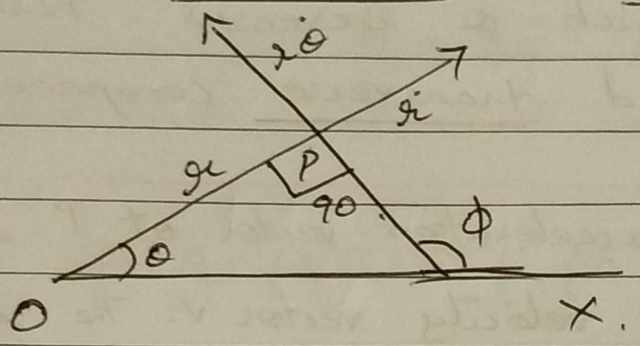
UNIT - IV

Motion Under The Action Of Central Forces.

Velocity and Acceleration in Polar Co-ordinates:



Let P be the position of a moving particle at time t.

Taking O as the pole and Ox as the initial line, let the polar coordinates of P be (r, θ).

$\vec{OP} = r$ is the position vector of P.

Hence Velocity of $P = \dfrac{d}{dt}(r)$.

Since r has modulus r and amplitude θ,

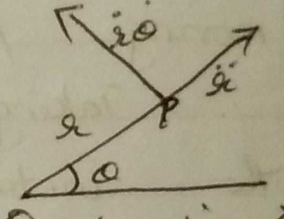
$\frac{d}{dt}(r)$ will have components $\underline{\dot{r}}$ along OP

and $\underline{r\dot{\theta}}$ $\perp^r$ to OP.

Hence the velocity vector v at P has components $\dot{r}$ along OP in the direction in which r increases and $r\dot{\theta}$ $\perp^r$ to OP in the direction in which θ increases. These are called the <u>radial</u> and <u>transverse</u> components of v.

The acceleration vector at P is the derivative of the velocity vector v. The radial component of v is a vector with modulus $\dot{r}$ and amplitude θ.

Hence the derivative of $\dot{r}$ will have components



derivative of $\dot{r}$

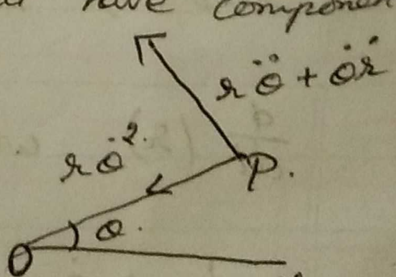
(i) $\frac{d}{dt}(\dot{r}) = \ddot{r}$ along OP in the direction in which r ↑ses

(ii) $\dot{r}\frac{d}{dt}(\theta) = \dot{r}\dot{\theta}$ $\perp^r$ to OP in the direction in which θ ↑ses.

The transverse component of v is a vector with modulus $r\dot{\theta}$ and amplitude $\phi = \frac{\pi}{2} + \theta$.

Hence the derivative of $r\dot{\theta}$ will have components

(i) $\frac{d}{dt}(r\dot{\theta}) = r\ddot{\theta} + \dot{r}\dot{\theta}$ along the line of $r\dot{\theta}$ (ie) in the direction $\perp^r$ to OP.



Derivative of $r\dot{\theta}$.

(ii). $r\dot{\theta}\frac{d}{dt}\left(\frac{\pi}{2} + \theta\right) = r\dot{\theta}^2$ in the direction $\perp^r$ to the line of $r\dot{\theta}$ (ie) in the direction PO. (This component is towards O, as it is in the direction in which ϕ ?ses)

Hence the totals of the components of acceleration are $\ddot{r} - r\dot{\theta}^2$ in the direction OP and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ in the $\perp^r$ direction.

Now $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta})$.

$= r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$\therefore$ Acceleration $\perp^r$ to OP is also $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

Corollary:

* Suppose the particle P is describing a circle of radius 'a'. Then $r = a$ throughout the motion.

Here $\ddot{r} = 0$ and radial acc. $= \ddot{r} - r\dot{\theta}^2$

$= -a\dot{\theta}^2$.

The acc. $\perp^r$ to OP $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{a}a^2\ddot{\theta} = a\ddot{\theta}$.

Hence for a particle describing a circle of radius a, the acceleration at any point P has the components $a\ddot{\theta}$ along the tangent at

P and $a\dot{\theta}^2$ along the radius to the centre

* The magnitude of the resultant velocity of $P = \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$ and

the magnitude of the resultant acceleration

$$= \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]^2} .$$

<u>Equations</u> of <u>Motion in</u> <u>Polar Coordinates</u>

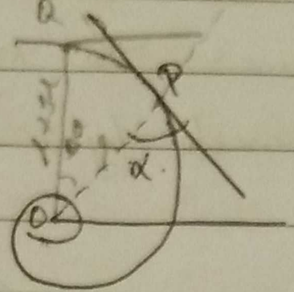
If R and S are the components of the external forces acting on a particle of mass m in the radial and transverse directions, we have the equations

$$R = m(\ddot{r} - r\dot{\theta}^2) \quad \text{——— (1)}$$

$$S = m\left(\frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta})\right) \quad \text{——— (2)}$$

If R and S are known functions of the coordinates r, θ and the time t, eqns (1) & (2) can be solved to find r and θ as functions of t and by eliminating t, the polar eqn. to the path is got.

