COMPLEX ANALYSIS

UNIT-I

Analytic Functions:

Complex - Function - Limit of function - Continuity of a function - Uniform Continuity - Differentiability and analyticity of a function - Necessary condition for differentiability - Sufficient condition for differentiability - problems.

(Chapter 4 : Sections: 4.1 to 4.7).

UNIT-II.

C.R. Equation in polar co-ordinates - Harmonic function - Conformal mapping.

Bilinear Transformations:

Transformation $w = z^2$, $w = e^z$, $w = \sin z$, $w = \cos z$

(chapter 4: Sections 4.8 ; chapter 6: 6.12 ; chapter 7: 7.4 to 7.8) chapter

UNIT - III.

Complex Integration :-

Simple rectifiable positively oriented curves - Simple integrals using definition - Definite integrals - Interior and exterior of a closed curve - Simply connected region - Cauchy Goursat's theorem (without proof) Integrals along an arc joining two points - problems.

(Chapter 8 : Sections : 8.1, 8.3 to 8.8).

UNIT - IV.

cauchy's integral formula - cauchy's formula for derivative - Cauchy's formula for higher derivatives (statement only) - Morera's theorem - problems.

Taylors and Laurent's series :

Taylor's series - Zeros of an analytic function - Laurent's series.

(chapter 8 : section 8.9 ; chapter 9 : Sec 9.1 to 9.3)

UNIT - V.

Singular points and types of singularities Residues - Calculus of residues - problems.

(chapter 9 : Sec 9.5 to 9.13 ; Chapter 10 : Sec 10.1, 10.2)

TEXT BOOK :

Complex Analysis P. Duraipandian and Laxmi Duraipandian, D. Muhilan - Emerald publisher, second Edition, 1984.

REFERENCE BOOK :

THE ELEMENTS OF COMPLEX ANALYSIS: B. Choudhary, Wiley Eastern Limited.

COMPLEX ANALYSIS

UNIT - I

ANALYTIC FUNCTIONS

SECTION : 4 : 1

Definition

Complex Function :-

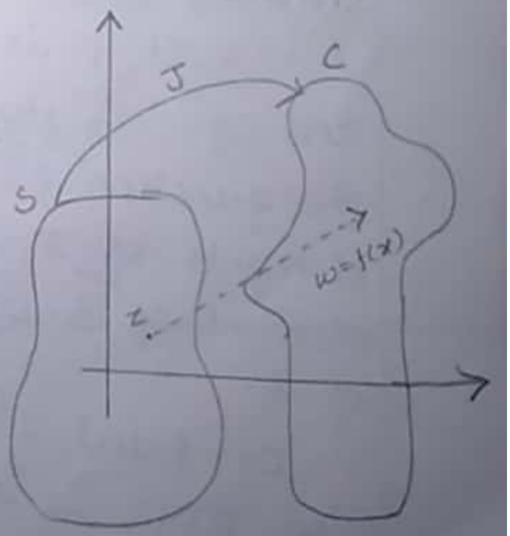
If S is any set of complex number then a rule $\boxed{F : S \to C}$ which associates each complex number of S which a complex number in C, then F is said to be a complex function.

If $z \in S$ is associated by f with $w \in C$, then $\boxed{w = f(z)}$.

Definition :-

Single - valued and Multi - Valued Function

If each $z \in S$ has only one value $f(z)$ of f then f is said to be single valued. Otherwise, f is said to be multi-valued function.



Example :-

Single - Valued Function :

$f(z) = z^2$

let $z = 1+i$, then

$f(z) = f(1+i) = (1+i)^2$

$= 1 - 1 + 2i$

$= 2i$

Multi-Valued Function:

$$f(z) = z^{1/2}$$

If $z = re^{i\theta}$ then $f(z) = r^{1/2}(\cos\theta + i\sin\theta)^{1/2}$

In analytical geometry, the angle θ is not unique we can add any multiple of 2π radians to θ.

$$\therefore f(z) = r^{1/2}\left[\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)\right]^{1/2}$$

$$f(z) = r^{1/2}\left[\cos\left(\frac{2n\pi+\theta}{2}\right) + i\sin\left(\frac{2n\pi+\theta}{2}\right)\right] \forall\ n = 0, 1, 2,$$

$\therefore$ De moivre's theorem,

It has two values for each z.

Because if,

$$z = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{ then}$$

$$f(z) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

In general,

$$f(z) = r^{1/2}(\cos\theta/2 + i\sin\theta/2) \cdot r^{1/2}\left(\cos\left(\frac{2\pi+\theta}{2}\right) + i\sin\left(\frac{2\pi+\theta}{2}\right.\right.$$

Inverse Function :-

If 'f' is a function defined in S. and if S' is the set of values of f, then the function f^{-1} which associates each elements of S' with the corresponding element of S is called the inverse function of f.

If f and f^{-1} are single-valued functions, then there is a 1-1 correspondance between S and S'.

$f \to S \to S'$ set of values of f

$S' \in f$. $f^{-1} \to$ inverse function of f

Polynomial Function :-

Function of the form.

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0.$$

where 'n' is a positive integer and $a_0, a_1, a_2, \ldots a_n$ are complex numbers is called a polynomial in z of degree n.

Rational function :-

The quotient of two polynomials is called a rational function. That is, the function

$$R(z) = \frac{P(z)}{Q(z)}, \quad Q(z) \neq 0.$$

where P(z) and Q(z) are polynomials in z, is a rational function. In this we assume that P(z) and Q(z) have no common factors.

Bounded Function:-

Suppose f(z) is a function defined in a set S. If there exists a finite number M such that $|f(z)| < M$ for all z in S, then f(z) is said to be bounded in S.

Example :

$f(z) = z$

$$f(z) = \begin{cases} 1/z, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

$S = [f(z)]$ → M → finite

$|f(z)| < M$

z in S

$|f(z)| < M \; \forall \; z$ in S

S [f(z)] ... M ∈ ... S

Section : 4.2

Limit of a Function

Definition:

Distance Between Two complex No's

If z_1 and z_2 are any two complex no's, then the distance between z_1 and z_2 is given by $|z_1 - z_2|$.

Definition:-

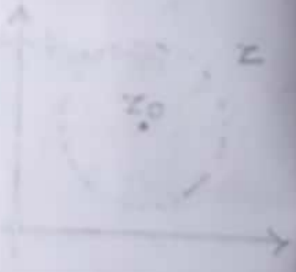
$|z - z_0| = \rho$, where 'ρ' is a positive number is referred as a circle with centre at z_0 and radius 'ρ'.

Definition:

Neighbourhood of z_0:

The set of all points $z \ni |z-z_0| < \rho$ is called neighbourhood of z_0. It is generally denoted by $N(z_0, \rho)$.

$z \ni$ $|z-z_0| < \rho$



Definition:-

Limit point of a set S:

A point z_0 is said to be a limit point of a set S if every neighbourhood of z_0 contains an infinite number of points of the set S.

Examples:-

Every point of the set of all points on a circle is a limit point of the set.

If z_0 is a point of the set, then every neighbourhood of z_0 will contain an infinite number of points of the set lying on the arc of the circle, which lies in the neighbourhood.

Definition:-

Limit of a Function (or) Limit of $f(z)$

$z_0 \to$ limit
f func

Let $\boxed{z_0}$ be a limit point of a set S and $\boxed{f}$ be a function defined on S.

A complex value A is said to be limit point of f at z_0 if for any $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$

$$|f(z) - A| < \epsilon, \quad \forall z \in S \text{ with}$$

$$0 < |z - z_0| < \delta$$

It is denoted by $\lim\limits_{z \to z_0} f(z) = A$.

Diagram

NOTE:
The limit of a function is unique.

Example:-

Indeterminate form

$$\lim_{z \to i} \frac{z^2+1}{z-i} = \frac{i^2+1}{i-i} = \frac{-1+1}{i-i} = \frac{0}{0} \quad [\text{I.D form}].$$

Substitute $z = i+h$, $h \to 0$.

$$\lim_{h \to 0} \frac{(i+h)^2+1}{(i+h)-i} = \lim_{h \to 0} \frac{i^2+h^2+2hi+1}{i+h-i}$$

$$= \lim_{h \to 0} \frac{-1+h^2+2hi+1}{h}$$

$$= \lim_{h \to 0} \frac{h(h+2i)}{h}$$

$$= \lim_{h \to 0} (h+2i)$$

$$\lim_{z \to i} \frac{z^2+1}{z-i} = 2i$$

Hence $\boxed{\text{limit of a function is unique}}$.

Example :-

Suppose that $f(z) = \begin{cases} \frac{Re\, z}{|z|} & , z \neq 0 \\ 0 & , z = 0. \end{cases}$

$z = x + iy$

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \cdot \lim_{y \to 0} f(x+iy)$$

$$= \lim_{x \to 0} \cdot \lim_{y \to 0} \frac{x}{\sqrt{x^2+y^2}}$$

Another the straight line, $\boxed{y = mx}$

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x}{\sqrt{x^2+m^2x^2}} = \lim_{x \to 0} \frac{x}{x\sqrt{1+m^2}}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{1+m^2}}$$

Thus for different straight line [ie different value of m]

$\lim_{z \to 0} f(z)$ is not unique

Hence

$\lim_{z \to 0} f(z)$ does not exist.

Uniqueness of a limit :-

Theorem : 4.1

The limit of a function is unique.

Proof :-

Suppose, a function $f(z)$ has two limits as

$$\lim_{z \to z_0} f(z) = A_1 \ , \quad \lim_{z \to z_0} f(z) = A_2$$

Then, for any small positive ϵ, $\exists$ two positive numbers δ_1 and δ_2 depending on ϵ $\ni$

$$|f(z) - A_1| < \epsilon \text{ if } |z - z_0| < \delta_1$$

$$|f(z) - A_2| < \epsilon \text{ if } |z - z_0| < \delta_2$$

Let 'δ' be the smaller of δ_1, δ_2. Then, if

$|z - z_0| < \delta$,

we have.

$$|f(z)-A_1| < \epsilon,$$

$$|f(z)-A_2| < \epsilon,$$

and using the result $|z_1+(-z_2)| \leq |z_1| \pm |z_2|$

we get,

$$|(f(z)-A_1)-(f(z)-A_2)| \leq |f(z)-A_1|+|f(z)-A_2|$$

$$= \epsilon+\epsilon$$

$$\text{or } |A_1-A_2| \leq 2\epsilon$$

But 'ϵ' is arbitrary and so we can choose any small value for ϵ. This requires that

$$A_1-A_2=0 \text{ (or) } A_1=A_2$$

i.e, the limit of $f(z)$ as $z \to z_0$ is unique.

It is important to note that the limit, when it exists, is unique whatever be the mode for approach of z to z_0 though the points in S. If the value is different for different approach of z to z_0 then the limit does not exist. The following illustration exhibits non-uniqueness and hence non-existence of the limit.

** 10 marks

Theorem : 4.2 :-

Suppose $f(z)=u(x,y)+iv(x,y)$ is a function defined in a set S and $z_0=x_0+iy_0$ is a limit point of S. Then $f(z)$ has a limit $h+ik$ at $z_0=x_0+iy_0$. if and only if $u(x,y)\to h$, $v(x,y)\to k$ as $x\to x_0$ and $y\to y_0$.

proof :-

Necessary part :- Given $f(z)=u(x,y)+iv(x,y)$

Hypothesis :-

Assume that $f(z)\to h+ik$ as $z\to x_0+iy_0$.

