UNIT-II

C-R equations in polar co-ordinates

Section: 4.8

C-R equations in polar co-ordinates

*** 10m **Theorem: 4.10:-**

The C-R equations in polar coordinates are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$; $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Proof:-

Let (u,v) be functions of (x,y) and let (x,y) be functions of polar coordinates (r,θ)

Then,

$$x = r\cos\theta \ ; \ y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \ ; \ \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \ , \ \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial r} = u_x(\cos\theta) + u_y(\sin\theta) \longrightarrow (1)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$u_x = v_y$

$u_y = -v_x$

$$= v_x(-r\sin\theta) + v_y(r\cos\theta)$$

$$= -u_y(-r\sin\theta) + u_x(r\cos\theta)$$

$$\frac{\partial v}{\partial \theta} = r\left[u_y\sin\theta + u_x\cos\theta\right] \longrightarrow (2)$$

From (1), (2)

$$\frac{\partial u}{\partial r} = u_y\sin\theta + u_x\cos\theta$$

$$\therefore \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Similarly,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial r} = V_x(\cos\theta) + V_y(\sin\theta) \longrightarrow (3)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= U_x[-r\sin\theta] + U_y[r\cos\theta]$$

$$= V_y[-r\sin\theta] + (-V_x)[r\cos\theta]$$

$$\frac{\partial u}{\partial \theta} = -r[V_x\cos\theta + V_y\sin\theta] \longrightarrow (4)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Hence the theorem.

Corrollary :-

If the C-R eqns in polar coordinates

** are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \cdot \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ then

$$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

proof :-

Let (u,v) be a function of (x,y) and (x,y) be function of polar Co-ordinates (r,θ) then

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial u}{\partial r} = U_x\cos\theta + U_y\sin\theta$$

$$\frac{\partial v}{\partial r} = V_x\cos\theta + V_y\sin\theta$$

$$r\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)=r\left[(u_x\cos\theta+u_y\sin\theta)+i(v_x\cos\theta+v_y\sin\theta)\right]$$

$$=r\cos\theta(u_x+iv_x)+r\sin\theta(u_y+iv_y)$$

$$=x(u_x+iv_x)+y(u_y+iv_y)$$

$$=x f'(z)+iy f'(z)$$

$$=(x+iy)f'(z)$$

$$=z f'(z)$$

$$\therefore f'(z)=\frac{r}{z}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right).$$

Definition:-

power series :-

A series having the form

$$a_0+a_1(z-a)+a_2(z-a)^2+\cdots=\sum_{n=0}^{\infty}a_n(z-a)^n$$

is called a power series in $z-a$.

Definition :-

Geometric Series :-

The power series with unit co-efficient $1+z+z^2+\cdots$, $1+(z-a)+(z-a)^2+\cdots$ are called geometric series.

Sec : 6.12 : Harmonic functions :-

Definition :-

Harmonic function :-

A real valued function $u(x,y)$ defined region is said to be harmonic. if the following hold.

i) The second order partial derivatives $u_{xx}, u_{yy}, u_{xy}, u_{yx}$ exist and are continuous and

ii) The laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ is satisfied.

$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

Definition :-

Conjugate Harmonic functions :-

Two harmonic functions are said to be conjugate harmonic functions if they satisfy the C-R equations.

NOTE :-

If u and v are conjugate harmonic functions. Then the C-R equations are true. i.e $u_x = v_y$, $u_y = -v_x$.

But u and v are not conjugate harmonic functions because they do not satisfy the C-R equations.

i.e $v_x \neq u_y$, $v_y \neq -u_x$

$-v$ and u are conjugate harmonic functions

Theorem : 6.10 :-

A function $f(z) = u(x,y) + i\,v(x,y)$ defined in a region D is analytic in it if and only if $u(x,y)$ and $v(x,y)$ are conjugate harmonic functions.

proof :-

Necessary part :-

Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in D then $f'(z)$, $f''(z)$, $f'''(z)$, ... are analytic in D

To prove :-

$u(x,y)$ and $v(x,y)$ are conjugate harmonic function :-

We know the theorem,

For a function to be analytic, the necessary and Sufficient condition is that u_x, u_y, v_x and v_y exist such that they are continuous and satisfy C-R equations.

i.e, $u_x = v_y$, $u_y = -v_x \rightarrow (1)$

As $f(z)$ is analytic.