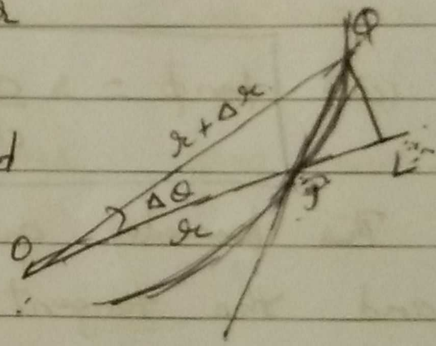
Note on the equiangular spiral



This curve has the important property that the tangent at any point P on it makes a constant angle with the radius vector OP.

Let $OP = r$ and $OQ = r + \Delta r$ be two consecutive radii vectors such that ~~the included angle~~ $\angle POQ = \Delta\theta$.

Draw QL $\perp^r$ to OP.



Then $\cos\Delta\theta = \dfrac{OL}{OQ}$ $\Rightarrow OL = r+\Delta r \cos\Delta\theta$

$= r + \Delta r$ approx.

Hence $PL = OL - OP = r + \Delta r - r = \Delta r$.

and $\sin\Delta\theta = \dfrac{QL}{OQ}$ $\Rightarrow QL = r + \Delta r \sin\Delta\theta$.

$= (r+\Delta r)\Delta\theta$ (nearly)

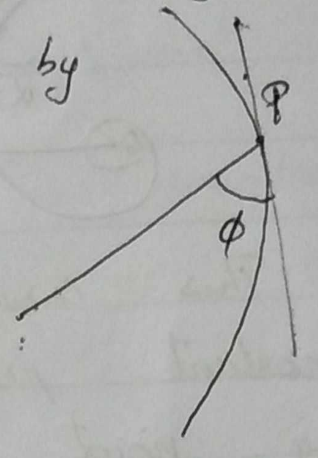
$= r\Delta\theta$.

Hence $\tan\angle QPL = \dfrac{QL}{PL} = r\dfrac{\Delta\theta}{\Delta r}$.

In the limit as Δr and $\Delta\theta$ both $\to 0$, the point Q tends to coincide with P.

The chord QP becomes the tangent at P.

Let ϕ be the angle ~~made~~ made by the tangent at P with OP.



Then $\phi = \underset{Q \to P}{Lt} \angle QPL$.

Hence $\tan\phi = \underset{\Delta r \to 0}{Lt} \tan \angle QPL$.

$$= \underset{\Delta r \to 0}{Lt} r \cdot \frac{\Delta\theta}{\Delta r} = r\frac{d\theta}{dr}.$$

(ie) $\boxed{\tan\phi = r\dfrac{d\theta}{dr}.}$

This formula gives the angle b/w the radius vector and the tangent.

Now for the equiangular spiral, at any point P on it the angle ϕ is a constant.

Let $\phi = \alpha$, then $\tan\phi = \tan\alpha$.

(ie) $r\dfrac{d\theta}{dr} = \tan\alpha$.

(or) $\dfrac{1}{\tan\alpha} d\theta = \dfrac{dr}{r}$. (ie) $\dfrac{dr}{r} = \cot\alpha \, d\theta$.

Integrating, ~~[illegible]~~ $\log r = \cot\alpha \, \theta + a$ (constant)

(ii) $\boxed{r = a e^{\theta \cot\alpha}}$

This is the polar equation ~~of~~ to the equiangular spiral.

Problems

① The velocities of a particle along and $\perp^r$ to a radius vector from a fixed origin are λr^2 and $\mu \theta^2$ where μ and λ are constants. Show that the equation to the path of the particle is $\frac{\lambda}{\theta} + c = \frac{\mu}{2r^2}$ where c is a con

Show also that the accelerations along and ⊥ to the radius vector are

$2\lambda^2 r^3 - \frac{\mu^2 \theta^4}{r}$ and $\mu\left(\lambda r \theta^2 + \frac{2\mu\theta^3}{r}\right)$

Radial velocity $= \dot{r} = \frac{dr}{dt} = \lambda r^2$ ——— ①.

Transverse " $= r\dot{\theta} = r\frac{d\theta}{dt} = \mu\theta^2$ ——— ②.

To solve ① + ②,

$\frac{②}{①} \Rightarrow \quad r\frac{d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2}$.

(i) $\lambda \frac{d\theta}{\theta^2} = \mu \frac{dr}{r^3}$

Integrating, $-\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} + c$.

(ie) $\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + c$. ——— (8).

(8) is the eqn. to the path.

differentiating (1). $\frac{d^2r}{dt^2} = \lambda \cdot 2r\frac{dr}{dt}$

$= 2\lambda^2 r^3$ (from (1))

Radial acceleration $= \ddot{r} - r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$= 2\lambda^2 r^3 - r\left(\frac{\mu\theta^2}{r}\right)^2$

$= 2\lambda^2 r^2 - \frac{\mu^2\theta^4}{r}$.

Transverse acceleration $= \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta})$.

$= \frac{1}{r} \cdot \frac{d}{dt}\left(r^2 \frac{\mu\theta^2}{r}\right)$.

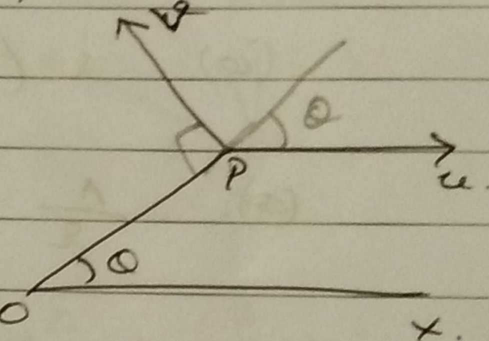
$= \frac{1}{r} \cdot \frac{d}{dt}(\mu r\theta^2)$.

