

(ii) $\frac{d^2x}{dt^2} + \mu x = 0$.

$(D^2+\mu)x = 0$ $\qquad D = \frac{d}{dt}$.

Auxillary eqn is $m^2+\mu = 0$.

$m^2 = -\mu$.

$m = \pm\sqrt{-\mu} = \pm i\sqrt{\mu}$.

$\therefore x = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t$ —— (2).

where A and B are constants

Let $A = c\cos\epsilon$ $\qquad B = -c\sin\epsilon$.

Now (2) $\Rightarrow x = c\cos\sqrt{\mu}t\cos\epsilon - c\sin\sqrt{\mu}t\sin\epsilon$.

$= c[\cos(\sqrt{\mu}t+\epsilon)]$.

(ie) $x = c\cos(\sqrt{\mu}t+\epsilon)$ —— (3).

Let $A = D\sin\alpha$ $\qquad B = D\cos\alpha$.

(2) $\Rightarrow x = D\cos\sqrt{\mu}t\sin\alpha + D\sin\sqrt{\mu}t\cos\alpha$.

$= D[\sin(\sqrt{\mu}t+\alpha)]$

(ie) $x = D\sin(\sqrt{\mu}t+\alpha)$ —— (4).

The constants $A, B, c, \epsilon, D, \alpha$ are known if we know the values of x and $\frac{dx}{dt}$

corresponding to a given time.

From ③ & ④ maximum value of $x = C$ or D.

If a is the amplitude of motion, then

$a = C$ or D.

$\therefore$ ③ $\Rightarrow$ $x = a\cos(\sqrt{\mu}t + \varepsilon)$ —— ⑤

④ $\Rightarrow$ $x = a\sin(\sqrt{\mu}t + \alpha)$ —— ⑥.

Epoch

When the solution of S.H.M. is expressed as $x = a\cos(\sqrt{\mu}t + \varepsilon)$, the quantity ε is called the epoch.

Phase

The phase of S.H.M. at any instant is the time that has elapsed since the particle was at its maximum distance in the +ve direction.

If t_0 is the time when x is maximum

⑤ $\Rightarrow$ $a = a\cos(\sqrt{\mu}t_0 + \varepsilon)$.

$$\cos(\sqrt{\mu}t_0 + \epsilon_0) = 1.$$

$$\sqrt{\mu}t_0 + \epsilon_0 = 0 . \Rightarrow t_0 = -\epsilon_0/\sqrt{\mu}$$

Phase at time $t = t - t_0$.

$$= t + \frac{\epsilon_0}{\sqrt{\mu}} . = \frac{\sqrt{\mu}t + \epsilon_0}{\sqrt{\mu}} .$$

2. Two S.H.M. of the same period can be represented by

$$x_1 = a_1 \cos(\sqrt{\mu}t + \epsilon_{01})$$

$$x_2 = a_2 \cos(\sqrt{\mu}t + \epsilon_{02}) .$$

The difference in phase $= \frac{\sqrt{\mu}t + \epsilon_{01}}{\sqrt{\mu}} - \frac{\sqrt{\mu}t + \epsilon_{02}}{\sqrt{\mu}} .$

$$= \frac{\epsilon_{01} - \epsilon_{02}}{\sqrt{\mu}} .$$

If $\epsilon_{01} = \epsilon_{02}$, the difference in phase $= 0$.

$\therefore$ If $\epsilon_{01} = \epsilon_{02}$ the motions are in the same phase.

If $\epsilon_1 = \epsilon_2 = \pi$, they are in opposite phases.

Problems

1. A particle is moving with S.H.M and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are x_1, x_2, x_3. Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$.

we know that $x = a\cos\sqrt{\mu}\, t$. —— (1)

Let at 3 consecutive seconds t_1, t_1+1, t_1+2 the displacements be x_1, x_2, x_3.