To prove:-

$u(x,y) \to h$, $v(x,y) \to k$ as $x \to x_0$ and $y \to y_0$.

From assumption,

$$\lim_{z \to z_0} f(z) = h + ik$$

By the definition of limit function,

for any $\epsilon > 0$ $\exists$ a $\delta \ni$

$|f(z) - (h+ik)| < \epsilon$, $\forall z \in S$ with $0 < |z - z_0| < \delta$.

$\therefore |u(x,y) - h| = |Re\{f(z) - (h+ik)\}|$

[$Re\ z = |z|$]

$$|u(x,y) - h| \leq ||f(z) - (h+ik)||$$
$$\leq |f(z) - h + ik|$$
$$< \epsilon \quad \forall z \in S \text{ with } 0 < |z - z_0| < \delta$$

$|u(x,y) - h| < \epsilon \ \forall x, y \ni |z - z_0| < \delta$

$$\left[\because \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\right]$$

$$\therefore \lim_{x \to x_0, y \to y_0} u(x,y) = h$$

$|||^{ly}$, it can be proved

$$\lim_{x \to x_0, y \to y_0} v(x,y) = k.$$

Sufficient part:-

Hypothesis:- Assume that $u(x,y) \to h$, $v(x,y) \to k$ as $x \to x_0$, $y \to y_0$.

To prove:-

$f(z) \to h + ik$ as $z \to z_0 = x_0 + iy_0$.

Proof:-

$$\lim_{x \to x_0, y \to y_0} u(x,y) = h; \quad \lim_{x \to x_0, y \to y_0} v(x,y) = k.$$

$\therefore |u(x,y) - h| < \epsilon/2 \quad , \forall x, y \ni$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \longrightarrow (1).$$

$|v(x,y) - k| < \epsilon/2 \quad , \forall x, y \ni$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \longrightarrow (2)$$

Let $\delta = \min\{\delta_1, \delta_2\}$

Then both (1) & (2) are true for δ.

Now,

$$\begin{aligned} |f(z) - (h+ik)| &= |u(x,y) + iv(x,y) - (h+ik)| \\ &= |(u(x,y) - h) + i(v(x,y) - k)| \\ &\leq |u(x,y) - h| + |v(x,y) - k| \\ &\leq \epsilon/2 + \epsilon/2 \quad \text{for } z \in S \text{ with} \\ &= \epsilon \qquad\qquad 0 < |z - z_0| < \delta \end{aligned}$$

ie $\lim_{z \to z_0} f(z) = h + ik$ as $z \to z_0 = x_0 + iy_0$

$\Rightarrow z \to x_0 + iy_0$.

THEOREM : 4.3 [Finite Limit at ∞

If $\lim_{z \to \infty} f(z) = A$ then $\lim_{z \to 0} f\left(\frac{1}{z}\right) = A$.

Proof :- Assume that $\lim_{z \to \infty} f(z) = A$

From the hypothesis, we get that, for any small positive number ϵ, $\exists$ an $M(\epsilon) \ni$

if $|z| > M(\epsilon)$ then $|f(z) - A| < \epsilon$

$\frac{1}{z} = t$

$z = \frac{1}{t}$

i.e, if $\lim_{z \to 0} f(z) =$

if $\left|\frac{1}{z}\right| < \frac{1}{M(\epsilon)}$ then $|f(z) - A| < \epsilon \longrightarrow (1)$

Denote the point $\frac{1}{z}$ by t in (1)

Then z is the point $\frac{1}{t}$ and (1) becomes,

$(1) \Rightarrow |t - 0| < \delta(\epsilon)$ then $\left|f\left(\frac{1}{t}\right) - A\right| < \epsilon$

where, $\delta(\epsilon) = \frac{1}{M(\epsilon)}$

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$$\Rightarrow \lim_{t \to 0} f\left(\frac{1}{t}\right) = A$$

Here, 't' is a dummy symbol.

changing it to z, we have

$$\lim_{z \to 0} f\left(\frac{1}{z}\right) = A.$$

Theorem 4.4 :-

If $\lim_{z \to 0} f(z) = A$ then $\lim_{z \to \infty} f\left(\frac{1}{z}\right) = A$

proof :- Assume that $\lim_{z \to 0} f(z) = A$

From the hypothesis, we get that for any small positive number ϵ, $\exists$ an $\delta(\epsilon) \ni$ if $|z-0| < \delta(\epsilon)$ then $|f(z)-A| < \epsilon$.

i.e,

if $\left|\frac{1}{z}\right| > \frac{1}{\delta(\epsilon)}$ then $\boxed{|f(z)-A| < \epsilon}$.

Let us denote the point $\frac{1}{z}$ of 't'.

Then z is the point $\frac{1}{t}$ and (1) becomes

(1) $\Rightarrow$ If $|t| > \frac{1}{\delta(t)}$ then $\left|f\left(\frac{1}{t}\right) - A\right| < \epsilon$

$$\therefore \lim_{t \to \infty} f\left(\frac{1}{t}\right) = A$$

Here 't' is a dummy symbol

Changing it to z, we have.

$$\lim_{z \to \infty} f\left(\frac{1}{z}\right) = A.$$

i.e, $\lim_{z \to \infty} f\left(\frac{1}{z}\right) = A.$

REMARK :-

If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$ then

i) $\lim_{z \to z_0} [f(z) \pm g(z)] = A \pm B$

ii) $\lim_{z \to z_0} f(z) \cdot g(z) = AB$

iii) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$, $B \neq 0$

Definition :-

Finite limit at ∞ :-

Let $f(z)$ be a function defined in a set S and let ∞ be a limit point of S. Then A is said to be the limit of $f(z)$ at ∞ if for any, $\epsilon > 0$ $\exists$ $R(\epsilon) > 0$ $\ni$

$|f(z) - A| < \epsilon'$ $\forall$ $z \in S$ with $|z| > R$.

Example :-

Let $f(z) = \frac{1}{z}$ then $\lim_{z \to \infty} \frac{1}{z} = 0$.

proof :-

consider $|f(z) - 0| < \epsilon$

$\Rightarrow \left|\frac{1}{z} - 0\right| < \epsilon$

$\Rightarrow \left|\frac{1}{z}\right| < \epsilon$

$\Rightarrow \frac{1}{\epsilon} < |z|$

$\Rightarrow |z| > \frac{1}{\epsilon}$.

S

$f(z)$ S

∞

A

$f(z)$ at ∞ $\epsilon > 0$ $R(\epsilon) > 0$ $\ni$ $|f(z) - A| < \epsilon'$ $\forall z \in S$ with $|z| > R$

Let $\delta = \frac{1}{\epsilon}$ then $|z| > \delta = \frac{1}{\epsilon}$ then

$\Rightarrow \frac{1}{|z|} < \epsilon$

i.e, $|f(z) - 0| < \epsilon$

$\lim_{z \to \infty} f(z) = 0$

$\Rightarrow \lim_{z \to \infty} \frac{1}{z} = 0$.

Definition :-

Infinite limit at a Finite point :-

Let $f(z)$ be a function defined in a set S and let z_0 be a limit point of S. Then ∞ is said to be the limit of $f(z)$ at z_0 if for any $M>0$ there exists a $\delta(M)>0$ $\ni$

$$|f(z)|>M, \forall z \in S \text{ other than } z_0 \text{ with } |z-z_0|<\delta.$$

Example :-

Let $f(z)=\frac{1}{z}$, $z \neq 0$ then $\lim_{z\to 0} f(z)=\infty$.

proof :-

Let $f(z)=\frac{1}{z}$, $z\neq 0$.

Let $M>0$ be given set $\delta=\frac{1}{M}$

If $0<|z-z_0|<\delta$ finite point $\boxed{z_0=0}$.

ie $|z|<\delta=\frac{1}{M}$

$\Rightarrow |z|<\frac{1}{M}$

$\Rightarrow \frac{1}{|z|}>M$

Thus, $|f(z)|=\frac{1}{|z|}>M, \forall z\in S$ with $0<|z-z_0|<\delta$

$\therefore \lim_{z\to 0} f(z)=\infty$ (Finite point $z_0=0$).

Definition :-

Infinite limit at ∞

Let $f(z)$ be a function defined in a set S and let ∞ be a limit of S. Then ∞ is said to be the limit of $f(z)$ at ∞ if for any $M>0$ $\exists$ $R(M)>0$ $\ni$.

$$|f(z)|>M, \forall z\in S \text{ other than } \infty \text{ with } |z|>R.$$

Example :-

Let $f(z)=z^2$ be a function defined in the finite plane. then $\lim_{z\to\infty} f(z)=\infty$.

proof :-

$|f(z)| = |z^2| = |z|^2$

Suppose $M>0$ is given.

$|z^2|>M$, $\forall z$ other form than ∞ with

$|z|>\sqrt{M}$

$\Rightarrow |z|>\sqrt{M}$

$\therefore \lim_{z\to\infty} f(z)=\infty$ $\quad \because z^2\to\infty$ as $z\to\infty$.

Section : 4-3

Definition :-

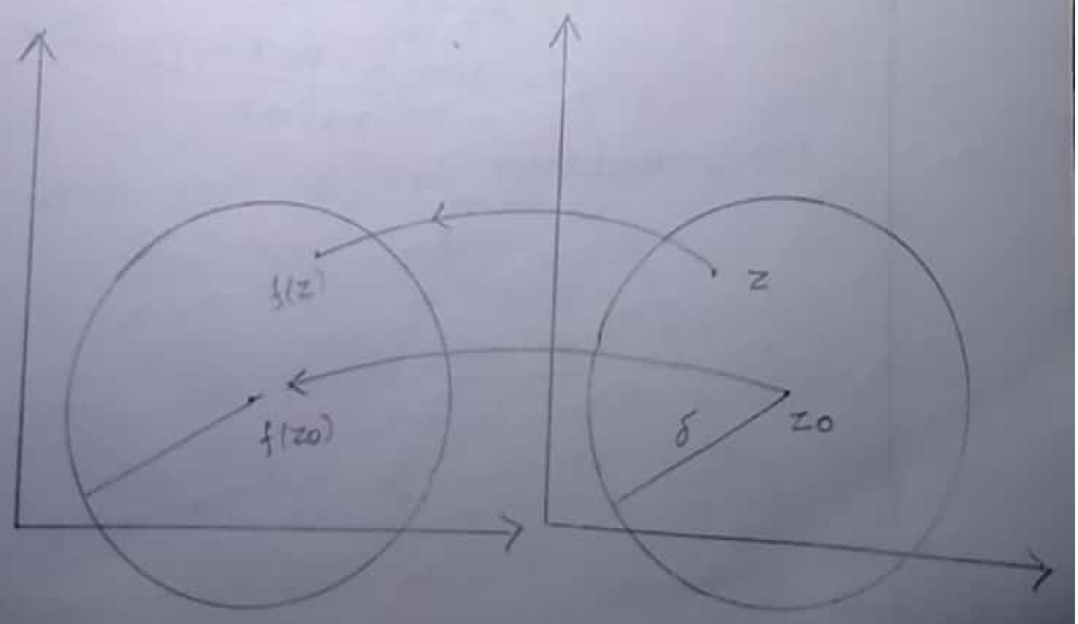
Continuity of a Function :-

Let $f(z)$ be a function defined in S and z_0 be a limit point of S contained in S. Then $f(z)$ is said to be continuous at z_0, if for any $\epsilon>0$ $\exists$ a $\delta>0$ $\ni$.