$u_{xx}, u_{yy}, v_{xx}, v_{yy}, u_{xy}, u_{yx}, v_{xy}, v_{yx}$ exist and are continuous.

The order of differentiation is immaterial.

$$\therefore u_{xy} = u_{yx} \text{ and } v_{xy} = v_{yx}$$

From (1) differentiating w.r.t 'x',

$$u_{xx} = v_{yx} \text{ and } u_{yx} = -v_{xx}$$

Differentiating w.r.t 'y',

$$u_{yy} = -v_{xy} = -v_{yx} \cdot \quad u_{xy} = v_{yy}$$

$$\Rightarrow u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence 'u' is harmonic.

Again consider (1).

Differentiating $u_x = v_y$ w.r.t 'y' we have

$$u_{xy} = v_{yy} \text{ and}$$

Differentiating $u_y = -v_x$ with respect to 'x'.

$$u_{yx} = -v_{xx} \rightarrow -u_{yx} = v_{xx}$$

Consider,

$$v_{xx} + v_{yy} = u_{xy} - u_{xy} = 0.$$

$$\Rightarrow v_{xx} + v_{yy} = 0$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\therefore v$ is harmonic

Thus, 'u' and 'v' are harmonic and satisfy C-R eqns.

Hence $u(x,y)$ and $v(x,y)$ are conjugate harmonic functions.

conversely,

Sufficient part:-

Assume that $u(x,y)$ and $v(x,y)$ are conjugate harmonic functions.

To prove:-

$f(z)$ is analytic

$u(x,y)$ and $v(x,y)$ are conjugate harmonic functions.

$\Rightarrow$ The second order partial derivatives.

$u_{xx}, u_{yy}, v_{xx}, v_{yy}, u_{xy}, u_{yx}, v_{xy}, v_{yx}$ are exists and satisfy C-R equations.

$\Rightarrow$ u_x, u_y, v_x, v_y exist continuous and satisfy C-R equations.

$\Rightarrow$ $f(z)$ is analytic.

Definition:-

Harmonic conjugate:-

If $u+iv$ is analytic, then 'u' and 'v' are harmonic functions and of them, 'v' is said to be harmonic conjugate of 'u'. of two conjugate harmonic functions, one is called the harmonic conjugate of the other.

Remark :-

1. Suppose $u+iv$ is analytic. Then it may be noted that $v+iu$ is not analytic. Since v, u do not satisfy the C-R equations. So 'u' is not the conjugate of v.

For example,

$$f(z) = z^2 = (x+iy)^2 = \overset{u}{(x^2-y^2)} + i\,\overset{v}{2xy}$$

considering z^2, we can see that

$$u = x^2-y^2, \quad v = 2xy \qquad u_x = 2$$

satisfy the C-R equations but

$$u = 2xy, \quad v = x^2-y^2$$

do not satisfy the C-R equations.

2. Now we shall show that, if 'u' and 'v' are any two harmonic functions, then $u+iv$ may or may not be analytic.

If $u = x^2-y^2$, $v = 4xy$ then

$$u_{xx}+u_{yy} = 0, \quad v_{xx}+v_{yy} = 0$$

So 'u' and 'v' are harmonic functions. But $u_x = 2x$, $v_y = 4x$

Hence the first equation of the C-R equation $u_x = v_y$ is not satisfied. Therefore, $u+iv$ is not analytic.

$u = x^2-y^2$
$u_x = 2x$.
$u_{xx} = 2$
$u_y = -2y$
$u_{yx} = 0$
$u_{yy} = 0$
$v = 4xy$
$v_x = 4y \to v_{xx} = 0$
$v_y = 4x \to v_{yx} = 4$

Here note that x^2-y^2 is the real part of z^2 and $4xy$ is the imaginary part of a different function $2z^2$.