Then $x_1 = a\cos\sqrt{\mu}\, t_1$. —— (2)

$x_2 = a\cos\sqrt{\mu}\,(t_1+1) = a\cos(\sqrt{\mu}\, t_1 + \sqrt{\mu})$. —— (3)

$x_3 = a\cos\sqrt{\mu}\,(t_1+2) = a\cos(\sqrt{\mu}\, t_1 + 2\sqrt{\mu})$. —— (4)

(2) + (4) $\Rightarrow x_1 + x_3 = a\left[\cos(\sqrt{\mu}\, t_1 + 2\sqrt{\mu}) + \cos\sqrt{\mu}\, t_1\right]$.

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$= a\left[2\cos\left(\frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2}\right)\cos\left(\frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}\right)\right]$$

$$= 2a\cdot\cos(\sqrt{\mu}t_1 + \sqrt{\mu})\cos\sqrt{\mu}.$$

$$= 2x_2\cos\sqrt{\mu}.$$

$$\therefore \frac{x_1 + x_3}{2x_2} = \cos\sqrt{\mu} \quad \text{or} \quad \sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$

$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}$$

(2) If the displacement of a moving point at any time be given by an equation of the form $x = a\cos\omega t + b\sin\omega t$, show that the motion is a simple harmonic motion.

If $a = 3$, $b = 4$, $\omega = 2$ determine the period, amplitude, maximum velocity and max. acceleration of the motion.

$$x = a\cos\omega t + b\sin\omega t \quad \text{——— (1)}$$

$$\frac{dx}{dt} = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$\frac{d^2x}{dt^2} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t.$$

$$= -\omega^2\left(a\cos\omega t + b\sin\omega t\right)$$

$$= -\omega^2 x.$$

$$\frac{d^2x}{dt^2} \propto x.$$

$\therefore$ The motion is Simple Harmonic.

$\therefore \mu = \omega^2.$

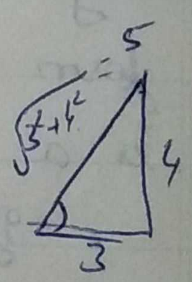
$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \text{ secs}.$$

Amplitude is the greatest value of x.

When x is maximum $\frac{dx}{dt} = 0$.

$\therefore -a\omega\sin\omega t + b\omega\cos\omega t = 0.$

(or) $a\cancel{\omega}\sin\omega t = b\cancel{\omega}\cos\omega t$



and $\tan\omega t = \frac{b}{a} = \frac{4}{3}.$

$\therefore \sin\omega t = \frac{4}{5}$ $\cos\omega t = \frac{3}{5}.$

Sub these values in ① we get max value of x.

(b) $x = 3 \times \frac{3}{5} + 4 \times \frac{4}{5} = \frac{25}{5} = 5.$

Hence amplitude $= 5$.

Max. acceleration $= \mu \cdot$ amplitude $= \omega^2 (a) = 4 \times 5 = 20$.

Max. velocity $= \sqrt{\mu}$ amplitude $= \omega (a) = 2 \times 5 = 10$.

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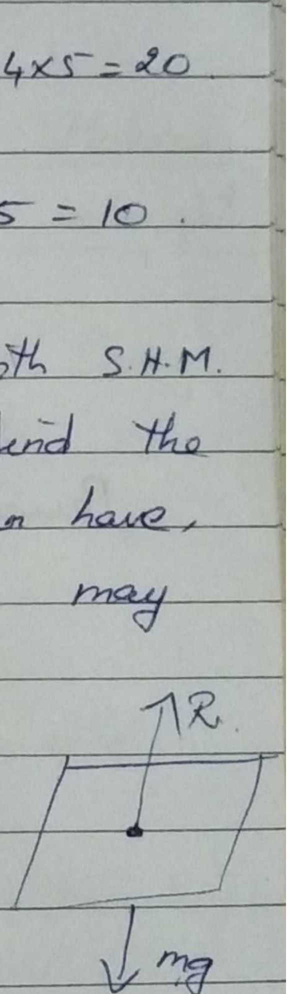
3. A horizontal shelf moves vertically with S.H.M. whose complete period is one second; find the greatest amplitude in centimeters, it can have, so that an object resting on the shelf may always remain in contact.



Let m be the mass of an object lying on the shelf.

O be the centre of S.H.M.

and P be the position of m at time t.

Let $OP = x$.

The forces acting on the mass at P are

(i) its weight mg vertically downwards

(ii) Reaction R due to the shelf acting upwards.