$|f(z)-f(z_0)|<\epsilon$, $\forall z\in S$ with $|z-z_0|<\delta$

In this, δ depends on ϵ and z_0.

ie $\lim_{z\to z_0} f(z) = f(z_0) \neq \infty$.



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Remark:

When $f(z_0)=\infty$, the function $f(z)$ is not countinuous at $z=z_0$.

$f(z)$ is said to be continuous at ∞, if for every $\epsilon>0$ $\exists$ an R $\ni$

$|f(z)-f(\infty)|<\epsilon$, $\forall z \in S$ with $|z|>R$.

In this, R depends on ϵ.

Continuity in a set - If a function is continuous at all points of a set S, then the function is said to be continuous in S.

In particular, S may be a region or a closed region or a curve.

Continuous real functions. polynomial function in x and y, e^x, $\sin x$, $\cos x$ etc. are some simple continuous real functions.

Continuity in a set:-

If a function is continuous at all points of a set S, then the function is said to be continuous in S. In particular, S may be a region or a closed region or a curve.

Example : 1 :- Discontinuity :-

Let
$$f(z)=\begin{cases}\dfrac{\operatorname{Re} z}{|z|}, & z\neq 0\\ 0, & z=0\end{cases}$$

then $f(z)$ is not continuous at $z=0$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{x}{\sqrt{x^2+y^2}}$$

let $y=mx$ $$=\frac{x}{\sqrt{x^2+x^2m^2}}=\frac{x}{\sqrt{x^2(1+m^2)}}$$

$$\lim_{x\to 0},\ \lim_{y\to 0}\frac{1}{\sqrt{1+m^2}}\neq 0. \quad =\frac{x}{x\sqrt{1+m^2}} \qquad y^2=m^2x^2$$

$\therefore f(z)$ is not continuous at $z=0$ $$=\frac{1}{\sqrt{1+m^2}} \qquad x^2+m^2x^2 \qquad x^2(1+m^2)$$

Example : 2 :- Discontinuity.

Suppose
$$f(z) = \begin{cases} z^2, & z \neq i \\ 0, & z = i \end{cases}$$ then

$\lim_{z \to i} f(z) = z^2 = i^2 = -1$

If $z = i$ then $f(i) = 0$ but $\lim_{z \to i} f(z) \neq f(i)$

$\therefore$ The given function is not continuous at $z = i$.

Example :3 : Continuous :-

Suppose $f(z) = z^2$, $\forall z$ then $\lim_{z \to i} f(z) = f(i)$

$\therefore f(z)$ is continuous at $z = i$. $f(z) = z^2 = -1$

$\lim_{z \to i} f(z) = z^2 = -1$

$f(z_0) = f(i) = -1$.

Book theorem 4.2

Theorem : 4.5 :-

If $f(z) = u(x,y) + i\,v(x,y)$ is a function defined in a set S and $z_0 = x_0 + iy_0$ is a limit point of S contained in S. Then $f(z)$ is continuous at z_0 if and only if

$$\lim_{x \to x_0,\, y \to y_0} u(x,y) = u(x_0, y_0) \neq \infty$$

$$\lim_{x \to x_0,\, y \to y_0} v(x,y) = v(x_0, y_0) \neq \infty$$

Remark :-

1. If $f(z)$, $g(z)$ are continuous at z_0, then $f(z) \pm g(z)$, $f(z)\,g(z)$ and $\frac{f(z)}{g(z)}$ $[g(z) \neq 0]$ are continuous at z_0.

2. A function $f(z)$ which is continuous in a closed region D, which is bounded in D.

3. A function which is continuous in a region which is not closed may or may not be bounded in it.

For example, $f(z) = \frac{1}{z}$ is continuous in $0 < |z| \leq 1$ but not bounded.

Theorem: 4.6.

Continuity of a function of a function.

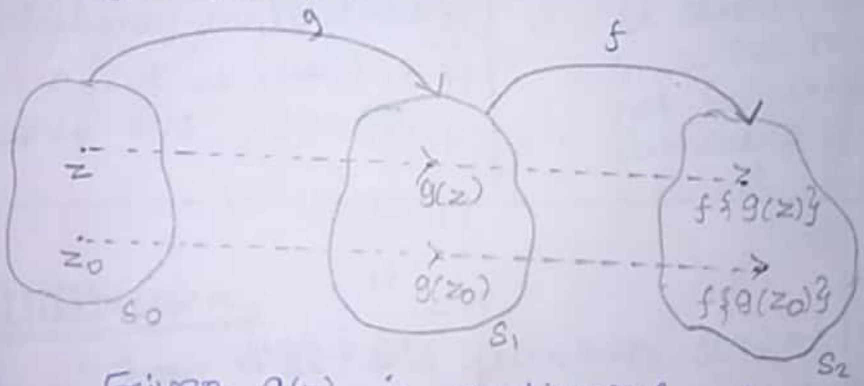
Suppose $g(z)$ is a function defined in a set S_0 and is continuous at z_0 in S_0. Let S_1 be the range of S_0 under the mapping g. If $f(z)$ is a function defined in the set S_1 and is continuous at $g(z_0)$ then the function $(fg)(z)$ is continuous at z_0.

Proof :-

Given $f(x)$ is continuous at $g(z_0) \in S_1$.

By the definition of continuity of function for any $\epsilon > 0$ $\exists$ a, $\delta > 0$ $\ni$

$$|f\{g(z)\} - f\{g(z_0)\}| < \epsilon, \ \forall\ g(z) \in S_1 \text{ with}$$
$$|g(z) - g(z_0)| < \delta \longrightarrow (1)$$



Given $g(z)$ is continuous at $z_0 \in S_0$, by the definition of continuity of function,

Given $\forall\ \epsilon_1 > 0$ $\exists$ a $\delta_1 > 0$ $\ni$

$$|g(z) - g(z_0)| < \epsilon_1, \ \forall\ z \in S_0 \text{ with } |z - z_0| < \delta_1$$

In particular,

for $\epsilon_1 = \delta$ $\exists$ a $\delta' > 0$ $\ni$

$$|g(z) - g(z_0)| < \delta, \ \forall\ z \in S_0 \text{ with } |z - z_0| < \delta' \longrightarrow (2)$$

From (1) and (2) we have

$$|f(g(z)) - f(g(z_0))| < \epsilon, \ \forall\ z \in S_0 \text{ with } |z - z_0| < \delta'$$

$\therefore (fg)(z)$ is continuous at $z_0 \in S_0$.

$|g(z) - g(z_0)| < \epsilon$. $|z - z_0| < \delta'$

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Theorem : 4.7

If $f(z)$ is continuous as $z=z_0$ and if for any $M>0$ there exists a 'd' such that $|f(z)|>M$ for all z in the disc $|z-z_0|<d$. then $\frac{1}{f(z)}$ is continuous at $z=z_0$.

Proof :-

Given that $f(z)$ is continuous at $z=z_0$.

By the definition of continuity of function, for every $\epsilon_1>0$ $\exists$ a $\delta>0$ $\ni$

$$|f(z)-f(z_0)|<\epsilon_1, \forall z \text{ in } |z-z_0|<\delta \rightarrow (1)$$

Also given that,

$$|f(z)|>M, \forall z \in |z-z_0|<d \rightarrow (2)$$

Consider,

$$\left|\frac{1}{f(z)}-\frac{1}{f(z_0)}\right| = \left|\frac{f(z_0)-f(z)}{f(z)\cdot f(z_0)}\right|$$

$$= \frac{|f(z_0)-f(z)|}{|f(z)\cdot f(z_0)|}$$

$$= \frac{|f(z_0)-f(z)|}{|f(z)||f(z_0)|}$$

$$< \frac{\epsilon_1}{M|f(z_0)|} \quad [\because \text{From } (1),(2)$$

$|f(z)|>M$

$\frac{1}{|f(z)|}<\frac{1}{M}$

$$\therefore \left|\frac{1}{f(z)}-\frac{1}{f(z_0)}\right| < \frac{\epsilon_1}{M|f(z_0)|}, \forall z \in |z-z_0|<\delta'$$

where $\delta' = \min(\delta, d)$

Let $\epsilon_1' = \frac{\epsilon_1}{M|f(z_0)|}$ then

$$\left|\frac{1}{f(z)}-\frac{1}{f(z_0)}\right| < \epsilon_1', \forall z \in |z-z_0|<\delta'$$

$\therefore \frac{1}{f(z)}$ is continuous at $z=z_0$.

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Section : 4.4 :-

Uniform continuity :-

A function $f(z)$ defined in a set S is said to be uniformly continuous in S if for every $\epsilon > 0$ $\exists$ a $\delta > 0$, depending on ϵ alone, $\ni$.

$$|f(z_1) - f(z_2)| < \epsilon, \ \forall z_1, z_2 \text{ whenever } |z_1 - z_2| < \delta$$

Continuity and Uniform continuity :-

Continuity	Uniform continuity
i) It is point wise property	It is setwise property.
ii) A function which is uniformly continuous in a set is also continuous in the set.	A function which is continuous in a set need not be uniformly continuous in the set.

Illustration :-

Show that the function $f(z) = \frac{1}{z}$ is continuous in the region $S: 0 < |z| < 1$ but not uniformly continuous in it.

Solution:-

If $z_0 \in S$ then 1 and z are continuous at z_0. The function $f(z) = \frac{1}{z}$ is continuous at z_0. $[\because z_0 \neq 0]$

Let $\lambda > 2$,

$$\alpha_1 = \frac{1}{\lambda}, \quad \alpha_2 = \frac{2}{\lambda}$$

$$\Rightarrow \alpha_1, \alpha_2 \in 0 < |z| < 1$$

consider,

$$|\alpha_1-\alpha_2| = \left|\frac{1}{\lambda}-\frac{2}{\lambda}\right| = \frac{1}{\lambda} \text{ and}$$

$$|f(\alpha_1)-f(\alpha_2)| = \left|\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right|$$

$$= \left|\lambda-\frac{\lambda}{2}\right| = \left|\frac{2\lambda-\lambda}{2}\right|$$

$$= \frac{\lambda}{2}.$$

$\therefore$ It is clear that the distance between α_1 and α_2 decreases, $|f(\alpha_1)-f(\alpha_2)|$ increases.

$\therefore$ For any $\epsilon>0$ we cannot find a $\delta>0$ $\ni$ $|f(z_1)-f(z_2)|<\epsilon$, $\forall z_1, z_2 \in S$ $\ni |z_1-z_2|<\delta$

$\therefore f(z)=\frac{1}{z}$ is not uniformly continuous in S.

Theorem: 4.8 :-

In a compact set every continuous function is uniformly continuous. [i.e. In a bounded closed region a continuous function is uniformly continuous].

proof :-

Let S be compact, let $a \in S$

Let $f(z)$ be continuous in S at a.

To prove :-

$f(z)$ is uniformly continuous :-

By the definition of continuity, for any $\epsilon>0$ $\exists$ a $\delta>0$ $\ni$.