3. If 'u' or 'v' is not harmonic, then $u+iv$ is not analytic.

Sufficient part :-

Now the hypothesis is that $u(x,y)$ and $v(x,y)$ are conjugate harmonic functions. Therefore

The second partials of u, v are continuous and u and v satisfy the C-R equations.

Since the second partials exist, the first partials u_x, u_y, v_x, v_y exist and are continuous. Now, 'u' and 'v' satisfy the C-R equations.

Therefore, by theorem 4.9 $f(z)$ is analytic in D.

Harmonic Functions In polar co-ordinates

*** 2m

The Laplacian operator in cartesian co-ordinate is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In polar coordinates it is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Thus, if $\phi(r,\theta)$ is harmonic function, then

$$\left[\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0\right]$$

Thearem : 6.12

If $f(z) = u(r,\theta) + i v(r,\theta)$ is an analytic function then $u(r,\theta)$ and $v(r,\theta)$ are harmonic functions.

proof:-

The C-R equations in polar co-ordinates are

$$u_r = \frac{1}{r} v_\theta , \quad u_\theta = -r v_r$$

$(-r)(v_r)$ $(uv)' = vu' + v'u$

Now differentiating u_r and u_θ partially w.r.t 'θ' and 'r'.

$$u_{\theta r} = \frac{1}{r} v_{\theta\theta} , \quad u_{r\theta} = -(v_r + r v_{rr})$$

Since, $u_{\theta r} = u_{r\theta}$, we get.

$$\frac{1}{r} v_{\theta\theta} = -(v_r + r v_{rr})$$

Dividing this by r,

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 .$$

Thus, 'v' is a harmonic function. Again from the C-R eqns.

$$v_r = -\frac{1}{r} u_\theta , \quad v_\theta = r u_r .$$

Differentiating v_r and v_θ partially w.r.t 'θ' and 'r' and equating $v_{\theta r}$ and $v_{r\theta}$, we get

$$-\frac{1}{r} u_{\theta\theta} = u_r + r u_{rr}$$

$v_{\theta r} = r u_{rr} + u_r$

$v_{r\theta} = -\frac{1}{r} u_{\theta\theta}$

Dividing by 'r', we get

$$\therefore u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

So 'u' is also a harmonic function.//

Analytic Function when its real or Imaginary part is given:-

When $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a region and when one of $u(x,y)$ and $v(x,y)$ is given, we can find the other and then $f(z)$. In other words, when a harmonic function $u(x,y)$ is given we can find another harmonic function $v(x,y)$ so that $u+iv$ is analytic.

Method : 1

Exact Differential Method :-

Suppose the harmonic function $u(x,y)$ is given. Now $dv = v_x\,dx + v_y\,dy$ is an exact differential where v_x and v_y are known from 'u' through the C-R equations.

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Therefore,

$v = \int v_x\,dx$ treating 'y' as a constant + $\int$(terms of v_y not containing x) dy.

Illustration : 1

consider the harmonic function $u = x^2 - y^2$

Then $v_x = -u_y = 2y$, $v_y = u_x = 2x$.

$u_x = v_y$

$u_y = -v_x$

$\int M\,dx + \int N\,dy = c$

$\therefore v = \int 2y\,dx + \int (0)\,dy = 2xy + 0 = 2xy$ $\quad u_x = 2x$

$f(z) = x^2 - y^2 + i(2xy) + c$

$f(z) = (x+iy)^2 + c$

$f(z) = z^2 + c$

Method : 2

Substituting Method :-

This method provides us with a formal formula to write down at once $f(z)$ itself when 'u' or 'v' or 'u+v' or '2u+v' or 'u-2v' is given. We know that, if $f(z)$ is analytic, then $\frac{\partial f}{\partial \bar{z}} = 0$, that is $f(z)$ is independent of $\bar{z}$.