$\therefore$ Resultant force $= mg - R$ —— ①

w.k.T. $F = ma$.

$\therefore \quad a = \frac{F}{m} = \frac{mg - R}{m}$ and it acts towards (

Acceleration at $P = \mu\,(OP) = \mu x$.

$$\therefore \quad \mu x = \frac{mg - R}{m}$$

$$m\mu x = mg - R.$$

$$R = m(g - \mu x) \quad \text{——} \; (2).$$

Period $= \frac{2\pi}{\sqrt{\mu}} = 1$ (given)

$\therefore \quad 2\pi = \sqrt{\mu}$. or $\mu = 4\pi^2$

sub μ in (2).

$$R = m(g - 4\pi^2 x).$$

For the mass to remain in contact with the shelf. R must be ≥ 0. (ie) R must not be -ve

(ie) $m(g - 4\pi^2 x) \geq 0$.

$$g - 4\pi^2 x \geq 0.$$

$$g \geq 4\pi^2 x$$

$$4\pi^2 x \leq g$$

$$x \leq \frac{g}{4\pi^2}.$$

$\therefore$ Max value of $x = \frac{g}{4\pi^2}$

$= \frac{98\cancel{1}}{4(3.14)^2} = 24.8$ cm.

Composition of two Simple Harmonic Motions of the same period and in the same straight line:

Since the period is dependent only on the constant μ, the two separate S.H.M. are expressed by the same differential equation $\frac{d^2x}{dt^2} = -\mu x$.

Let x_1 and x_2 be the displacements for the separate motions.

Then $x_1 = a_1 \cos(\sqrt{\mu}t + \varepsilon_1)$; $x_2 = a_2 \cos(\sqrt{\mu}t + \varepsilon_2)$.

Let x be the resultant displacement.

$\therefore x = x_1 + x_2$.

$= a_1 \cos(\sqrt{\mu}t + \varepsilon_1) + a_2 \cos(\sqrt{\mu}t + \varepsilon_2)$

$= a_1(\cos\sqrt{\mu}t \cos\varepsilon_1 - \sin\sqrt{\mu}t \sin\varepsilon_1) + a_2(\cos\sqrt{\mu}t \cos\varepsilon_2 - \sin\sqrt{\mu}t \sin\varepsilon_2)$.

$= \cos\sqrt{\mu}t\,(a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2) - \sin\sqrt{\mu}t\,(a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2)$.

(a) $x = \cos\sqrt{\mu}t \cdot A\cos\epsilon - \sin\sqrt{\mu}t \cdot A\sin\epsilon$ —— (1).

where $A\cos\epsilon = a_1\cos\epsilon_1 + a_2\cos\epsilon_2$ —— (2).

and $A\sin\epsilon = a_1\sin\epsilon_1 + a_2\sin\epsilon_2$ —— (3).

To find A and ϵ

$(2)^2 + (3)^2 \Rightarrow A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\epsilon_1 - \epsilon_2)$ —— (4)

$\frac{(3)}{(2)} \Rightarrow \tan\epsilon = \frac{a_1\sin\epsilon_1 + a_2\sin\epsilon_2}{a_1\cos\epsilon_1 + a_2\cos\epsilon_2}$ —— (5).

Now (1) $\Rightarrow$ $x = A\left[\cos\sqrt{\mu}t\cos\epsilon - \sin\sqrt{\mu}t\sin\epsilon\right]$.

$x = A\cos(\sqrt{\mu}t + \epsilon)$. —— (6).

From eq-(6), the resultant displacement also represents a S.H.M. of the same period as the individual motions.

A, the new amplitude is the diagonal of the $\|^{gram}$ whose sides are the original amplitudes a_1 and a_2 inclined at an angle $\epsilon_1 - \epsilon_2$, the difference of the epochs.

Composition of two S.H.M.s of the same period in two perpendicular directions.

If a particle possesses two SHM in $\perp^r$ directions and of same period, we can prove that its path is an ellipse. Take the two $\perp^r$ lines as the x and y axis.