$|f(z)-f(a)| < \epsilon/2$, $\forall z \in S$ with $|z-a|<\delta$

i.e $\forall z \in N(a,\delta)$

consider all the half-neighbourhoods

i.e $N(a_k, \frac{1}{2}\delta_k)$, $a_k \in S$

This is to cover S.

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As S is compact, by Heine-Borel theorem, A finite number of these half-neighbourhood can cover S.

Let these half-neighbourhood be

$N(a_1, \frac{1}{2}\delta_1), N(a_2, \frac{1}{2}\delta_2), \ldots, N(a_n, \frac{1}{2}\delta_n)$

Let $d = \min\left\{ \frac{\delta_1}{2}, \frac{\delta_2}{2}, \ldots, \frac{\delta_n}{n} \right\}$

Let $z_1 \in N(a_k, \frac{1}{2}\delta_k)$ and z_2 be a point $\ni$

$|z_1 - z_2| < d$ [Some $\frac{\delta_k}{2}$]

consider,

$$|z_2 - a_k| = |z_2 - z_1| + |z_1 - a_k|$$
$$\leq |z_1 - z_2| + |z_1 - a_k|$$
$$< d + \frac{1}{2}\delta_k$$
$$\leq \frac{1}{2}\delta_k + \frac{1}{2}\delta_k$$
$$|z_2 - a_k| = \delta_k$$

By (1),

$\therefore z_1$ and z_2 lie in $|z_1 - a_k| < \delta$

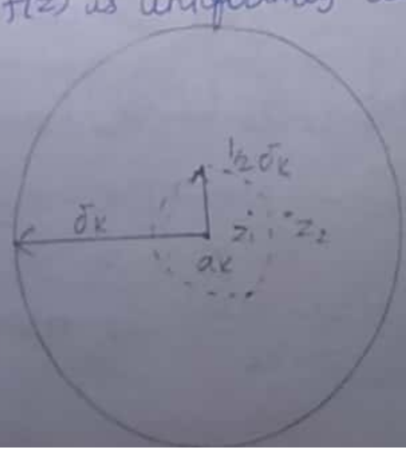
(1) $\rightarrow$ i.e by definition of continuity of $f(z)$ at a_k.

$|f(z_1) - f(a_k)| < \epsilon/2$, $|f(z_2) - f(a_k)| < \epsilon/2$.

$$\therefore |f(z_1) - f(z_2)| = |f(z_1) - f(a_k) + f(a_k) - f(z_2)|$$
$$\leq |f(z_1) - f(a_k)| + |f(a_k) - f(z_2)|$$
$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon.$$

$\therefore |f(z_1) - f(z_2)| < \epsilon$, $\forall z_1, z_2 \in S$ with $|z_1 - z_2| < d$.

$\therefore f(z)$ is uniformly continuous in S.



Section: 4.5:-

Differentiability is a property Analyticity of a Function.

Definition:-

Differentiability is a property of a function stronger than continuity. That is, a function which is differentiable, is continuous while a function which is continuous, need not be differentiable.

Differentiability at a point:-

$D \to C$

$C \not\to D$

Suppose $f(z)$ is a Single-valued function defined in a region and z_0 is a point in that region. Then the function is said to be differentiable at z_0.

If the following hold:-

i) if $f(z_0) \neq \infty$

ii) if $\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exist and

iii) i.e $\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0} \neq \infty$.

Remark:-

$\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ is called the derivative of $f(z)$ at z_0 and is denoted by

$$f'(z_0) = \left[\frac{d}{dz}(f(z))\right]\Big|_{z=z_0}$$

i.e $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$

$\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ is also written as

$$\lim_{a \to 0} \frac{f(z_0+a) - f(z_0)}{a}.$$

$z = z_0 + a$

$z_0 + a \to z_0$

$a \to 0$

Definition :-

Differentiability in a region:-

If a function is differentiable at all points in a region, then the function is said to be differentiable in that region.

Definition :-

Analyticity at a point :-

If a function is differentiable at all points in some neighbourhood of a point, then the function is said to be analytic at that point.

Example :-

The function $f(z) = |z^2|$ is differentiable at $z=0$ but it is not differentiable at all other points in any neighbourhood of $z=0$. So it is not analytic at $z=0$.

Synonyms for analyticity :-

The word used synonymous with the word analytic are homomorphic and regular.

Entire function:-

A function which is analytic everywhere in the finite plane is called an entire function. An entire function is analytic everywhere except at $z=\infty$.

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Examples of Entire Functions :-

* The polynomial in z $a_0 z^n + a_1 z^{n-1} + \dots + a$
* z^2
* The exponential function e^z
* The circular and hyperbolic functions $\sin z, \cos z, \sinh z, \cosh z$ [$\because$ defined using e^z]
* $(1+z)\sin z$.

Differentiation :-

The derivative of $f(z)$ at z is denoted by $\frac{d}{dz} f(z)$ or $f'(z)$.

If $f(z)$ and $g(z)$ are differentiable at z. then the following are true.

i) $\frac{d}{dz}[f(z)+g(z)] = f'(z) + g'(z)$.

ii) $\frac{d}{dz}[c\, f(z)] = c f'(z)$ [$\because$ c is a constant]

iii) $\frac{d}{dz}[f(z)\, g(z)] = f'(z)\, g(z) + f(z)\, g'(z)$

iv) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z) f'(z) - f(z) g'(z)}{[g(z)]^2}$ if $g(z) \neq 0$.

v) $\frac{d}{dz}[f\{g(z)\}] = f'\{g(z)\}\, g'(z)$ if $g(z)$ is analytic in a region D_1 and $f(z)$ is analytic in D_2 which is the range of D_1 under the mapping $g(z)$.

If $w = f(z)$ has a single valued inverse f^{-1} then $z = f^{-1}(w)$ and $\frac{dw}{dz}$ and $\frac{dz}{dw}$ are related by $\frac{dw}{dz} = \frac{1}{(dz/dw)}$.

If $z = f(t)$ and $w = g(t)$, where t is a parameter, then

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{g'(t)}{f'(t)}.$$

Derivatives of Elementary Functions

1. $\frac{d}{dz}(c) = 0$
2. $\frac{d}{dz}(z^n) = n z^{n-1}$
3. $\frac{d}{dz}(e^z) = e^z$
4. $\frac{d}{dz}(\sin z) = \cos z$
5. $\frac{d}{dz}(\cos z) = -\sin z$
6. $\frac{d}{dz}(\tan z) = \sec^2 z$
7. $\frac{d}{dz}(\cot z) = -\operatorname{cosec}^2 z$
8. $\frac{d}{dz}(\sec z) = \sec z \cdot \tan z$
9. $\frac{d}{dz}(\operatorname{cosec} z) = -\operatorname{cosec} z \cot z$
10. $\frac{d}{dz}(\log z) = \frac{1}{z}$
11. $\frac{d}{dz}(\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$
12. $\frac{d}{dz}(\cos^{-1} z) = \frac{-1}{\sqrt{1-z^2}}$
13. $\frac{d}{dz}(\tan^{-1} z) = \frac{1}{1+z^2}$
14. $\frac{d}{dz}(\cot^{-1} z) = \frac{-1}{1+z^2}$
15. $\frac{d}{dz}(\sec^{-1} z) = \frac{1}{z\sqrt{z^2-1}}$
16. $\frac{d}{dz}(\operatorname{cosec}^{-1} z) = \frac{-1}{z\sqrt{z^2-1}}$
17. $\frac{d}{dz}(\sinh z) = \cosh z$
18. $\frac{d}{dz}(\cosh z) = -\sinh z$
19. $\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z$
20. $\frac{d}{dz}(\coth z) = -\operatorname{cosech}^2 z$
21. $\frac{d}{dz}(\operatorname{sech} z) = \operatorname{sech} z \tanh z$
22. $\frac{d}{dz}(\operatorname{cosech} z) = -\operatorname{cosech} z \coth z.$

Differentiability of an Inverse Function:

Theorem 4.9:-

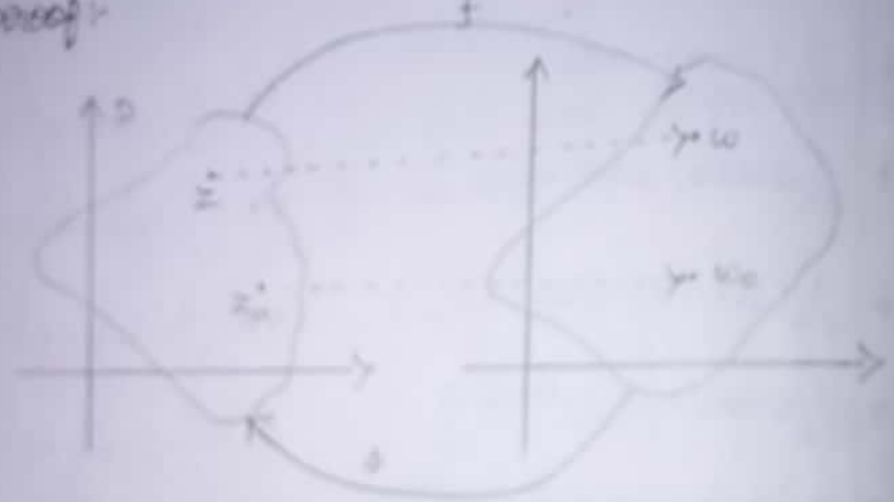
Suppose $f(z)$ is a function differentiable in a region D and the mapping $w = f(z)$ is 1-1 on the inverse mapping is $z = \phi(w)$.

If $z_0 \in D \ni f'(z_0) \neq 0$ then

The inverse function $\phi(w)$ is differentiable at w_0, where $w_0 = f(z_0)$ and

$$\phi'(w_0) = \frac{1}{f'(z_0)}$$

proof:-



Given $f(z)$ is differentiable at $z_0 \in D$.

To prove:-

$\phi(w)$ is differentiable at w_0

ie to prove:

$$\lim_{w \to w_0} \frac{\phi(w) - \phi(w_0)}{w - w_0} \text{ exists.}$$

As $f(z)$ is differentiable at $z_0 \in D$.

$\Rightarrow f(z)$ is continuous at z_0.

By the defn of continuity,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

ie $\lim_{z \to z_0} w = w_0$ $\left[\because \text{As } z \to z_0, f(z) \to f(z_0) \quad w \to w_0\right]$

consider, $\phi'(w_0) \Rightarrow$

$$\lim_{w \to w_0} \frac{\phi(w) - \phi(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} \quad \begin{matrix} w = f(z) \\ \cdot\ w_0 = f(z_0) \end{matrix}$$

$$= \lim_{z \to z_0} \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}$$

$$= \frac{\lim_{z \to z_0} \cdot 1}{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

$$\lim_{w \to w_0} \frac{\phi(w) - \phi(w_0)}{w - w_0} = \frac{1}{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'(z_0)}$$

$$= \frac{1}{f'(z_0)} \quad \left[\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)\right]$$

$$\phi'(w_0) = \frac{1}{f'(z_0)} \quad [\because f(z) \text{ is differentiable at } z_0 \text{ and } z_0 \neq 0]$$

Hence the inverse function. $\phi(w)$ is differentiable at w_0.

Definition :-

Singular point or singularity :-

A singular point or singularity of a function is a point at which the function Ceases to be analytic not analytic or the function is not defined.