$z = x + iy$
$\bar{z} = x - iy$
$z + \bar{z} = 2x$

So, if we write,

$$** \; f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \quad (6.12.1)$$

then $f(z)$ is independent of $\bar{z}$ which means that $\bar{z}$ can be replaced by any number without altering the value of $f(z)$. Also conjugation of (6.12.1) gives

$$\overline{f(\bar{z})} = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \quad (6.12.2)$$

Since $(z+\bar{z})/2$, $(z-\bar{z})/2i$ being real, remain unchanged. Now addition of (6.12.1) and (6.12.2) gives.

$$** \; f(z) = 2u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - \overline{f(\bar{z})}. \quad (6.12.3)$$

If $\bar{z}$ is replaced by any number 'a', then (6.12.3) becomes.

$$f(z) = 2u\left(\frac{z+a}{2}, \frac{z-a}{2i}\right) - \overline{f(\bar{a})}.$$

provided $\overline{f(\bar{a})}$ exists or becomes.

$$** \; f(z) = 2u\left(\frac{z+a}{2}, \frac{z-a}{2i}\right) \longrightarrow (4) \quad (6.12.4).$$

apart from an arbitrary additive constant.

This formula can be simplified as

$$** \; \boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) = 2u\left(\frac{1}{2}z, -\frac{1}{2}iz\right)} \quad (6.12.5) \longrightarrow (5)$$

By changing $a = 0$ provided $\overline{f(0)}$ exists, that is $u(0,0) - iv(0,0)$ or in particular, $u(0,0)$ exists apart from an arbitrary additive constant.

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Remark :-

If $u(0,0)$ is either infinity or undefined, then (6.12.5) fails to give $f(z)$. In such a case (6.12.4) may be used with a particular choice for 'a' other than '0'.

Illustration : 2 :-

If $u(x,y) = x^2 - y^2$, from (6.12.5), setting $\frac{1}{2}z$ for x and $-\frac{1}{2}iz$ for y.

$$u\left(\frac{1}{2}z, \frac{-1}{2}iz\right) = \left(\frac{1}{2}z\right)^2 - \left(\frac{-1}{2}iz\right)^2 = \frac{1}{2}z^2$$

$$\frac{1}{4}z^2 + \frac{1}{4}z^2 = \frac{2}{4}z^2 = \frac{1}{2}z^2$$

From (5) $\Rightarrow f(z) = 2u\left(\frac{1}{2}z, -\frac{1}{2}iz\right) = z^2$

$$= \cancel{2}\left(\frac{1}{\cancel{2}}z^2\right) = z^2$$

[$\because$ By using eqn (5) $f(z) = 2u(x,y)$]

Illustration : 3

Consider the harmonic function $u = \frac{1}{2}\log(x^2+y^2)$. In this case $u(0,0) = -\infty$.

So we cannot use the formula (6.12.5)

Alternatively using (6.12.4) with a choice $a = 1$

$$f(z) = \cancel{2}\cdot\frac{1}{\cancel{2}}\log\left[\left\{\frac{1}{2}(z+1)\right\}^2 + \left\{-\frac{1}{2}i(z-1)\right\}^2\right]$$

$$f(z) = \log z.$$

$$f(z) = 2u\left(\frac{z+a}{2}, \frac{z-a}{2i}\right)$$

Corollary to Method 2 :-

Suppose $f(z) = u + iv$ is an analytic function. Now

$$-if(z) = v + i(-u)$$

$$(1-i)f(z) = (u+v) + i(v-u),$$

$$(1+i)f(z) = (u-v) + i(u+v).$$

So if v, $u+v$, $u-v$ are given, we can find $(-u)$, $v-u$, $u+v$, the imaginary parts on the left, using (6.12.4) if $v(a)$, $u(a)+v(a)$,

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$u(a) - v(a)$ exist. Thus the corresponding analytic functions $f(z)$'s are

$$v \quad f(z) = 2iv\left(\frac{z+a}{2}, \frac{z-a}{2i}\right) \quad (6.12.6)$$

$$u+v \quad f(z) = \frac{2}{1-i}\left[(u+v)\left(\frac{z+a}{2}, \frac{z-a}{2i}\right)\right] \quad (6.12.7)$$

$$u-v \quad f(z) = \frac{2}{1+i}\left[(u-v)\left(\frac{z+a}{2}, \frac{z-a}{2i}\right)\right] \quad (6.12.8)$$

Illustration : 4 :-

If $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ in (6.12.7) we cannot choose $a=0$ because $u(0,0)+v(0,0)$ is not defined. So, choosing $a = \frac{1}{2}\pi$.