$= \frac{\mu}{r}\left[2r\theta\frac{d\theta}{dt} + \theta^2\frac{dr}{dt}\right]$

$= \frac{\mu}{r}\left[2\cancel{r}\theta \cdot \frac{\mu\theta^2}{\cancel{r}} + \theta^2 \cdot \lambda r^2\right]$

$$= \mu\left[\frac{2\mu\theta^3}{r} + \lambda r\theta^2\right] //.$$

2. Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the second of which is $\perp^r$ to the radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is u/v.



Take O as the pole and the line Ox parallel to the given direction as the initial line.

P has two velocities u $\parallel^{el}$ to Ox and v $\perp^r$ to OP.

Resolving the velocities along and $\perp^r$ to OP, we have.

$$v\cos 90 + u\cos(360-\theta)$$

$$\dot{r} = \frac{dr}{dt} = u\cos\theta \quad \text{———} (1)$$

$$v\sin 90 + u\sin(360-\theta)$$

$$r\dot{\theta} = r\frac{d\theta}{dt} = v - u\sin\theta \quad \text{———} (2).$$

$$\frac{(2)}{(1)} \Rightarrow r\frac{d\theta}{dr} = \frac{v - u\sin\theta}{u\cos\theta}.$$

(ie) $\frac{u\cos\theta}{v-u\sin\theta}\,d\theta = \frac{dr}{r}$.

(ie) $\frac{d(u\sin\theta)}{v-u\sin\theta} = \frac{dr}{r}$.

Integrating, $-\log(v-u\sin\theta)$ ~~log~~ $+\log A = \log r$

(ie) $\log r + \log(v-u\sin\theta) = \log A$.

(ie) $r(v-u\sin\theta) = A$.

(ie) $\frac{A}{r} = v-u\sin\theta$.

(ie) $\frac{A}{v}\cdot\frac{1}{r} = 1-\frac{u}{v}\sin\theta = 1+\frac{u}{v}\cos(90+\theta)$.

(ie) $\left(\frac{A}{v}\right)\cdot\frac{1}{r} = 1+\frac{u}{v}\cos(90+\theta)$. —— (3).

This is of the form $\frac{l}{r} = 1+e\cos(\theta+\alpha)$ —— (4)

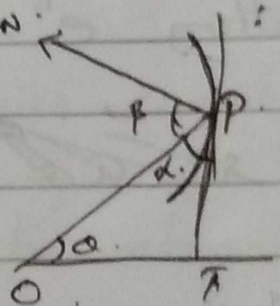
Comparing (3) & (4) we have

$l = \frac{A}{v}$, $e = \frac{u}{v}$ and $\alpha = 90^\circ$.

Thus (3) is a conic whose focus is at O, semi-latus rectum is $\frac{A}{v}$, eccentricity is $\frac{u}{v}$

and whose major axis is $\perp^r$ to the initial line.

(2). A point P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as the distance from O; show that the curve is an equiangular spiral whose pole is O and that the acceleration of the point is along the normal at P and varies inversely as OP.



Taking O as the pole, let P be (r, θ).

Resultant vel. of P $= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$ = const.

If this velocity $= k$, then. $\dot{r}^2 + r^2\dot{\theta}^2 = k^2$ — (1)

Also angular velocity about $O = \dot{\theta} = \frac{\lambda}{r}$ (given) — (2)

From (1) & (2).

$$\dot{r}^2 + \cancel{r^2}\frac{\lambda^2}{\cancel{r^2}} = k^2$$

(ie) $\dot{r}^2 = k^2 - \lambda^2$ or $\dot{r} = \sqrt{k^2 - \lambda^2}$ — (3)

To Eliminate t from ② & ③

$$\frac{③}{②} \Rightarrow \frac{\dot{r}}{\dot{\theta}} = \frac{dr/dt}{d\theta/dt} = \frac{dr}{d\theta} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}\, r.$$

(ie) $$\frac{dr}{r} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}\, d\theta.$$

Integrating, $$\log r = \frac{\sqrt{k^2-\lambda^2}}{\lambda}\,\theta + B.$$

(or) $$r = e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta + B} = e^{B}\cdot e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta}$$

Putting $e^{B} = a$ and $\frac{\sqrt{k^2-\lambda^2}}{\lambda} = \cot\alpha$, we get

$$r = a e^{\theta\cot\alpha} \quad \text{——— ④}.$$

④ is an equiangular spiral whose pole is O.

Differentiating ② $\ddot{r} = 0$.

Radial acceleration $= \ddot{r} - r\dot{\theta}^2 = -r\frac{\lambda^2}{r^2} = -\frac{\lambda^2}{r}$

Transverse acceleration $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{\lambda}{r}\right) = \frac{\lambda}{r}\frac{d}{dt}(r).$$

$$= \frac{\lambda}{r}\dot{r} = \frac{\lambda}{r}\sqrt{k^2-\lambda^2} \quad (\text{from } 3)$$

$\therefore$ Resultant acc $= \sqrt{\left(\frac{-\lambda^2}{r}\right)^2 + \left(\frac{\lambda}{r}\sqrt{k^2-\lambda^2}\right)^2}$

$$= \sqrt{\frac{\lambda^4}{r^2} + \frac{\lambda^2}{r^2}(k^2-\lambda^2)}$$

$$= \sqrt{\frac{\lambda^4+\lambda^2k^2-\lambda^4}{r^2}} = \sqrt{\frac{\lambda^2k^2}{r^2}}$$

$$= \cancel{\sqrt{\lambda k}} = \frac{\lambda k}{r}$$

Thus the resultant acc varies inversely as (ie) as OP.