The displacements of the particle ~~due~~ ~~to~~ can be taken as

$x = a_1 \cos\sqrt{\mu}\, t$ — ① $\qquad y = a_2 \cos(\sqrt{\mu}\, t + \varepsilon)$ — ②.

The path is obtained by eliminating t b/w ① & ②.

② $\Rightarrow$ $y = a_2 \left[\cos\sqrt{\mu}\, t \cos\varepsilon - \sin\sqrt{\mu}\, t \sin\varepsilon\right]$.

$$= a_2 \left[\frac{x}{a_1}\cos\varepsilon - \sin\varepsilon \sqrt{1 - \frac{x^2}{a_1^2}}\right].$$

(ie) $\dfrac{y}{a_2} - \dfrac{x}{a_1}\cos\varepsilon = -\sin\varepsilon \sqrt{1 - \dfrac{x^2}{a_1^2}}$.

squaring we get

$$\frac{y^2}{a_2^2} + \frac{x^2}{a_1^2}\cos^2\varepsilon - \frac{2xy}{a_1 a_2}\cos\varepsilon = \sin^2\varepsilon\left(1 - \frac{x^2}{a_1^2}\right).$$

(ie) $\frac{y^2}{a_2^2} + \frac{x^2}{a_1^2}(\cos^2\epsilon + \sin^2\epsilon) - \frac{2xy\cos\epsilon}{a_1 a_2} = \sin^2\epsilon.$

(ie) $\frac{x^2}{a_1^2} - \frac{2xy}{a_1 a_2}\cos\epsilon + \frac{y^2}{a_2^2} = \sin^2\epsilon$ —— (3)

This is of the form $ax^2 + 2hxy + by^2 = \lambda$ — (4)

where $a = \frac{1}{a_1^2}$ $h = -\frac{\cos\epsilon}{a_1 a_2}$ $b = \frac{1}{a_2^2}$

(4) represents a conic with centre at the origin

Also $ab - h^2 = \frac{1}{a_1^2 a_2^2} - \frac{\cos^2\epsilon}{a_1^2 a_2^2} = \frac{\sin^2\epsilon}{a_1^2 a_2^2} = +ve.$

Hence (4) represents an ellipse.

If $\epsilon = 0$, (3) gives $\frac{x^2}{a_1^2} - \frac{2xy}{a_1 a_2} + \frac{y^2}{a_2^2} = 0.$

(ie) $\left(\frac{x}{a_1} - \frac{y}{a_2}\right)^2 = 0.$

(ie) $\frac{x}{a_1} - \frac{y}{a_2} = 0$ which is a straight line.

* If $\epsilon = \pi$ (5) gives $\frac{x^2}{a_1^2} + \frac{2xy}{a_1 a_2} + \frac{y^2}{a_2^2} = 0$.

(ie) $\frac{x}{a_1} + \frac{y}{a_2} = 0$ which is also a straight line.

* If $\epsilon = \frac{\pi}{2}$ (5) gives $\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$ which is an ellipse whose principal axes are along the axes of x and y.

* If $\epsilon = \frac{\pi}{2}$ and $a_1 = a_2$, the path is the circle $x^2 + y^2 = a_1^2$.

Problems

(1) Show that the resultant of two ~~simple~~ S.H.Ms in the same direction and of equal period time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a S.H.M. of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by $\frac{\tan^{-1} 2}{2\pi}$ of a period.

Let the displacements be

$x_1 = a_1 \cos(\sqrt{\mu}\, t + \epsilon_1)$ —— (1)

$$x_2 = a_2 \cos(\sqrt{\mu}\, t + \varepsilon_2) \quad \text{——— (2).}$$

Here $a_2 = 2a_1$ and $\frac{\varepsilon_2 - \varepsilon_1}{\sqrt{\mu}}$ = phase difference

$$= \frac{1}{4} \times \frac{2\pi}{\sqrt{\mu}}.$$

$\therefore \ \varepsilon_2 - \varepsilon_1 = \frac{\pi}{2}$ or $\varepsilon_2 = \frac{\pi}{2} + \varepsilon_1$.

we know that resultant displacement is

$$x = A\cos(\sqrt{\mu}\, t + \varepsilon).$$

where $A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2)$

$$= a_1^2 + 4a_1^2 + \cancel{2}4 a_1^2 \cos(-90)$$

$$= 5a_1^2$$

$\therefore$ Amplitude of the resultant motion $= A = a_1\sqrt{5}$.