$x = \frac{z+a}{2}$

$y = \frac{-i(z-a)}{2}$

$$f(z) = \frac{2}{1-i} \; \frac{\sin 2\left\{\frac{1}{2}\left(z+\frac{1}{2}\pi\right)\right\}}{\cosh 2\left\{-\frac{1}{2}i\left(z-\frac{1}{2}\pi\right)\right\} - \cos 2\left\{\frac{1}{2}\left(z+\frac{1}{2}\pi\right)\right\}}$$

$$= \frac{2}{1-i} \cdot \frac{\sin\left(z+\frac{1}{2}\pi\right)}{\cos\left(z-\frac{1}{2}\pi\right) - \cos\left(z+\frac{1}{2}\pi\right)}$$

180° $(\cos z)$

$\sin z - (-\sin z)$ $(2\sin z)$

$$= \frac{\cancel{2}}{1-i} \cdot \frac{\cos z}{\cancel{2}\sin z}$$

$$= \frac{\cos z}{1-i}$$

$$\therefore f(z) = \frac{\cos z}{1-i}.$$

Method 3 :-

Milne-Thomson Method :-

Now, $f'(z)$ is analytic and it is

$$f'(z) = u_x(x,y) + i v_x(x,y) = u_x(x,y) - i u_y(x,y)$$

$$= u_x\left[\tfrac{1}{2}(z+\bar{z}), -\tfrac{1}{2}i(z-\bar{z})\right]$$

$$- i u_y\left[\tfrac{1}{2}(z+\bar{z}), -\tfrac{1}{2}i(z-\bar{z})\right]$$

Replacing $\bar{z}$ by z,

$$f'(z) = u_x(z,0) - i\, u_y(z,0)$$

$$\therefore f(z) = \int [u_x(z,0) - i\, u_y(z,0)]\, dz.$$

Similarly, if V is known

$$f(z) = \int [V_y(z,0) + i V_x(z,0)]\, dz.$$

Illustration : 5

Let $u(x,y) = x^2 - y^2$. Then

$u_x(x,y) = 2x$, $u_y(x,y) = -2y$, $u_x(z,0) = 2z$,

$u_y(z,0) = 0$.

$$\therefore f(z) = \int [2z - i(0)]\, dz.$$

$$\therefore f(z) = z^2.$$

× Velocity potential and stream Function :-

Suppose the complex function

$f(z) = u(x,y) + i v(x,y)$ is analytic.

Then the real part $u(x,y)$ is the velocity potential and the imaginary part $V(x,y)$ in the stream function in some two dimentional steady state irrotational flows in Fluid Dynamics. Thermodynamics and Electronics. In this f(z) is called the complex potential of the flow.

The family of equipotential lines

$u(x,y) = C_1$

are orthogonal to the family of Stream lines

$V(x,y) = C_2$

where c_1 and C_2 are parameters.

Notation :-

In general, the velocity potential and the stream function are denoted by $\phi(x,y)$, $\psi(x,y)$. So that in two dimensional steady state flows the complex potential.

$f(z) = \phi(x,y) + i\psi(x,y)$ is analytic.

Illustration : 5

In a two dimentional flow, the stream function is $\psi = \tan^{-1} y/x$. Find the velocity potential ϕ.

We shall denote ϕ, ψ by u, v.

Then $f(z) = u + iv$ is an analytic function.

$$V_y = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$V_x = \frac{1}{1+\frac{y^2}{x^2}} \left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

consider the exact equation,

$$du = u_x dx + u_y dy$$

$$= V_y dx + (-V_x) dy \quad \text{by C-R equations.}$$

$u = \int V_y dx$ treating y as a constant $+$

$\int$ (terms of $(-V_x)$ not containing x) dy

$$= \int \frac{x}{x^2+y^2} dx + \int 0 \, dy$$

$$= \frac{1}{2} \log(x^2+y^2)$$

So, the velocity potential ϕ is

$$\phi = \frac{1}{2} \log(x^2+y^2).$$

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Bilinear Transformation :-

Section : T.4 :-

Definition :-

Bilinear (or) Mobius (or) Linear Fractional Transformation.