Let this acceleration be along PN making an angle β with PO.

$$\tan\beta = \frac{\text{component } \perp^r \text{ to PO}}{\text{component along PO}}$$

$$= \frac{\frac{\lambda}{r}\sqrt{k^2-\lambda^2}}{\lambda^2/r} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}$$

But $\cot\alpha = \frac{\sqrt{k^2-\lambda^2}}{\lambda}$ from 2.

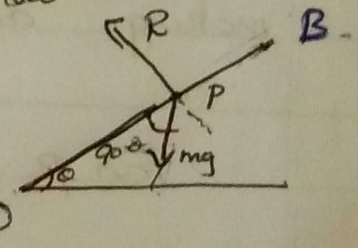
$\therefore \tan\beta = \cot\alpha = \tan(90^\circ - \alpha)$.

(or) $\beta = 90^\circ - \alpha$ (or) $\alpha + \beta = 90^\circ$.

Hence $\underline{|NPT} = 90^\circ$ where PT is the tangent at P. Hence PN is the normal to P //.

4\. A smooth straight thin tube revolves with uniform velocity ω in a vertical plane about one extremity which is fixed; if at zero time the tube be horizontal and a particle inside it be at a distance 'a' from the fixed end, and be moving with velocity V along the tube, show that the distance at time t is

$$a \cosh \omega t + \left(\frac{V}{\omega} - \frac{g}{2\omega^2}\right) \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t.$$



Let at time t, P be the position of the particle of mass m on the tube OB. The forces acting at P are

(i) weight mg vertically downwards

(ii) normal reaction R $\perp^r$ to OB.

Angular velocity $= \dot{\theta} = \frac{d\theta}{dt} = \omega$ (given).

Integrating, $\theta = \omega t + A$

Initially when $t=0,\ \theta=0$

$\therefore A=0.$

Hence $\theta = \omega t$ —— ①.

Resolving along the radius vector OB.

$R\cos 90 + mg\cos(270-\theta)$

$0 + mg\sin\theta$

$$m(\ddot{r} - r\dot{\theta}^2) = -mg\cos(90-\theta) = -mg\sin\theta$$

$$\therefore \ddot{r} - r\omega^2 = -g\sin\omega t.$$

(ie) $(D^2-\omega^2)r = -g\sin\omega t$ —— ② where $D=\dfrac{d}{dt}$

A.E. is $m^2-\omega^2=0.$

$m^2=\omega^2 \Rightarrow m=\pm\omega.$

$\therefore$ C.F $= Ae^{\omega t}+Be^{-\omega t}.$ —— ③

$$P.I = \frac{-g\sin\omega t}{D^2-\omega^2} = -\frac{g\sin\omega t}{-\omega^2-\omega^2}$$

$$= \frac{g\sin\omega t}{2\omega^2}. \quad \text{—— ④}$$

Hence general soln. $r = C.F + P.I$

(ie) $r = Ae^{\omega t} + Be^{-\omega t} + \dfrac{g}{2\omega^2}\sin\omega t$ —— ⑤

The initial conditions are : when $t=0$, $r=a$ and $\dot{r}=V$.

Hence (5) $\Rightarrow$ $a = A+B$ ——— (6).

differentiating (5) we get

$$\dot{r} = A\omega e^{\omega t} - B\omega e^{-\omega t} + \frac{g}{2\omega^2}\omega\cos\omega t \quad \text{——— (7)}$$

sub $t=0$ & $\dot{r}=V$ we get

$$V = A\omega - B\omega + \frac{g}{2\omega}.$$

(ie) $A\omega - B\omega = V - \frac{g}{2\omega}$.

(or) $A - B = \frac{V}{\omega} - \frac{g}{2\omega^2}$ ——— (8).

solving (6) & (8) we get

$$A = \frac{1}{2}\left[a + \frac{V}{\omega} - \frac{g}{2\omega^2}\right]$$

and $B = a - \frac{1}{2}\left[a + \frac{V}{\omega} - \frac{g}{2\omega^2}\right]$

$$= \frac{1}{2}\left[a - \frac{V}{\omega} + \frac{g}{2\omega^2}\right].$$

Sub A and B in (5).

$$r = \frac{1}{2}\left(a + \frac{v}{\omega} - \frac{g}{2\omega^2}\right)e^{\omega t} + \frac{1}{2}\left(a - \frac{v}{\omega} + \frac{g}{2\omega^2}\right)e^{-\omega t}$$

$$+ \frac{g}{2\omega^2}\sin\omega t.$$

$$= \frac{a\left(e^{\omega t} + e^{-\omega t}\right)}{2} + \frac{v}{\omega}\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right).$$

$$- \frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) + \frac{g}{2\omega^2}\sin\omega t.$$

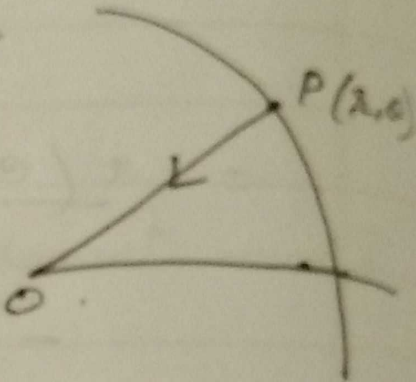
$$= a\cosh\omega t + \left(\frac{v}{\omega} - \frac{g}{2\omega^2}\right)\sinh\omega t + \frac{g}{2\omega^2}\sin\omega t$$

Motion under a Central force.

Suppose a particle describes a path, acted on by an attractive force F towards a fixed point O. Such a force is called central force and the path described by the particle is called ~~the~~ a central orbit. The fixed point is known as the centre of the force.