Also $\tan\varepsilon = \dfrac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}$.

$$= \frac{a_1 \sin\varepsilon_1 + 2a_1 \sin(90 + \varepsilon_1)}{a_1 \cos\varepsilon_1 + 2a_1 \cos(90 + \varepsilon_1)}.$$

(or) $\dfrac{\sin\epsilon}{\cos\epsilon} = \dfrac{\sin\epsilon_1 + 2\cos\epsilon_1}{\cos\epsilon_1 - 2\sin\epsilon_1}$

$\sin\epsilon\cos\epsilon_1 - 2\sin\epsilon\sin\epsilon_1 = \sin\epsilon_1\cos\epsilon + 2\cos\epsilon_1\cos\epsilon$

$\sin\epsilon\cos\epsilon_1 - \cos\epsilon\sin\epsilon_1 = 2(\cos\epsilon_1\cos\epsilon + \sin\epsilon\sin\epsilon_1)$.

$\sin(\epsilon - \epsilon_1) = 2\cos(\epsilon - \epsilon_1)$.

(or) $\tan(\epsilon - \epsilon_1) = 2$. or $\epsilon - \epsilon_1 = \tan^{-1} 2$.

$$\frac{\epsilon - \epsilon_1}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{2\pi} \cdot \left(\frac{2\pi}{\sqrt{\mu}}\right),$$

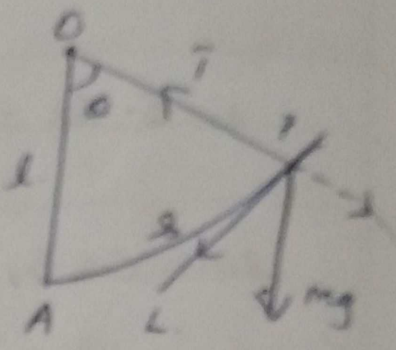
$$= \frac{\tan^{-1} 2}{2\pi} \text{ of a period.}$$

This is the phase difference of the resultant S.H.M.)

<u>Simple Pendulum:</u>

A simple pendulum is a small heavy particle or bob suspended from a fixed point by means of a light inextensible string and oscillating in a vertical plane.

Period of Oscillation of a simple pendulum



Let O be the point of suspension, OA - the vertical position of the string,

l - the length of the string

m - mass of the particle

Let at time t the particle be at P whose arc AP = s and $\angle AOP = \theta$ (a small angle).

The forces acting at P are

(i) weight mg acting vertically downwards

(ii) tension T acting along PO.

The force mg can be resolved into two components

(i) $mg\cos\theta$ along OP

(ii) $mg\sin\theta$ along the downward tangent PL.

The component $mg\cos\theta$ ~~bl~~ balances the tension T.

Hence we have

$m\dfrac{d^2s}{dt^2}$ = external force along the tangent at P in the direction in which $s \uparrow$es

(or) $\cancel{m}\dfrac{d^2s}{dt^2} = -\cancel{m}g\sin\theta$ ——— (1)

(we have put -ve sign as the component $mg\sin\theta$ is towards A and opposite to the direction in which $s \uparrow$es).

be replaced by θ in (1)

$\therefore$ (1) $\Rightarrow \dfrac{d^2s}{dt^2} = -g\theta$ —— (2) (θ is small enough)

From the sector OAP

But arc $AP = s = l\theta \Rightarrow \theta = \dfrac{s}{l}$

(2) $\Rightarrow \dfrac{d^2s}{dt^2} = -\dfrac{g}{l}s$ —— (3)

$\therefore$ The motion of P is SH about A

Here $\mu = \dfrac{g}{l}$

$\therefore$ Periodic time $\dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\dfrac{l}{g}}$

Equivalent simple Pendulum

Whatever be the shape of the pendulum that simple pendulum which oscillates in the same time as the given pendulum is called simple equivalent Pendulum

Any simple harmonic motion may be compared with the motion of a simple pendulum and such motions can be taken as equivalent if their periods are the same.