The transformation

$$w = \frac{az+b}{cz+d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

where a,b,c,d are complex numbers is called a bilinear transformations.

This transformation was first studied by A.F. Mobius.

Bilinear Transformation - Meaning :-

When the fraction is cleared

$$cwz + dw - az - b = 0.$$

$$\Rightarrow w(cz+d) - az - b = 0$$

which is linear in w and also linear in z. Hence the transformation is called a bilinear transformation.

Transformations :-

$w = z^2$, $w = z^{1/2}$, $w = e^z$, $w = \sin z$, $w = \cos z$

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① (5m) (or) (10m) Discuss the transformation $w = z^2$.

Solution :-

Case (i) :-

In terms of polar coordinates :-

Let $z = re^{i\theta}$ and $w = Re^{i\phi}$ then $w = z^2 \Rightarrow Re^{i\phi} = (re^{i\theta})^2$

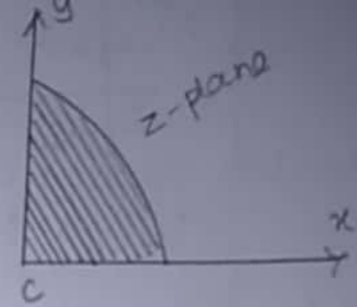
$$= r^2 e^{2i\theta}$$

Equating modulus and argument,

$$R = r^2 \text{ and } \phi = 2\theta \Rightarrow \theta = \frac{\phi}{2}.$$

Case(a):-

Consider the first quadrant of a circular disc.



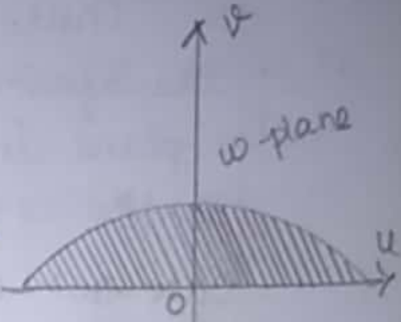
$r \geq 0$, $0 \leq \theta \leq \pi/2$

under the transformation

$w = z^2$, we get

$R \geq 0$, $0 \leq \frac{\phi}{2} \leq \frac{\pi}{2}$

$\Rightarrow R \geq 0$, $0 \leq \phi \leq \pi$



Thus, the transformation $w = z^2$ transforms the first quadrant of a circular disc $r \geq 0$.

$0 \leq \theta \leq \pi/2$ into an upper half of a circular disc.

$R \geq 0$, $0 \leq \theta \leq \pi$.

Case(ii):- In terms of Cartesian coordinates:-

Let $z = x+iy$ and $w = u+iv$ then

$w = z^2 \Rightarrow u+iv = (x+iy)^2 = (x^2-y^2) + i2xy$

$u = x^2-y^2$, $v = 2xy$

Case (a):- Consider the line $x = h$.