Differential Equation of Central Orbits:

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane; to obtain the differential equation of its path.



Take O as the pole and a fixed line through O as the initial line. Let $P(r,\theta)$ be the polar coordinates of the particle at time t and m be its mass.

Let P be the magnitude of the central acceleration along PO.

The eqns of motion of the particle are

$$m(\ddot{r} - r\dot{\theta}^2) = -mP \quad \text{(i.e)} \quad \ddot{r} - r\dot{\theta}^2 = -P \quad \text{--- (1)}$$

$$m\left(\frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta})\right) = 0 \,. \quad \text{(i.e)} \quad \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \text{--- (2)}$$

as transverse component of the acceleration is zero throughout the motion.

From (2) $\quad r^2\dot{\theta} = \text{constant} = h \text{ (say)} \quad \text{--- (3)}.$

To get the polar equation of the path, we have to eliminate the element of time between ① and ②.

For this put $u = \frac{1}{r}$

$$\therefore \text{③} \Rightarrow \dot{\theta} = \frac{h}{r^2} = hu^2$$

Also $\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\frac{du}{dt}$

$$= \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= -\frac{1}{u^2} \cdot \frac{du}{d\theta} \cdot hu^2$$

$$= -h\frac{du}{d\theta}$$

$$\therefore \ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) \cdot \frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2} \cdot hu^2 = -h^2u^2\frac{d^2u}{d\theta^2}$$

Sub $\ddot{r}$, r and $\dot{\theta}$ in ①

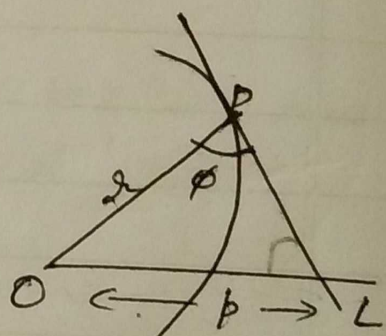
$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -P$$

(i.e) $h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = P$.

(or) $$\boxed{u+\frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}}$$

This is the differential equation of a central orbit, in polar coordinates.

Perpendicular from the pole on the tangent Formulae in polar Coordinates



Let ϕ be the angle made by the tangent at P with the radius vector OP.

we know that $\tan\phi = r\frac{d\theta}{dr}$ — ①.

From O draw OL $\perp^r$ to the tangent at P.

let $OL = p$.

Then $\sin\phi = \frac{OL}{OP} = \frac{p}{r}$ (or) $p = r\sin\phi$ — ②.

Let us eliminate ϕ b/w ① & ②.

② $\Rightarrow$ $\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}\operatorname{cosec}^2\phi$.

(i) $\dfrac{1}{p^2} = \dfrac{1}{r^2}(1+\cot^2\phi)$.

$= \dfrac{1}{r^2}\left(1+\dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)^2\right)$

(i) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ —— (3)

Using $r = \dfrac{1}{u}$, $\dfrac{dr}{d\theta} = \dfrac{dr}{du}\cdot\dfrac{du}{d\theta}$.

$= \dfrac{d}{du}\left(\dfrac{1}{u}\right)\cdot\dfrac{du}{d\theta}$.

$= -\dfrac{1}{u^2}\dfrac{du}{d\theta}$.

$\therefore$ (3) $\Rightarrow$ $\dfrac{1}{p^2} = u^2 + u^4\left\{\dfrac{1}{u^4}\left(\dfrac{du}{d\theta}\right)^2\right.$

(i) $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$ —— (4).

Pedal equation of the Central Orbit:

In certain curves the relation between p (the $\perp^r$ from the pole to the tangent) and r (the radius vector) is very simpl

Such a relation is called the <u>pedal eqn</u> or (p, r) equation.

(ie) The relation b/w p, the ⊥r from the pole on the tangent and r, the radius vector is called <u>pedal equation</u> or (p, r) eqn.

<u>(p, r) equation to a Central Orbit</u>

we know $$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \text{--- (1)}$$

differentiating both sides of (1) with respect to θ,

$$-\frac{2}{p^3} \cdot \frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2}$$

(ie) $$-\frac{1}{p^3} \cdot \frac{dp}{d\theta} = \frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right).$$

But the differential eqn in polars is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

$$\therefore -\frac{1}{p^3} \cdot \frac{dp}{d\theta} = \frac{P}{h^2u^2} \cdot \frac{du}{d\theta}.$$

(ie) $-\frac{1}{p^3} \cdot dp = \frac{P}{h^2u^2} du$

$$= \frac{P}{h^2} r^2 \, d\left(\frac{1}{r}\right)$$

$$= \frac{P\cancel{r^2}}{h^2}\left(-\frac{1}{\cancel{r^2}}\right) dr$$

$$= -\frac{P}{h^2} dr$$

(or) $\boxed{\frac{h^2}{p^3} \cdot \frac{dp}{dr} = P}$

This is the (p,r) equation or pedal eqn. to the central orbit.

<u>Two-fold problems in central orbits:</u>

There are two types of problems in connection with central orbit. They are

(i)* Given the orbit, to find the law of force to the pole.

(ii)* Given the law of force, to find the path.

We shall take up (i).

The differential equation to the central orbit in polar coordinates is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}.$$

Hence $P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$.

Since the orbit is given, u is known as a function of θ.

Hence by differentiation, P can be got from the above equation.

In few cases we may know the (p, r) equation to the path.