Thus consider the two motions

$$\frac{d^2x}{dt^2} = -\mu x \quad \text{—} ① \quad \text{and} \quad \frac{d^2s}{dt^2} = -\frac{g}{l}s \quad \text{—} ③$$

we know ① & ② are S.H.M and ③ is the motion of simple pendulum.

They represent equivalent motion if

$$\mu = \frac{g}{l}.$$

(ie) if $l = \dfrac{g}{\mu}$.

So with regard to the motion ① we say that the length of the simple equivalent pendulum is $\dfrac{g}{\mu}$.

Seconds Pendulum.

A seconds pendulum is one whose period of oscillation is 2 seconds.

Hence if l is its length, we have

$$2 = 2\pi\sqrt{\frac{l}{g}}$$

$$\therefore \quad l = \frac{g}{\pi^2}.$$

Since the time of oscillation of a seconds pendulum is 2 secs, it should make 43,200 oscillations per day.

If it gains n seconds a day, it means that it makes $43,200 + \frac{n}{2}$ oscillations in 86,400 secs.

Hence its time of oscillation will be

$$(T) = \frac{86400}{43200 + \frac{n}{2}} \text{ seconds} \quad \text{—— (1)}$$

If it loses n seconds a day, it makes

$43,200 - \frac{n}{2}$ oscillations in 86400 secs.

So its period will be $(T) = \dfrac{86,400}{43200 - \frac{n}{2}}$ seconds. —— (2).

<u>Loss or gain in the number of oscillations made by a pendulum.</u>

The period of oscillation T of a simple pendulum of length l is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{———} \; (1)$$

Since T depends on both l and g, a change in either l or g will affect the value of T.

Hence the pendulum will gain or lose in number of oscillations in any given time.

The change in the number of oscillations in a given time can be investigated as follows.

Taking log on both sides of (1) we get

$$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g \quad \text{———} \; (5)$$

Taking differentials of (5).

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l} - \frac{1}{2}\frac{dg}{g} \quad \text{———} \; (6)$$

Let n be the number of oscillations made in a given interval of time t sec.

Then $T = \frac{t}{n}$ or $n = \frac{t}{T}$ ——— (7)

Taking log on both sides of (7).

$\log n = \log k - \log T$ —— (5)

Taking differentials of (5).

$$\frac{dn}{n} = -\frac{dT}{T} \quad \text{—— (6)}$$

sub the value of $\frac{dT}{T}$ from (2) in (6)

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l} + \frac{1}{2}\frac{dg}{g} \quad \text{—— (7)}$$

Deductions

(i) If l alone varies (ie) g is constant, then $dg = 0$.

Hence (7) $\Rightarrow$ $\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}$.

(ie) $dn = -\frac{n}{2} \cdot \frac{dl}{l}$.

(ie) the change in the number of oscillations made in a given interval of time $= -\frac{n}{2}\frac{dl}{l}$.

The -ve sign shows that if dl is +ve, dn is -ve.

So if the length l is increased, n is decreased, giving a loss in number of oscillations

If l is decreased, n is increased, resulting in a gain in number of oscillations.

(ii) If g ~~along~~ alone varies l remaining constant, then $dl = 0$.

In this case, from ⑦ $\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}$ — ⑨

If dg is +ve, dn is also +ve and if dg is -ve, so also is dn.

So if g increases there is a gain in the number of oscillations and if g ↓ses, there is a loss.

We shall now consider two special cases when g alone varies, l remaining constant.

Case I :-

Let the pendulum be taken to the top of a mountain at a height h above the earth's surface.

There, the acceleration due to gravity decreases. According to Newton's law of attraction, the earth regarded as a sphere of radius a exerts on a particle of mass m distant x $(x > a)$ from its centre an attraction of magnitude $\frac{m\mu}{x^2}$.