(or)

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Definition:-

Entire Function : [$f(z)$ is analytic $\forall z$]

An entire function is one which is analytic in the entire finite complex plane, that is, analytic at every point of the extended complex plane except at ∞.

Example:-

1. The derivative of a polynomial in z exists at every point in the finite plane. So a polynomial is analytic in the finite plane and hence it is an entire function.

2. e^z, $\sin z$, $\cos z$, ze^z and $(1+z)\sin z$ are entire function.

Section : 4.6 :-

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Necessary conditions for differentiability

Theorem : 4.10 :-

A necessary conditions for a function to be differentiable at a point is the continuity of the function at the point.

proof :-

Suppose that $f(z)$ is differentiable at a point say z_0.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exist}$$

Now,

To prove :-

$f(z)$ is continuous at z_0.

$$\lim_{z \to z_0} f(z) = f(z_0) \Rightarrow \lim_{z \to z_0} f(z) = \lim_{z \to z_0} f(z_0)$$

$\lim_{z \to z_0} f(z) = f(z_0)$

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0).$$

$\lim_{z \to z_0} (z_0 - z_0)$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot (0)$$

$$= 0.$$

$$\therefore \lim_{z \to z_0} f(z) - f(z_0) = 0.$$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$\therefore$ $f(z)$ is continuous at z_0.

Remarks :-

1. By the above theorem it is clear that if $f(z)$ is not continuous at a point, then it is not differentiable at that point.

2. Continuity is not a sufficient condition of differentiability.

Example :-

Proof :-

Consider $f(z) = \bar{z} = x - iy$.

It is continuous at $z = 0$, because for a given $\epsilon > 0$,

$$|\bar{z} - 0| < \epsilon \quad \forall z \text{ with } |z - 0| < \epsilon$$

i.e $|f(z)| < \epsilon$ whenever $|z| < \epsilon$

$$\Rightarrow \lim_{z \to 0} f(z) = 0.$$

$$\lim_{z \to 0} f(z) = f(0).$$

Hence, $f(z)$ is continuous at 0.

Now, $z_0 = 0$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\bar{z}-\bar{0}}{z-0} = \lim_{z\to 0} \frac{\bar{z}}{z}.$$

$$= \lim_{xy\to 0} \frac{x-iy}{x+iy}$$

$$= \lim_{x\to 0} \frac{x-imx}{x+imx} \quad [\because y = mx]$$

$$= \lim_{x\to 0} \frac{(1-im)x}{(1+im)x}$$

Thus,

$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ is not unique as 'm' varies

i.e, $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist

$\therefore f(z)$ is not differentiable at 0.

Stronger necessary conditions for differentiability :-

Let $f(z) = u(x,y) + i v(x,y)$.

Then the eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$ are Called

Cauchy - Riemann equations. [C-R eqn's]

NOTE :-

C-R equations are simply denoted by $u_x = v_y$ and $u_y = -v_x$.

*** Important

Theorem : 4.11 :-

Cauchy - Riemann Theorem or Equations :-

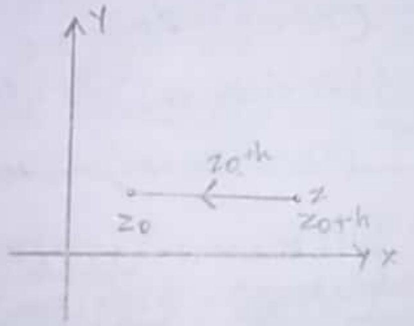
Suppose $f(z) = u(x,y) + iv(x,y)$ is a single valued functions defined in a neighbourhood of $z_0 = x_0 + iy_0$. Then the necessary condition for the differentiability of $f(z)$ at z_0 is the existance of the partial derivatives u_x, u_y, v_x, v_y at (x_0, y_0) which satisfys the relations $u_x = v_y$, $u_y = -v_x$.

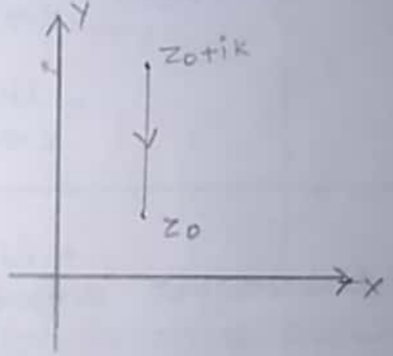
proof :-

Given $f(z) = u(x,y) + iv(x,y)$.

Suppose that $f(z)$ is differentiable at z_0, definition of differentiability

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exists.}$$





let z approach z_0 travelling along the line parallel to the x-axis. Then, $z = z_0 + h$ where 'h' is real and tends to 0, (zero).

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{(z_0+h) - z_0}$$

$$= \lim_{h \to 0} \frac{f[(x_0+h) + iy_0] - f(x_0 + iy_0)}{h}$$

$$= \lim_{h \to 0} \frac{[u(x_0+h, y_0) + iv(x_0+h, y_0)] - [u(x_0, y_0) + iv(x_0, y_0)]}{h}$$

$$= \lim_{h\to 0} \frac{[u(x_0+h, y_0) - u(x_0, y_0)] + i[v(x_0+h, y_0) - v(x_0, y_0)]}{h}$$

$$= \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} +$$

$$i \lim_{h\to 0} \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h}$$

$$= u_x(x_0, y_0) + i\, v_x(x_0, y_0).$$

$$f'(z_0) = u_x + i v_x \text{ at } (x_0, y_0) \longrightarrow (1)$$

If z approaches z_0 travelling along a line parallel to the y-axis.

Then, $z = z_0 + ik$, where 'k' real and tends to 0

$$f'(z_0) = \lim_{k\to 0} \frac{f(z_0+ik) - f(z_0)}{(z_0+ik) - z_0}$$

$$= \lim_{k\to 0} \frac{f(x_0+i(y_0+k)) - f(x_0+iy_0)}{(x_0+i(y_0+k)) - (x_0+iy_0)}.$$

$$= \lim_{k\to 0} \frac{[u(x_0, y_0+k) + i v(x_0, y_0+k)] - [u(x_0, y_0) + i v(x_0, y_0)]}{ik}.$$

$$= \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{ik}. +$$

$$\lim_{k\to 0} \frac{i[v(x_0, y_0+k) - v(x_0, y_0)]}{ik}$$

$$= \frac{1}{i}\left[\left[\lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{k}\right] + i\left[\lim_{k\to 0} \frac{v(x_0, y_0+k) - v(x_0, y_0)}{k}\right]\right]$$

$$= \frac{1}{i} u_y(x_0, y_0) + i v_y(x_0, y_0)$$

$$= \frac{1}{i}(u_y + i v_y) \text{ at } (x_0, y_0).$$

$$= \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-(i)^2} \qquad = -i\,(u_x + i v_y)$$

$$f'(z_0) = -i\,(u_y + i v_y) \text{ at } (x_0, y_0) \longrightarrow (2) \quad = -i u_y + v_y$$

Equating real and imaginary parts eqns (1) and (2) at (x_0, y_0) we get

$$u_x = v_y, \quad u_y = -v_x$$

Hence the C-R equations.

Remark :-

If $f(z)$ does not satisfy C-R equations then it is not differentiable.

Composite from of C-R eqns is :-

$$u_x + i v_x = \frac{u_y + i v_y}{i}$$

$$\frac{\partial f}{\partial x} = \frac{-i}{i} \times \frac{1}{-}\frac{\partial f}{\partial y} \quad (\text{or}) \quad \frac{\partial f}{\partial x} = -i\,\frac{\partial f}{\partial y}$$

and $f'(z) = \dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$.

Section: 4.7 :-

Sufficient conditions for differentiability:

***

Theorem : 4.12 :-

Suppose that $f(z) = u(x, y) + i v(x, y)$ is a single-valued function defined in a neighbourhood of $z_0 = x_0 + i y_0$. If the partial derivatives u_x, u_y, v_x, v_y exist and are continuous at (x_0, y_0) and if $u_x = v_y$, $u_y = -v_x$ at (x_0, y_0) then $f(z)$ is differentiable at z_0.

proof :-

Given u_x, u_y, v_x, v_y exist at (x_0, y_0) and are continuous at (x_0, y_0) and satisfy C-R eqn's.

To prove :-

$f(z)$ is differentiable at z_0.

Let $a = h+ik$, h, k are real numbers.

Let z_0+a be a point in the neighbourhood of z_0. $z = z_0+a$

$f(z)-f(z_0) \Rightarrow$

Then, $f((z_0)+a)-f(z_0) = f\{(x_0+iy_0)+(h+ik)\} - f(x_0+iy_0)$

$= f\{(x_0+h)+i(y_0+k)\} - f(x_0+iy_0)$

$= \{u(x_0+h, y_0+k)+iv(x_0+h, y_0+k)\} - \{u(x_0,y_0)+iv(x_0,y_0)\}$

$f(z_0+a)-f(z_0) = u(x_0+h, y_0+k)+iv(x_0+h, y_0+k) - u(x_0+h y_0) - iv(x_0, y_0)$

$f(z_0+a)-f(z_0) = \{u(x_0+h, y_0+k) - u(x_0,y_0)\} + i\{v(x_0+h, y_0+k) - v(x_0,y_0)\} \rightarrow (1)$

Consider,

$u(x_0+h, y_0+k) - u(x_0,y_0) = [u(x_0+h, y_0+k) - u(x_0, y_0+k)] + [u(x_0, y_0+k) - u(x_0,y_0)] \rightarrow (2)$

By <u>mean value</u> theorem

"If $f(x)$ is continuous in $a \le x \le b$ and differentiable in $a < x < b$. Then,

$f(a+b) - f(a) = h f(a+\theta h)$ where $0 < \theta < 1$".

$\therefore u(x_0+h, y_0+k) - u(x_0, y_0+k) = h u_x(x_0+\theta_1 h, y_0+k)$.

$u(x_0, y_0+k) - u(x_0,y_0) = k u_y(x_0, y_0+\theta_2 k)$

From (2), we have

$u(x_0+h, y_0+k) - u(x_0, y_0) = \{u(x_0+h, y_0+k) - u(x_0, y_0+k)\} + \{u(x_0, y_0+k) - u(x_0, y_0)\}$.

$= h u_x(x_0+\theta_1 h, y_0+k) + k u_y(x_0, y_0+\theta_2 k)$

$0 < \theta_1, \theta_2 < 1 \rightarrow (3)$

As u_x, u_y are:-

$$\lim_{\substack{h\to 0\\ k\to 0}} u_x(x_0+\theta_1 h, y_0+k) = u_x(x_0, y_0).$$

$$\lim_{\substack{h\to 0\\ k\to 0}} u_y(x_0, y_0+\theta_2 k) = u_y(x_0, y_0)$$

i.e, $u_x(x_0+\theta_1 h, y_0+k) = u_x(x_0, y_0) + \epsilon_1$ where $\epsilon_1 \to 0$ as $h\to 0, k\to 0$.