To find P, we can use the equation

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}.$$

1). Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described.

$$r^n = a^n \cos n\theta.$$

$\because$ $r = \frac{1}{u}$, the equation becomes.

$$\frac{1}{u^n} = a^n \cos n\theta.$$

(or) $\quad u^n a^n \cos n\theta = 1$ —— ①.

Taking log

$n \log u + n \log a + \log \cos n\theta = \log 1 = 0$ —— ②

diff. ② with respect to θ

$$n \cdot \frac{1}{u}\frac{du}{d\theta} + 0 - \frac{n \sin n\theta}{\cos n\theta} = 0$$

(ie) $\frac{1}{u}\frac{du}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$.

(ie) $\frac{du}{d\theta} = u \tan n\theta$. —— ③.

diff ③ with respect to θ.

$$\frac{d^2u}{d\theta^2} = u \cdot n \sec^2 n\theta + \tan n\theta \cdot \frac{du}{d\theta}.$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta \quad (\text{from } ③).$$

$$u + \frac{d^2u}{d\theta^2} = u + nu \sec^2 n\theta + u \tan^2 n\theta.$$

$$= nu \sec^2 n\theta + u(1 + \tan^2 n\theta).$$

$$= nu \sec^2 n\theta + u \sec^2 n\theta$$

$$= (n+1)\, u \sec^2 n\theta.$$

$$= (n+1)\, u \cdot u^{2n} a^{2n} \quad \left(\text{from } (1)\right).$$

$$= (n+1)\, a^{2n} u^{2n+1}.$$

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2 (n+1) a^{2n} \cdot u^{2n+1}.$$

$$= (n+1) a^{2n} h^2 u^{2n+3}$$

$$= (n+1) a^{2n} h^2 \cdot \frac{1}{r^{2n+3}}. \quad \text{--- } (4).$$

(ie) $P \propto \frac{1}{r^{2n+3}}$ which means that the central acceleration varies inversely as the $(2n+3)$rd power of the distance.

r.

<u>Note :</u>

From (4) P is true only when $n+1 > 0$ (ie) $n > -1$. For values of $n < -1$, P will be negative and in ~~ct~~ such cases the central force will be a repulsive one.

(i) when $n=1$, the eqn is $r = a\cos\theta$. The curve is a circle and $P \propto 1/r^5$.

(ii) when $n=2$, the eqn is $r^2 = a^2\cos 2\theta$. This is the Lemniscate and $P \propto 1/r^7$.

(iii) When $n = 1/2$, the eqn is $r^{1/2} = a^{1/2}\cos\theta/2$.
(ie) $r = a\cos^2\theta/2$. $= \frac{a}{2}(1+\cos\theta)$
This is a cardioid and $P \propto 1/r^4$.

(iv) when $n = -1/2$ The eqn is $r^{-1/2} = a^{-1/2}\cos\theta/2$.
(ie) $a^{1/2} = r^{1/2}\cos\theta/2$.
So $r = a/\cos^2\theta/2$. $= \frac{2a}{1+\cos\theta}$

(ie) $\frac{2a}{r} = 1+\cos\theta$.

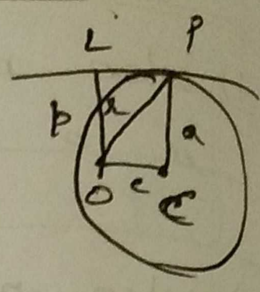
This is a parabola and $P \propto 1/r^2$.

(v) when $n=-2$, the eqn is $r^{-2} = a^{-2}\cos 2\theta$.
(ie) $a^2 = r^2\cos 2\theta$. This is a rectangular hyperbola.

In this case the actual value of $P = -a^{-4}h^2 r$. The $-ve$ sign of P shows that the central force is a repulsive one.

Pedal eqn of some well-known curves.

(i) Circle - pole at any point.



$$c^2 = r^2 + a^2 - 2ap.$$

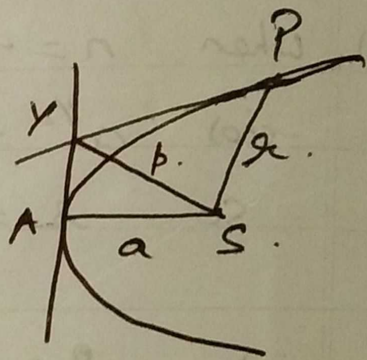
where C - centre.

radius CP = a.

O - pole.

$OC = c$, $OP = r$, $OL = p$

when $c = a$, the pole is on the circumference and the eqn is $r^2 = 2ap$.



(ii). Parabola - pole at focus.

$$p^2 = ar.$$

where S - focus.

A - vertex.

PY - tangent from P.

$AS = a$, $SP = r$ $SY = p$.

(iii) Ellipse and Hyperbola - pole at focus.

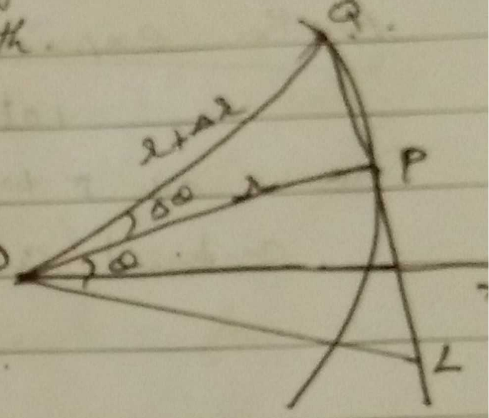
$\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is the $(p\ r)$ eqn to the ellipse

$\frac{b^2}{p^2} = \frac{2a}{r} + 1$ " " " " " " hyperbola

(iv) Equiangular spiral
$$p = r \sin\alpha = kr$$

Velocities in a central Orbit

In every central orbit the areal velocity is constant and the linear velocity varies inversely as the ⊥ from the centre upon the tangent to the path.