Hence if g_1 is the value of gravity at a height h above the earth's surface and g is its value at the earth's surface, we have

$$mg_1 = \frac{m\mu}{(a+h)^2} \quad \text{and} \quad mg = \frac{m\mu}{a^2}.$$

$$\therefore \frac{g_1}{g} = \frac{a^2}{(a+h)^2}.$$

Since $g_1 < g$, we can put $g_1 = g - dg$ where dg represents a small change in g.

$$\therefore \frac{g-dg}{g} = \frac{a^2}{(a+h)^2} = \frac{1}{\left(1+\frac{h}{a}\right)^2} = \left(1+\frac{h}{a}\right)^{-2}.$$

$$= 1 - \frac{2h}{a}$$, neglecting $\left(\frac{h}{a}\right)^2$ and higher powers of $\frac{h}{a}$ as h is small compared to a.

(or). $1 - \frac{dg}{g} = 1 - \frac{2h}{a}.$ (or)

$$\frac{dg}{g} = = \frac{2h}{a}.$$

From (9) $\frac{dn}{n} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}h}{a} = \frac{h}{a}$.

(ie) $dn = \frac{nh}{a}$.

$\therefore$ The number of oscillations lost $= \frac{nh}{a}$.

Case II :-

Let the pendulum be taken to the bottom of a mine of depth d.

According to Newton's law, attraction of the earth at an interior point varies directly as the distance from the centre of the earth.

If g_2 is the value of gravity at the above depth,

$$\frac{g_2}{g} = \frac{a-d}{a} = 1 - \frac{d}{a}.$$

Since g_2 is smaller than g, we can put

$g_2 = g - dg$, where dg is a small change in g

$$\therefore \frac{g-dg}{g} = 1 - \frac{d}{a} \quad \text{(or)} \quad \frac{dg}{g} = \frac{d}{a}.$$

From (9). $\frac{dn}{n} = \frac{1}{2} \cdot \frac{d}{a}$ or $dn = \frac{nd}{2a}$.

(ie) the loss in the number of oscillations $= \frac{nd}{2a}$ //

Problems

① A clock with a seconds pendulum loses 40 seconds per day at a place where the acceleration due to gravity is 981 cm/sec². Find what change in the length is necessary to make it accurate.

g being constant

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}.$$

where n = no. of oscillations per day = 43200.

$dn = -20$ as the pendulum loses 40 secs.

l = length of the correct seconds pendulu

$= g/\pi^2$.

$$\therefore \frac{-20}{43200} = -\frac{1}{2} \cdot \frac{dl}{g/\pi^2}.$$

$$\therefore dl = \frac{40g}{43,200\,\pi^2} = \frac{40 \times 981}{\pi^2 \times 43,200} = \cdot 092$$

∴ length of imperfect seconds pendulum $= l + dl$.

$= l + \cdot 092$ c

and so it must be shortened by 0·092 cms.

(5). A balloon ascends with constant acceleration and reaches a height of 360 metres in one minute. Show that during the ascent, a pendulum clock which has a seconds pendulum and which is carried in the balloon, will gain at the rate of about 37 seconds per hour.

Let f be the upward acceleration of the balloon.

It ascends a height of 36000 cms in 1 minute

$\therefore 36{,}000 = \frac{1}{2} f \times (60)^2$ (by $s = ut + \frac{1}{2}at^2$).

$\therefore f = \dfrac{36000 \times 2}{60 \times 60} = 20 \text{ cm/sec}^2$.

To bring the balloon to rest, we have to supply a downward acceleration equal to f cm/sec^2.

This means that the effective value of g is increased to $g + f$.

Hence there is a change in g which is equal to f

$\therefore dg = f = 20 \text{ cm/sec}^2$.

we know $\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}$

$\therefore dn = \frac{n}{2}\frac{dg}{g} = \frac{43,200}{2} \times \frac{20}{981} = \frac{48,000}{109}$.

(ii) number of oscillations gained per day $= \frac{48,000}{109}$.

$\therefore$ Number of seconds gained per day $= \frac{48,000}{109} \times 2$.

$= \frac{96,000}{109}$.

$\therefore$ Number of seconds gained per hour $= \frac{96,000}{109 \times 24}$.

$= 37$ nearly.

(3). A pendulum which beats seconds at the surface of the earth loses 10 seconds in 24hrs when taken to the summit of a hill. Find the height of the hill, taking the radius of the earth to be 6400 km.