$\therefore u = h^2 - y^2$, $v = 2hy$

$\Rightarrow u = h^2 - y^2$, $y = \frac{v}{2h}$

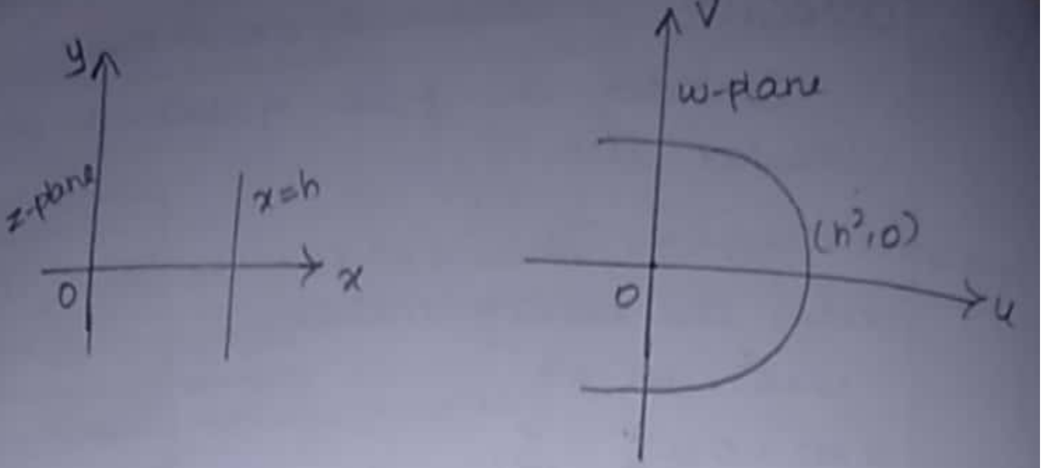
$\Rightarrow u = h^2 - \left(\frac{v^2}{4h^2}\right)$

$\frac{v^2}{4h^2} = h^2 - u$

$\Rightarrow v^2 = 4h^2(h^2-u)$

$[\because (v-k)^2 = 4a(u-h)$ vertex $(h,k)]$ $y^2 = 4ax$

This a parabola having vertex $(h^2, 0)$ and focus $w=0$ $\Rightarrow u = h^2$



Thus, the transformation $w=z^2$, transforms the line $x=h$ in the z-plane into the parabala $V^2=-(4h^2)(u-h^2)$ in the w-plane.

Case-(b):- Consider the line $y=k$.

$\therefore u=x^2-y^2 \,,\; V=2xy$

$\Rightarrow u=x^2-k^2 \,,\; V=2xk$

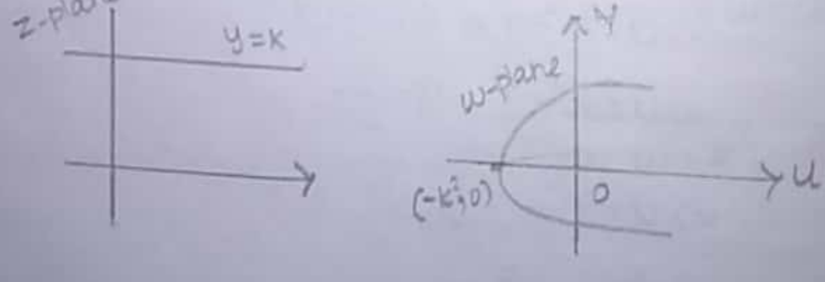
$\Rightarrow u=x^2-k^2 \,,\; x=\frac{V}{2k}$

$\therefore u=\frac{V^2}{4k^2}-k^2$

$\Rightarrow \frac{V^2}{4k^2}=u+k^2$

$V^2=4k^2(u+k^2)$

This is a parabola having vertex $(-k^2,0)$ and focus $w=0 \Rightarrow V^2=-k^2$.



Thus the transformation $w=z^2$ transforms the line $y=k$ in the z-plane into the parabala $V^2=4k^2(u+k^2)$ in the w-plane.

2. Discuss the transformation $w=z^{1/2}$, the concentric circles $|z-1|=e$ in the w-plane go onto leminscates with focal points $w=\pm 1$.

proof:-

The fuction $f(z)=z^{1/2}$ is a double valued function.

Example:-

If $z=4(\cos\alpha+i\sin\alpha)$ then the value of f(z) are

$w=z^{1/2} \Rightarrow w^2=z \Rightarrow w^2=4(\cos\alpha+i\sin\alpha)$

$w=2(\cos\alpha/2+i\sin\alpha/2)$,

$w=2\left[\cos(\pi+\alpha/2)+i\sin(\pi+\alpha/2)\right]$

$w=2c$.