Let the particle be at $P(r, \theta)$ at time t.

At time $t + \Delta t$, let the particle be at $Q(r + \Delta r, \theta + \Delta\theta)$.

Sectorial area OPQ = Area of ΔOPQ nearly.

$$= \frac{1}{2} OP \cdot OQ \sin \underline{|POQ}$$

$$= \frac{1}{2} r \cdot (r + \Delta r) \sin \Delta\theta$$

$$= \frac{1}{2} r^2 \Delta\theta \quad \text{(approx)}$$

The rate of description of the area traced out by the radius vector joining the partic to a fixed point is called the areal velo of the particle.

Hence in the central orbit,

$$\text{areal velocity of } P = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h \quad \text{———} (1).$$

since $r^2 \dot{\theta}$ = constant = h.

Hence h = twice the areal velocity and as h is a constant, the areal velocity is const.

In otherwords, equal areas are described by the radius vector in equal times.

Another expression for areal velocity.

Let Δs be the length of the arc PQ.

Draw OL ⊥ to PQ.

Sectorial area of POQ = Δ POQ nearly.

$$= \frac{1}{2} PQ \cdot OL.$$

As $\Delta t \to 0$, Q tends to coincide with P along the curve and the chord QP becomes the tangent at P.

Length PQ = Δs nearly.

OL is the ⊥r from O on the tangent at P

Let OL = p.

Hence areal velocity $= \lim_{\Delta t \to 0} \frac{1}{2} \frac{\Delta s}{\Delta t} \cdot p$.

$$= \frac{1}{2} p \cdot \frac{ds}{dt} = \frac{1}{2} pv. \quad \text{———} (2)$$

as $\frac{ds}{dt}$ is the rate of describing s and so is

The linear velocity of P.

from ① & ② areal vel. $= \frac{1}{2}h = \frac{1}{2}pv$

or $h = pv$ (ie) $v = \frac{h}{p}$

Hence linear velocity varies inversely as OL

② Find the law of force to an internal point under which a body will describe a circle.

the (p, r) eqn of a circle ~~for a general position of pole~~ is

$$c^2 = r^2 + a^2 - 2ap \quad \text{——} ①$$

$$\Rightarrow \cancel{p = \frac{r^2 + a^2 - c^2}{2a}}$$

diff. w.r.t. ~~t~~ r.

$$0 = 2r - 2a\frac{dp}{dr}$$

(ie) $\frac{dp}{dr} = \frac{r}{a}$

Pedal eqn to central orbit is

~~Central acc~~ $P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{p^3} \cdot \frac{r}{a}$

$$= \frac{8h^2 r a^2}{(r^2+a^2-c^2)^3} \quad \text{(from ①)}$$

(3). A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

The polar eqn. to the ellipse with the focus as the pole is $\frac{l}{r} = 1 + e\cos\theta$.

where e - eccentricity

l - semi-latus rectum.

we know $u = \frac{1}{r}$

$\therefore \quad lu = 1 + e\cos\theta$.

(or) $\quad u = \frac{1 + e\cos\theta}{l}$.

Hence $\frac{du}{d\theta} = -\frac{e\sin\theta}{l}$ and $\frac{d^2u}{d\theta^2} = -\frac{e\cos\theta}{l}$.

Diff eq. of central orbit is $\left(u + \frac{d^2u}{d\theta^2}\right) = P/h^2u^2$.

$$u + \frac{d^2u}{d\theta^2} = \frac{1 + e\cos\theta}{l} - \frac{e\cos\theta}{l} = \frac{1}{l}.$$

we know that $\quad u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$.

$$\therefore \quad \frac{1}{l} = \frac{P}{h^2u^2}$$

Hence $P = \dfrac{h^2u^2}{l} = \dfrac{\mu}{r^2}$ where $\mu = \dfrac{h^2}{l}$
$\Rightarrow P \propto 1/r^2$.

(ie) the force varies inversely as the square of the distance from the pole.

Velocity

We know that $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$.

$$= \left(\frac{1+e\cos\theta}{l}\right)^2 + \left(\frac{e\sin\theta}{l}\right)^2.$$

$$= \frac{1+2e\cos\theta+e^2}{l^2}.$$

Also $h = pv$ where v is the linear velocity.

Hence $v^2 = \dfrac{h^2}{p^2} = h^2\left(\dfrac{1+2e\cos\theta+e^2}{l^2}\right)$

$$= \frac{\mu l}{l^2}\left(1+e^2+2\left(\frac{l}{r}-1\right)\right).$$

[as $\mu = \dfrac{h^2}{l} \Rightarrow h^2 = \mu l$ and $\dfrac{l}{r} = 1+e\cos\theta \Rightarrow e\cos\theta = \dfrac{l}{r}-1$]

$$= \frac{\mu}{l}\left(1+e^2+\frac{2l}{r}-2\right).$$

$$= \frac{\mu}{l}\left(\frac{2l}{r}-(1-e^2)\right).$$

(ie) $v^2 = \mu\left(\dfrac{2}{r} - \dfrac{1-e^2}{l}\right)$ —— (1)

In an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

~~Now if~~ a and b ~~are~~ the semi-axes ~~of the~~

ellipse, ~~we~~ know that

$$l=\frac{b^2}{a}=\frac{a^2(1-e^2)}{a}=a(1-e^2)$$

put $l=a(1-e^2)$ in ① we get

$$v^2=\mu\left(\frac{2}{r}-\frac{\cancel{1-e^2}}{a\cancel{(1-e^2)}}\right)=\mu\left(\frac{2}{r}-\frac{1}{a}\right).$$

Areal velocity in the orbit $=\frac{h}{2}$ and thus is a constant.