Let h kms be the height of the hill

g - acceleration due to gravity on earth's surface

g_1 - " " " " at the top of the hill

we know $\frac{g_1}{g} = \frac{a^2}{(a+h)^2}$ —— ①.

For the correct clock, the time of a complete oscillation being 2 secs.

$$2 = 2\pi\sqrt{\frac{l}{g}} \text{ —— ①}$$

∵ the pendulum loses 10 seconds in a day of 86,400 seconds, it makes (43,200 - 5) oscillations

Hence $T = \frac{86400}{43195}$.

$$\therefore \frac{86400}{43195} = 2\pi\sqrt{\frac{l}{g_1}} \text{ —— ②}.$$

$\frac{①}{②} \Rightarrow$ ~~862~~ $\frac{2 \times 43195}{86,400} = \sqrt{\frac{g_1}{g}} = \frac{a}{a+h}$.

(or) $\frac{86390}{86400} = \frac{a}{a+h}$ or

$\frac{a+h}{a} = \frac{86400}{86390} = 1 +$ ~~86400~~ $\frac{10}{86390}$

(a) $\frac{h}{a} = \frac{1}{8639}$ or $h = \frac{a}{8639} = \frac{6600}{8639}$

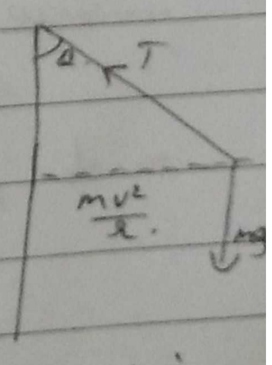
$= 761$ m.

(4) A pendulum suspended from the roof of a railway carriage travelling with speed v round a circular track of radius r makes n oscillations per sec. If it makes n' oscillations per sec when the carriage is stationary, show that $v^2 = rg\sqrt{\frac{n^4}{n'^4 - 1}}$.

Let l be the length of the pendulum and g the acceleration due to gravity.

When the carriage is stationary, number of oscillations made per sec $= n'$.

$\therefore$ Period of the pendulum $= \frac{1}{n'} = 2\pi\sqrt{\frac{l}{g}}$ —(1)



Now, when the carriage is moving round a curve of radius r, the bob (of mass m) is subject to a central force $\frac{mv^2}{r}$.

This force has to be supplied by the string of the pendulum which must assume a position inclined at some angle θ to the vertical.

Let T be the tension in the string.

Resolving vertically, $T\cos\theta = mg$. —— (2)

$T\cos\theta + mg\cos 180 + \frac{mv^2}{r}\cos 270 = 0$

Resolving horizontally, $T\sin\theta = \frac{mv^2}{r}$ —— (3).

$T\sin\theta + mg\sin 180 + \frac{mv^2}{r}\sin 270 = 0$

$(2)^2 + (3)^2 \Rightarrow T^2 = m^2g^2 + \frac{m^2v^4}{r^2} = m^2g^2\left(1 + \frac{v^4}{r^2g^2}\right).$

$= m^2g^2\lambda^2$ where

$\lambda^2 = 1 + \frac{v^4}{r^2g^2}$ —— (4)

$\therefore T = mg\lambda$ —— (5).

The result (5) shows that the pendulum can be replaced by another pendulum in whose universe the value of gravity is $g\lambda$.

In such situation, the number of oscillations made per sec $= n$.

$\therefore \frac{1}{n} = 2\pi\sqrt{\frac{l}{g\lambda}}$ —— (6).

$$\frac{①}{⑥} \Rightarrow \frac{n}{n'} = \sqrt{\lambda} \quad \text{or} \quad \frac{n^2}{n'^2} = \lambda$$

$$\therefore \frac{n^4}{n'^4} = \lambda^2 = 1 + \frac{v^4}{r^2 g^2} \quad (\text{from } ②)$$

$$\frac{v^4}{r^2 g^2} = \frac{n^4}{n'^4} - 1$$

$$v^4 = r^2 g^2 \left(\frac{n^4}{n'^4} - 1 \right)$$

$$v^2 = rg \sqrt{\frac{n^4}{n'^4} - 1} \quad //$$