$u_y(x_0, y_0+k\theta_2) = u_y(x_0, y_0) + \epsilon_2$ where $\epsilon_2 \to 0$ as $h \to 0, k\to 0 \rightarrow (4)$

From (3),

$$u(x_0+h, y_0+k) - u(x_0, y_0) = h[u_x(x_0,y_0)+\epsilon_1] + k[u_y(x_0, y_0+\epsilon_2] \rightarrow (5)$$

Similarly,

$$v(x_0+h, y_0+k) - v(x_0, y_0) = h[v_x(x_0,y_0)+\epsilon_3] + k[v_y(x_0,y_0)+\epsilon_4] \rightarrow (6)$$

Substituting (5), (6) in (1) we get

$$f(z_0+a) - f(z_0) = \{h[u_x(x_0,y_0)+\epsilon_1] + k[u_y(x_0,y_0)+\epsilon_2]\} + i\{h[v_x(x_0,y_0)+\epsilon_3] + k[v_y(x_0,y_0)+\epsilon_4]\}$$

$$= h u_x + k u_y + h\epsilon_1 + k\epsilon_2 + i v_x h + i h \epsilon_3 + i k v_y + i k \epsilon_4$$

$$= h(u_x + i v_x) + k(u_y + i v_y) + h(\epsilon_1 + i\epsilon_5) + k(\epsilon_2 + i\epsilon_4)$$

$$= h(u_x + i v_x) + k(-v_x + i u_x) + h(\epsilon_1 + i\epsilon_3) + k(\epsilon_2 + i\epsilon_4).$$

[By C-R equations]

$= u_x(h+ik) + iv_x(h+ik) + h\varepsilon_1 + k\varepsilon_2 + i[h\varepsilon_3 + k\varepsilon_4]$ $[\because i^2 = -1]$

$= (h+ik)(u_x + iv_x) + h\varepsilon_1 + k\varepsilon_2 + i(h\varepsilon_3 + k\varepsilon_4).$

$$\frac{f(z_0+a) - f(z_0)}{a} = (u_x + iv_x) + \frac{h\varepsilon_1 + k\varepsilon_2 + i(h\varepsilon_3 + k\varepsilon_4)}{a}$$

$$\lim_{a\to 0} \frac{f(z_0+a) - f(z_0)}{a} = u_x + iv_x = f'(z_0)$$

(i.e) $h \to 0$, $k \to 0$.

$\therefore$ So the limit exists and hence $f(z)$ is differentiable at z_0.

Section: 4.4 :-

Uniform continuity :-

problems :-

1. Show that the function $f(z) = z^2$ is uniformly continuous in the unit disc $S: |z| < 1$.

** 5 marks

Solution :-

Given function $f(|z|) = z^2$, $|z| < 1$.

To prove :-

$f(z)$ is uniformly continuous :-

Take any two points $z_1, z_2 \in S$ then

$|f(z_1) - f(z_2)| = |z_1^2 - z_2^2|$

$= |(z_1 - z_2)(z_1 + z_2)|$ $\quad [|z| < 1,\ |z_1| < 1,\ |z_2| < 1]$

$= |z_1 - z_2|\,|z_1 + z_2|$

$|z_1+z_2| \leq |z_1| + |z_2|$ $\quad \leq |z_1 - z_2|\{|z_1| + |z_2|\}$

$< |z_1 - z_2|(1+1) = 2|z_1 - z_2|$

$|f(z_1) - f(z_2)| \leq 2|z_1 - z_2|$

Given any $\epsilon > 0$.

$|f(z_1)-f(z_2)| < \epsilon$ if $2|z_1-z_2| < \epsilon \Rightarrow |z_1-z_2| < \epsilon/2$

Let $\epsilon/2 = \delta$, then $|z_1-z_2| < \delta$

$\therefore$ So given any $\epsilon > 0$, we are able to find a $\delta = \epsilon/2$.

Such that

$|f(z_1)-f(z_2)| < \epsilon, \forall z_1, z_2 \in \delta$ with $|z_1-z_2| < \delta$

Hence,

$f(z)$ is uniformly continuous $|z| < 1$.

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2. Prove that $f(z) = 3z-2$ is uniformly continuous in $|z| \leq 10$.

Solution:-

Given function is $f(z) = 3z-2$, $|z| \leq 10$.

To prove:-

$f(z)$ is uniformly continuous.

Take any two points $z_1, z_2 \in Z$ then,

$$|f(z_1)-f(z_2)| = |(3z_1-2)-(3z_2-2)|$$

$$= |3(z_1-z_2)-0|.$$

$$\therefore |f(z_1)-f(z_2)| = 3|z_1-z_2|$$

Given any $\epsilon > 0$,

$|f(z_1)-f(z_2)| < \epsilon$ if $3|z_1-z_2| < \epsilon \Rightarrow |z_1-z_2| < \epsilon/3$

Let $\delta = \epsilon/3 \Rightarrow |z_1-z_2| < \delta$

So given any $\epsilon > 0$ we are able to find a $\delta = \epsilon/3$, Such that

$|f(z_1)-f(z_2)| < \epsilon, \forall z_1, z_2 \in S$ with $|z_1-z_2| < \delta$

$\therefore f(z)$ is uniformly continuous in $|z| \leq 10$.

Section : 4.6 :-

Conditions for differentiability

3. Show that, at the region, the following functions are not continuous and not differentiable.

5 mark

i) $f(z) = \begin{cases} \dfrac{Re\, z}{|z|} , & z \neq 0 \\ 0 , & z = 0 \end{cases}$

ii) $f(z) = \begin{cases} \dfrac{z}{|z|} , & z \neq 0 \\ 0 , & z = 0 \end{cases}$

Solution:-

i) Given function is

$$f(z) = \begin{cases} \dfrac{Re\, z}{|z|} , & z \neq 0 \\ 0 , & z = 0. \end{cases}$$

To prove :-

f(z) is not continuous and also not differentiable.

By the definition of continuity

$$\lim_{z \to z_0} f(z) = f(z_0).$$

ie,

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{Re\, z}{|z|} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{\sqrt{x^2+y^2}}$$

Along the straight line $y = mx$

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{\sqrt{x^2+m^2x^2}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{x\sqrt{1+m^2}}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1}{\sqrt{1+m^2}}$$

Thus for different straight lines, we get different values of m.

$\lim\limits_{z \to 0} f(z)$ is not unique.

$\Rightarrow \lim\limits_{z \to 0} f(z)$ does not exist.

$\therefore f(z)$ is not continuous.

$\therefore$ By the conditions for differentiability,

A function is not continuous at a point than the function is not differentiable at the point,

$\therefore f(z)$ is not differentiable at $z=0$.

ii) Given function is

$$f(z) = \begin{cases} \dfrac{z}{|z|}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

By the definition of continuity

$$\lim_{z \to z_0} f(z) = f(z_0).$$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{|z|} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+iy}{\sqrt{x^2+y^2}}$$

Along the straight line $y = mx$

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+imx}{\sqrt{x^2+m^2x^2}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1+im}{\sqrt{1+m^2}}$$

we get different straight line for different values of m.

$\lim\limits_{z \to 0} f(z)$ is not unique.

$\lim_{z \to 0} f(z)$ is does not exist.

$\therefore f(z)$ is not continuous.

By the condition for differentiability,

$f(z)$ is not differentiable at $z=0$.

4. Show that the function $f(z)=\bar{z}$ is nowhere differentiable.

Solution:-

Given function is $f(z)=\bar{z}$.

To prove:-

$f(z)$ is no where differentiable.

$$f(z)=\bar{z}=\overline{x+iy}=x-iy$$

$$f(x+iy)=x-iy$$

$$u(x,y)=x \; ; \; v(x,y)=-y$$

$$u_x=1 \; ; \; u_y=0 \rightarrow (1)$$

$$v_x=0 \; ; \; v_y=-1 \rightarrow (2)$$

The C-R equations are

$$u_x=v_y \text{ and } u_y=-v_x \rightarrow (3)$$

$$u_x \neq v_y \, , \; u_y \neq -v_x$$

$$1 \neq -1 \, , \; 0=0.$$

The C-R equation's are not satisfied anywhere $f(z)=\bar{z}$ is nowhere differentiable.

5. Show that the function $Re\, z$ and $z-\bar{z}$ are no where differentiable.

Solution :-

i) $f(z) = Re\, z$.

Given function is $f(z) = Re\, z = x$.

$u(x,y) = x$, $v(x,y) = 0$

$u_x = 1$, $u_y = 0 \rightarrow (1)$

$v_x = 0$, $v_y = 0 \rightarrow (2)$

The C-R equations are $u_x = v_y$ and $u_y = -v_x$ From (1) and (2).

$u_x \neq v_y$

The C-R equations are not Satisfied on where $f(z) = Re\, z$ is no where differentiable.

ii) $f(z) = z - \bar{z}$.

$f(z) = (x+iy) - (x-iy) = 2iy$

$u(x,y) = 0$ $v(x,y) = 2y$.

$u_x = 0$, $u_y = 0 \rightarrow (1)$

$v_x = 0$, $v_y = 2 \rightarrow (2)$

The C-R equations are $u_x = v_y$, $u_y = -v_x$

From (1), (2)

$u_x \neq v_y$

$\therefore$ The C-R equations are not Satisfied

$\therefore$ The function $z-\bar{z}$ is no where differentiable.

6. Show that the function $f(z) = xy + iy$ is continuous everywhere but is not analytic anywhere.

Solution:-

Given function is $f(z) = xy + iy$ is continuous everywhere.

Since x and y are continuous then the product x and y are continuous in the finite plane. So, $xy + iy$ is continuous in the finite plane.

$u(x,y) = xy$ $\quad$ $v(x,y) = y$

$u_x = y$ $\quad$ $v_x = 0$

$u_y = x$ $\quad$ $v_y = 1$

As; $u_x \neq v_y$ and $u_y \neq -v_x$

the C-R eqn's are not satisfied anywhere. $f(z) = xy + iy$ is not analytic anywhere.

7. If $f(z)$ is a function analytic in a region D. prove that $f(z)$ is constant in D, if in D, either,

****

i) its real part is constant or

ii) its imaginary part is constant or

iii) $|f(z)|$ is constant or

iv) ~~arg~~ arg $f(z)$ is constant.

Solution:-

i) If $f(z) = u(x,y) + i\,v(x,y)$

let real part of $f(z)$ is constant.

$\Rightarrow u(x,y)$ is constant

$\therefore u_x = 0,\ u_y = 0$

Given that $f(z)$ is analytic in a region D

$\therefore$ It must be satisfy the C-R eqn's.

$u_x = v_y$ and $u_y = -v_x$.

If $u_x = 0 \Rightarrow v_y = 0$

If $u_y = 0 \Rightarrow v_x = 0$

$v_x = 0,\ v_y = 0$.

$\Rightarrow v(x,y)$ is constant

$\therefore u + iv$ is constant

$\therefore f(z)$ is constant.

ii) If $f(z) = u(x,y) + iv(x,y)$

let the imaginary part of $f(z)$ is constant.

i.e $v(x,y)$ is constant

$\rightarrow v_x = 0$ is constant

$\rightarrow v_x = 0,\ v_y = 0$.

Given that $f(z)$ is analytic in a region D.

It must be satisfy C-R eqn.

$u_x = v_y$ and $u_y = -v_x$

If $u_x = 0 \Rightarrow u_y = 0$.

$u_x = 0,\ u_y = 0$.