Total area of ellipse $=\pi ab$.

$$\therefore \text{Periodic time}=\frac{\pi ab}{h/2}=\frac{2\pi ab}{h}$$

$$=\frac{2\pi ab}{\sqrt{\mu l}} \text{ as } \mu=\frac{h^2}{l}.$$

$$=\frac{2\pi a\cancel{b}}{\sqrt{\mu}\cdot\cancel{b}}\sqrt{a} \text{ as } l=\frac{b^2}{a}.$$

$$=\frac{2\pi}{\sqrt{\mu}}a^{3/2}.$$

(4) A particle moves in a curve under a central attraction so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the path is an equiangular spiral and that the law of force is that of the inverse cube.

Let the central acceleration be P.

If v is the velocity in a circle at a distance r under the normal acceleration P, then

$$\frac{v^2}{r} = P \quad (\text{or}) \quad v^2 = Pr. \quad \text{—— (1)}$$

for a circle
$r_a = r\dot{\theta}^2$
$v = r\dot{\theta}$
$\therefore \dot{\theta} = v/r$
$r_a = r\frac{v^2}{r^2}$
$= v^2/r$

Since v is also the velocity in the central orbit,

$$h = pv \quad \text{or} \quad v = \frac{h}{p}.$$

sub. this in (1) we get $\frac{h^2}{p^2} = Pr$. —— (2)

we know $P = \frac{h^2}{p^3}\frac{dp}{dr}$ —— (3) (Pedal eqn)

sub (3) in (2).

$$\therefore \frac{h^2}{p^2} = \frac{h^2}{p^3}\cdot\frac{dp}{dr}\cdot r.$$

$$(\text{ie}) \quad 1 = \frac{r}{p}\frac{dp}{dr}. \quad \Rightarrow \frac{dr}{r} = \frac{dp}{p}.$$

(ie) $\frac{dp}{p} = \frac{dr}{r}$

Integ, $\log p = \log r + \log A$.

(ie) $p = rA$. ——— (4)

eq. (4) is the (p, r) eqn. to an equiangular spiral.

From (4) $\frac{dp}{dr} = A$.

sub this in (2), $P = \frac{h^2}{p^3} \cdot A$.

$= \frac{h^2 A}{A^3 r^3}$ as $p = Ar$.

$= \frac{h^2}{A^2} \cdot \frac{1}{r^3}$.

(ie) $P \propto \frac{1}{r^3}$ //

Areal Velocity: When a particle moves along a plane curve the rate of change of the area traced by the radius vector joining the particle to the centre of force is called the areal velocity of the particle.

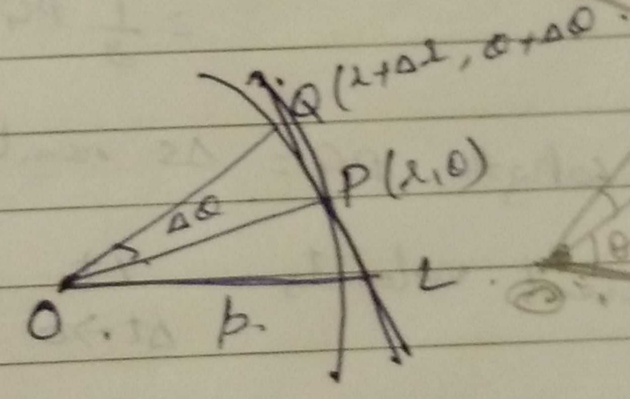
Result:- 1

In a central orbit areal velocity is a constant

(or)

In a central orbit equal areas are traced by the radius vector in equal times

Proof:



Sectorial area OPQ = The area of $\Delta^{le}\ OPQ$.

$= \frac{1}{2}\, OP \cdot OQ \sin \angle POQ$.

$= \frac{1}{2}\, r\,(r+\Delta r) \sin\Delta\theta$.

$= \frac{1}{2}\, r^2 \Delta\theta$.

$\therefore$ Areal vel $= \lim_{\Delta t\to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt}$

$= \frac{1}{2} r^2 \dot{\theta}$

$\therefore$ $\boxed{\text{Areal velocity} = \frac{h}{2}}$ where $h = r^2\dot{\theta}$

= constant.

Result :- 2.

In every central orbit the linear velocity of any point P of its path varies inversely as the ⊥r from the centre upon the tangent to the path P.

Proof :-

Sectorial area of POQ = Δ^{le} POQ.

$= \frac{1}{2} PQ \cdot PL$.

$= \frac{1}{2} PQ \cdot p$

Length PQ = Δs nearly.

Hence areal velocity $= \lim_{\Delta t \to 0} \frac{1}{2} \frac{\Delta s}{\Delta t} \cdot p$.

$= \frac{1}{2} p \cdot \frac{ds}{dt}$

$= \frac{1}{2} p \cdot v$.

as $\frac{ds}{dt}$ is the linear velocity of P.

but areal vel $= \frac{h}{2} = \frac{1}{2} pv$.

$\Rightarrow \boxed{v = \frac{h}{p}} \Rightarrow v \propto \frac{1}{p}$.

Note:

We know that $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

(ie) $\frac{v^2}{h^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

$\therefore \quad v^2 = h^2 \left\{ u^2 + \left(\frac{du}{d\theta}\right)^2 \right\}$.