$\Rightarrow u(x,y)$ is constant

$u + iv$ is constant

$\therefore f(z)$ is constant

iii) If $f(z) = u(x,y) + iv(x,y)$

Let $|f(z)|$ is constant

$\Rightarrow |u+iv|$ is constant

$\Rightarrow \sqrt{u^2+v^2}$ is constant

$\Rightarrow u^2+v^2$ is constant

$\Rightarrow u^2+v^2 = c \longrightarrow (1)$

Differentiating (1) with respect to 'x'

$2u \cdot u_x + 2v \cdot v_x = 0.$

$uu_x + vv_x = 0 \longrightarrow (2)$

Differentiating (1) w.r.t 'y'

$2u \cdot u_y + 2v v_y = 0.$

$u \cdot u_y + v \cdot v_y = 0 \longrightarrow (3)$

Eliminating u and v from (2), (3) we get

$(2) \times v_y \Rightarrow u u_x v_y + v v_x v_y = 0$

$(3) \times v_x \Rightarrow u u_y v_x + v v_x v_y = 0$

$\quad (-) \qquad (-)$

$u(u_x v_y - u_y v_x) = 0.$

$u_x v_y - u_y v_x = 0 \longrightarrow (4)$

but given $f(z)$ is analytic in D. If Satisfi

C-R eqn (i.e) $u_x = v_y$, $u_y = -v_x$.

Sub C-R eqn's in (4).

$u_x(u_x) - u_y(-u_y) = 0$

$(u_x)^2 + (u_y)^2 = 0.$

$\Rightarrow u_x = 0, \; u_y = 0.$

$\therefore$ u is constant.

$(2) \times u_y \Rightarrow u u_x u_y + v v_x u_y = 0$

$(3) \times u_x \Rightarrow u u_x u_y + v u_x v_y = 0.$

$\quad (-) \qquad (-)$

$$v(v_x u_y - u_x v_y) = 0$$

$$\therefore v_x u_y - u_x v_y = 0 \rightarrow (5)$$

By C-R eqns:-

$$v_x(-v_x) - (v_y)(v_y) = 0$$

$$\Rightarrow v_x^2 + v_y^2 = 0$$

$$\Rightarrow v_x = 0, \quad v_y = 0.$$

$\Rightarrow$ 'v' is constant.

$u + iv$ is constant

$\therefore f(z)$ is constant.

iv) Let arg $f(z)$ is constant

$\Rightarrow$ arg $(u+iv)$ is constant

$\Rightarrow \tan^{-1}\left(\frac{v}{u}\right)$ is constant

$\Rightarrow \frac{v}{u}$ is constant

consider

$$\frac{v}{u} = c \Rightarrow cu = v$$

Differentiating w.r to 'x' and 'y'.

$$cu_x = v_x \rightarrow (1)$$

$$cu_y = v_y \rightarrow (2).$$

Eliminating c from (1), (2)

$$\frac{u_x}{u_y} = \frac{v_x}{v_y} \Rightarrow u_x v_y = v_x u_y.$$

$$u_x v_y - u_y v_x = 0 \rightarrow (3)$$

Given $f(z)$ is analytic in 'D' it satisfies C-R eqn.

i.e $u_x = v_y$, $u_y = -v_x$

Substituting C-R eqns in (3)

$u_x^2 + u_y^2 = 0 \Rightarrow u$ is constant

By C-R eqn, $v_x = 0$, $v_y = 0$

$\therefore v$ is constant

$\therefore u + iv$ is constant

$f(z)$ is constant.

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(8) If $f(z)$ and $\overline{f(z)}$ are analytic in a region show that $f(z)$ is constant in that region.

Solution:-

Let $f(z) = u + iv$ and $\overline{f(z)} = (u - iv)$ are analytic.

$\therefore$ It satisfies the C-R eqns.

$$u_x = v_y \rightarrow (1)$$

$$u_y = -v_x \rightarrow (2)$$

To prove:-

$f(z)$ is constant.

$\overline{f(z)} = u - iv$ satisfies the C-R eqns.

$$\therefore u_x = -v_y \rightarrow (3)$$

$$u_y = v_x \rightarrow (4)$$

From (1), (3) $\Rightarrow v_y = -v_y \Rightarrow 2v_y = 0 \Rightarrow v_y = 0$.

From (1) $\Rightarrow u_x = 0$.

From (2) & (4)

$-v_x = v_x \Rightarrow 2v_x = 0 \Rightarrow v_x = 0$.

From (2) $\Rightarrow u_y = 0$ All the first derivatives

$$u_x = u_y = v_x = v_y = 0.$$

$\therefore f(z) = u + iv$ is constant in that region.

9. If $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$, $(z\neq 0)$, $f(0)=0$, show that $\dfrac{f(z)-f(0)}{z-0}$ as $z\to 0$ along any radius vector $y=mx$ but not as $z\to 0$ in any manner and hence $f(z)$ is not differentiable at $z=0$.

Solution:-

Given $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$, $z\neq 0$ $f(0)=0$.

As $z\to 0$ along $y=mx$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y(y-ix)}{(x^6+y^2)(x+iy)} \qquad \frac{f(z)}{z}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3(mx)(mx-ix)}{(x^6+m^2x^2)(x+imx)}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^4m(m-i)}{x^2(x^4+m^2)\cdot x(1+im)} \qquad \frac{x^5m(m-i)}{x^2(x^4+m^2)\,x(1+im)}$$

$$= \lim_{x\to 0}\frac{x^4m(m-i)}{x^3(x^4+m^2)(1+im)}$$

$$= \lim_{x\to 0}\frac{x^2m\cdot(m-i)}{(x^4+m^2)(1+im)} \qquad \frac{x^2m(m-i)}{(x^4+m^2)(1+im)}$$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = 0.$$

$\therefore f(z)$ is differentiable at $z=0$ at $y=mx$.

consider,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y}}\frac{x^3y(y-ix)}{(x^6+y^2)(x+iy)}$$

As $z\to 0$ along $z=mx^3$.

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$$\lim_{z\to 0}\left[\frac{f(z)-f(0)}{z-0}\right] = \lim_{x\to 0}\frac{x^3 m x^3 (mx^3 - ix)}{(x^6+m^2x^6)(x+imx^3)}$$

$$= \lim_{x\to 0}\frac{x^6 x m (mx^2-i)}{x^6 x(1+m^2)(1+imx^3)}$$

$$= \lim_{x\to 0}\frac{m(mx^2-i)}{(1+m^2)(1+imx^2)}$$

$$\lim_{z\to 0}\left[\frac{f(z)-f(0)}{z-0}\right] = \frac{-mi}{1+m^2}$$

Thus for different values of m, which is not unique.

$\therefore$ It does not exist.

$\therefore f(z)$ is not differentiable at $z=0$.

---

10. If $f(z) = \dfrac{xy^2(x+iy)}{(x^2+y^2)}$, $(z\neq 0)$, $f(0)=0$ then $f(z)$ is differentiable at the origin.

Solution:-

Given $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{(x^2+y^2)} & , z\neq 0 \\ 0 & , z=0. \end{cases}$

To prove:-

$f(z)$ is differentiable at the origin.

$\lim_{z\to 0}\dfrac{f(z)-f(0)}{z-0}$ exist.

consider,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{xy^2(x+iy)}{(x^2+y^2)(x+iy)}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{x^2+y^2}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{y^2\left(1+\frac{x^2}{y^2}\right)}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{x\left(\frac{1}{x}+\frac{x}{y^2}\right)}$$

$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0} = 0$ as $z \to 0$.

$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ exists.

$\therefore f(z)$ is differentiable at the origin.

---

(11) Show that, at the origin, the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$, $f(0)=0$ is continuous and the C-R equations are satisfied and yet $f'(0)$ does not exist.

Solution:-

Given function is $f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$.

$$= \frac{x^3+ix^3-y^3+iy^3}{x^2+y^2}.$$

$$f(z) = \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}$$

$$f(z) = \left(\frac{x^3-y^3}{x^2+y^2}\right) + i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

Now,

$$u(x,y) = \frac{x^3-y^3}{x^2+y^2} \begin{cases} u(x,y) \neq 0 \\ u(0,0) = 0 \end{cases}$$

$$v(x,y) = \frac{x^3+y^3}{x^2+y^2} \begin{cases} v(x,y) \neq 0 \\ v(0,0) = 0 \end{cases}$$

Consider,

$$\lim_{\substack{x\to 0\\ y\to 0}} \left|\frac{x^3}{x^2+y^2}\right| = \lim_{\substack{x\to 0\\ y\to 0}} \left|\frac{x^2}{x^2+y^2}\right| |x|$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{x^2+y^2} |x|.$$

$$< \lim_{\substack{x\to 0\\ y\to 0}} |x| \to 0.$$

$|x| \to 0$ as $(x,y) \to (0,0)$.

So,

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3}{x^2+y^2}$ is exist and it is zero.

Hence $\frac{x^3}{x^2+y^2}$ is continuous at $(0,0)$.

Similarly, $\frac{y^3}{x^2+y^2}$ is continuoust at $(0,0)$.

$\therefore$ Hence $u(x,y)$ and $v(x,y)$ are continuous at $(0,0)$ and $f(z)$ is continuous at $z=0$.

Further at $(0,0)$,

$$u_x = \lim_{z\to 0} \frac{u(z,0)-u(0,0)}{z-0} = \frac{0}{0} \text{ [I.D form]}$$

Substitute $z = 0+h$, $h\to 0$ as $z\to 0$.

$$u_x = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{(0+h)-0}.$$

$$= \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3}{h^2\cdot h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1.$$

$$\therefore u_x = 1 \to (1)$$

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$$u_y = \lim_{z \to 0} \frac{u(0,z) - u(0,0)}{z-0} = \frac{0}{0} \text{ [I.D. form]}$$

Substitute $z = 0 + k$, $k \to 0$ as $z \to 0$

$$u_y = \lim_{k \to 0} \frac{u(0, 0+k) - u(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{-k^3}{k^2 \cdot k}$$

$$= \lim_{k \to 0} \frac{-k^3}{k^3} = -1$$

$$\therefore u_y = -1 \longrightarrow (2)$$

$$v_x = \lim_{z \to 0} \frac{v(z,0) - v(0,0)}{z-0} = \frac{0}{0} \text{ [I.D form]}$$

Substitute $z = 0 + h$, $h \to 0$.

$$v_x = \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^3}{h^2 \cdot h} = \lim_{h \to 0} \frac{h^3}{h^3} = 1$$

$$v_x = 1 \longrightarrow (3)$$

$$v_y = \lim_{z \to 0} \frac{v(0,z) - v(0,0)}{z-0} = \frac{0}{0} \text{ [I.D form]}$$

Substitute $z = 0 + k$, $k \to 0$.

$$v_y = \lim_{k \to 0} \frac{v(0, 0+k) - v(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{k^3}{k^2 \cdot k} = \lim_{k \to 0} \frac{k^3}{k^3} = 1.$$

$$\therefore v_y = 1 \longrightarrow (4)$$

From (1), (4)

$$u_x = v_y.$$

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From (2), (3)

$u_y = -v_x$

C-R eqn are satisfied at the origin.

consider,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$$

If $z\to 0$ along $y=mx$

$$= \lim_{x\to 0} \frac{x^3(1+i)-m^3x^3(1-i)}{(x^2+m^2x^2)(x+imx)}$$

$$= \lim_{x\to 0} \frac{x^3[(1+i)-m^3(1-i)]}{x^3[(1+m^2)(1+im)]}$$

$$= \frac{(1+i)-m^3(1-i)}{(1+m^2)(1+im)}$$

which is not unique.

$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ is does not exist

Hence $f(z)$ is not differentiable at $z=0$.

12. Show that for the following function. the C-R equations are satisfied at $z=0$ but the functions are not differentiable there!

i) $f(z) = \sqrt{|xy|}$

ii) $f(z) = \begin{cases} e^{-1/z^4}, & (z\neq 0) \\ 0, & z=0. \end{cases}$

iii) $f(z) = \begin{cases} \frac{z^4}{|z|^4}, & z\neq 0 \\ 0, & z=0 \end{cases}$

Solution:-

i) Given function is $f(z) = \sqrt{|xy|}$

$u(x,y) = \sqrt{|xy|}$, $v(x,y) = 0$

$u(x,y) = \sqrt{xy}$, $v(x,y) = 0$

$u_x = \frac{1}{2}(xy)^{1/2-1} \cdot (y)$, $v_x = 0$.

$u_x = y/2\,(xy)^{-1/2}$, $v_y = 0$.

$y/2\,(xy)^{-1/2} = u_x$, $u_y = \frac{1}{2}(xy)^{1/2-1}(x)$

$$u_x = \frac{y}{2\sqrt{xy}}, \quad u_y = \frac{x}{2\sqrt{xy}}$$

At $(0,0)$; $u_x = 0$, $u_y = 0$, $v_x = 0$, $v_y = 0$.

$\Rightarrow u_x = v_y$, $u_y = -v_x$

The C-R eqns are satisfied at $z=0$.

But $$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|xy|}}{x+iy}$$

As $z \to 0$ along $y = mx$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{x\to 0} \frac{\sqrt{|x,m|}}{x|1+mi|}$$

$$= \lim_{x\to 0} \frac{\sqrt{|m|}}{(1+im)}$$

$$= \frac{\sqrt{|m|}}{(1+im)}$$

Thus for different values of m, the limit is not unique.

The function is not differentiable at $z=0$.

Given function is $f(z) = e^{-1/z^4}$, $(z \neq 0)$

But $f(0) = 0$

$u(0,0) = 0$, $v(0,0) = 0$

Further from

$$f(x+0i) = u(x,0) + i v(x,0) = e^{-\frac{1}{x^4}}$$

$$f(0+iy) = u(0,y) + iv(0,y) = e^{-1/(iy)^4} \quad [i^4 = 1]$$

we get

$$u(x,0) = e^{-1/x^4}, \quad v(x,0) = 0$$

$$u(0,y) = e^{-1/y^4}, \quad v(0,y) = 0$$

At the origin,

$$u_x = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{1}{x e^{1/x^4}} = 0.$$

$$u_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{e^{-1/y^4}}{y}$$

$$= \lim_{y \to 0} \frac{1}{y e^{1/y^4}} = 0.$$

Similarly,

At the origin.

$v_x = 0$, $v_y = 0$

$u_x = v_y$, $u_y = -v_x$

The C-R eqns are satisfied at the origin.

But $$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{e^{-1/z^4}}{z}$$

$$= \lim_{z \to 0} \frac{1}{z e^{\frac{1}{z^4}}}$$

In particular $z \to 0$ travelling along the ray.

$$\arg z = \frac{\pi}{4} \text{ (ie) along } y = x.$$

(ie) $z = r \exp\left(\frac{\pi}{4}\right)$ where $r \to 0$.

$$\frac{1}{z^4} = z^{-4} = \left[r \cdot \exp\left(\frac{i\pi}{4}\right)\right]^{-4}$$

$$= r^{-4} \exp\left[-4 \frac{i\pi}{4}\right]$$

$$= r^{-4} \exp[-\pi i]$$

$$= -r^4$$

$$\exp\left(-\frac{1}{z^4}\right) = r^4$$

Hence, $e^{1/z^4} \to 1$ as $z \to 0$ and

$\frac{1}{z e^{1/z^4}} \to \infty$ as $z \to 0$

$\therefore f(z)$ is not differentiable at $z = 0$.

---

13. Show that at the origin, the following functions are differentiable but not analytic there:

i) $f(z) = x^2 + y^2 = |z|^2$

ii) $f(z) = x(x + iy)$.

Solution:-

i) Given function is $f(z) = x^2 + y^2$

$$u(x,y) = x^2 + y^2, \quad v(x,y) = 0.$$

$$u_x = 2x, \quad v_x = 0.$$

$$u_y = 2y, \quad v_y = 0.$$

At $(0,0)$ the C-R eqns.

$$u_x = v_y, \quad u_y = -v_x \text{ are satisfied.}$$

At the origin the function $f(z)$ is differentiable at $z = 0$.

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But the C-R eqns are not true at the point other than $z=0$.

So $f(z)$ is not differentiable at points other than at $z=0$.

Hence there is no neighbourhood of $z=0$ in which $f(z)$ is differentiable.

$\therefore f(z)$ is not analytic at $z=0$.

ii) Given function is $f(z) = x(x+iy)$

$f(z) = x^2 + ixy$, $v(x,y) = xy$

$u(x,y) = x^2$, $v_x = y$

$u_x = 2x$ $\quad v_x = y$

$u_y = 0$ $\quad v_y = x$

At $(0,0)$ the C-R eqns are $u_x = v_y$, $u_y = -v_x$ are satisfied.

At the origin $f(z)$ is differentiable at $z=0$ but the C-R eqns are not true at the point other than $z=0$.

So, $f(z)$ is not differentiable at points other than $z=0$.

Hence there is an neighbourhood of $z=0$ in which $f(z)$ is differentiable.

$\therefore f(z)$ is not analytic at $z=0$.

4.7: Sufficient Condition for Differentiability

14. Show that the function $f(z) = e^x(\cos y + i \sin y)$ is analytic in the finite plane. Find its derivative.

Solution:-

Given function is $f(z) = e^x(\cos y + i\sin y)$

$\therefore f(z) = e^x \cos y + i e^x \sin y$

$\therefore u(x,y) = e^x \cos y$; $v(x,y) = e^x \sin y$

$u_x = e^x \cos y$; $v_x = e^x \sin y$

$u_y = -e^x \sin y$; $v_y = e^x \cos y$

$u_x = v_y$ $\quad u_y = -v_x$.

C-R eqns are satisfied and u_x, u_y, v_x, v_y are continuous.

By the sufficient condition for differentiability $f(z)$ is differentiable.

$\therefore f'(z)$ exists.

$\therefore f(z)$ is analytic in the finite plane.

$f'(z) = u_x + iv_x = (e^x \cos y) + i(e^x \sin y)$

$f'(z) = e^x(\cos y + i\sin y)$

$\therefore f'(z) = f(z)$.

15. Show that $f(z) = e^{-x}(\cos y - i\sin y)$ is differentiable everywhere in the finite plane and $f'(z) = -f(z)$.

Solution:-

$f(z) = e^{-x}(\cos y - i\sin y)$

$f(z) = e^{-x}\cos y - ie^{-x}\sin y$

$u(x,y) = e^{-x}\cos y$, $v(x,y) = -e^{-x}\sin y$

$u_x = -e^{-x}\cos y$, $v_x = e^{-x}\sin y$

$u_y = -e^{x}\sin y$, $v_y = -e^{-x}\cos y$

$\therefore u_x = v_y$, $u_y = -v_x$.

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$\therefore$ The C-R eqns are satisfied and

u_x, u_y, v_x, v_y are continuous.

By the sufficient condition everywhere

$f'(z) = u_x + i v_x$

$= -e^{-x}(\cos y + i e^{-y} \sin y)$ $(-e^{-x}\cos y) + i(e^{-x} \ldots$ $-ie^{-x}\cos y - ie^{-x} \ldots$ $-f(z)$

$= -(e^{-x}\cos y - ie^{-x}\sin y)$

$= -f(z)$

$f'(z) = -f(z)$.

16. Show that $f(z) = e^{-y}(\cos x + i\sin x)$ is differentiable everywhere in the finite plane and $f'(z) = if(z)$

Solution:-

Given $f(z) = e^{-y}(\cos x + i \sin x)$

$f(z) = e^{-y}\cos x + i e^{-y}\sin x$

$u_x = -e^{-y}\sin x, \quad v_x = e^{-y}\sin x$

$u_y = -e^{-y}\cos x, \quad v_y = -e^{-y}\sin x$

$u_x = v_y, \quad u_y = -v_x$

The C-R eqns are satisfied and u_x, u_y, v_x, v_y are exists and continuous. $\therefore$ By the sufficient condition for differentiability

$f(z)$ is differentiable

$f'(z) = u_x + i v_x$

$= (-e^{-y}\sin x) + i(e^{-y}\cos x)$

$= i[e^{-y}\cos x + ie^{-y}\sin x]$

$= ie^{-y}[\cos x + i\sin x]$

$f'(z) = i\, f(z)$.

17. Show that $e^x(\cos y - i\sin y)$ and $e^y(\cos x + i\sin x)$ are nowhere differentiable.

Solution:-

i) Let $f(z) = e^x(\cos y - i\sin y)$

$\therefore f(z) = e^x \cos y - i e^x \sin y$

$u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$

$u_x = e^x \cos y$, $v_x = -e^x \sin y$

$u_y = -e^x \sin y$, $v_y = -e^x \cos y$

$u_x \neq v_y$ and $u_y \neq -v_x$

$\therefore$ The C-R eqns are not satisfied.

$\therefore f(z)$ is no where differentiable.

ii) Given $f(z) = e^y(\cos x + i\sin x)$

$\therefore f(z) = e^y(\cos x) + i\, e^y(\sin x)$

$u(x,y) = e^y \cos x$, $v(x,y) = e^y \sin x$

$u_x = -e^y \sin x$, $v_x = e^y \cos x$

$u_y = e^y \cos x$, $v_y = e^y \sin x$

$\therefore u_x \neq v_y$, $u_y \neq -v_x$.

$\therefore$ The C-R eqns are not satisfied.

$\therefore f(z)$ is no where differentiable.

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18. ϕ and ψ are functions of x and y Satisfying laplace's equation, namely $\phi_{xx}+\phi_{yy}=0$, $\psi_{xx}+\psi_{yy}=0$. If $u=\phi_y-\psi_x$, $v=\phi_x+\psi_y$ show that $u+iv$ is analytic.

Solution :-

Given the laplace eqns are

$\phi_{xx}+\phi_{yy}=0$

$\psi_{xx}+\psi_{yy}=0$

let $u=\phi_y-\psi_x$

$u_x=\phi_{yx}-\psi_{xx}$ [$\because$ laplace eqn $\psi_{xx}+\psi_{yy}=0$]

But

$v=\phi_x+\psi_y$

$v_y=\phi_{yx}+\psi_{yy}$

$=\phi_{yx}-\psi_{xx}$

$u_y=\phi_{yy}-\phi_{xx}$

But

$u_x=\phi_{xx}+\psi_{xy}$

$=\phi_{yy}+\psi_{yx}$

$v_x=-(\phi_{yy}-\psi_{yx})$

$u_x=v_y$ and $u_y=-v_x$

$\therefore$ The C-R eqns are Satisfied u_x, u_y, v_x, v_y are continuous.

Hence $u+iv$ is analytic.