Theorem : 7.11:-

Under the transformation $w=z^{1/2}$, the concentric circles $\boxed{|z-1|=e}$ in the w-plane go onto leminscates with focal points $w=\pm 1$:

proof:-

consider the transformation

$\boxed{w=z^{1/2}}$ $w^2=((z)^{1/2})^2$

$\Rightarrow w^2=z$

$\Rightarrow w^2-1=z-1$

$(w-1)(w+1)=z-1$

$(w-1)(w+1)=\rho$, where $\rho=|z-1|$.

Hence the circles $|z-1|=\rho$ go onto the leminscates $|w-1||w+1|=\rho$ with foci at $w=\pm 1$.

Leminscate :-

It is the locus of a point which moves such that the product of its distances from two fixed points (called foci) is constant.

---

3. Discuss the transformation $w=e^z$.

** (10m) Solution :-

Consider the transformation $w=e^z$.

Let $w=u+iv$ and $z=x+iy$

$w=u+iv=e^{x+iy} \rightarrow (1)$

Now, $w=e^z$ is in polar coordinates

$w=Re^{i\phi} \rightarrow (2)$

comparing (1), (2)

$Re^{i\phi}=e^{x+iy}$

$\Rightarrow Re^{i\phi}=e^x \cdot e^{iy}$

Equating modulus and argument.

$R=e^x$ and $\phi=y \rightarrow (3)$

Case (i) : consider the line $x=a$.

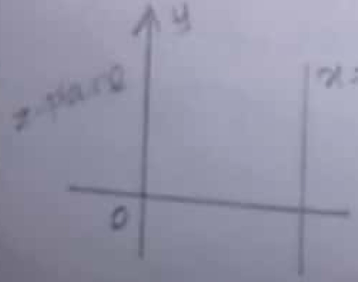
Then (3) $\Rightarrow R=e^a$ and $\phi=y$.

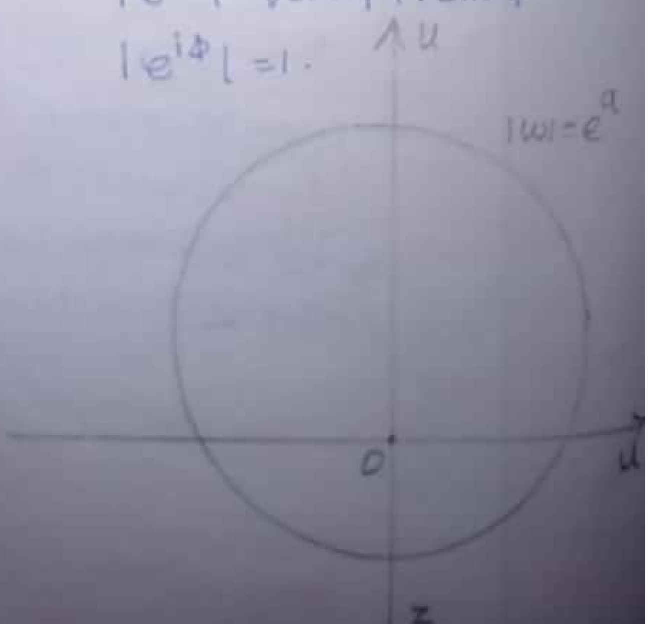
$\therefore w=Re^{i\phi}$

$\Rightarrow |w|=|R|\,|e^{i\phi}|$ $\quad \because e^{i\phi}=\cos\phi+i\sin\phi$

$|w|=|e^a|.$ $\quad |e^{i\phi}|=\sqrt{\cos^2\phi+i\sin^2\phi}$

$\Rightarrow |w|=e^a.$ $\quad |e^{i\phi}|=1.$

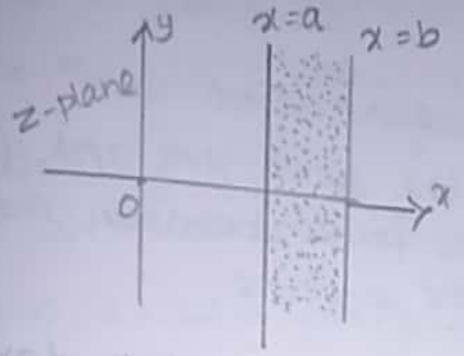


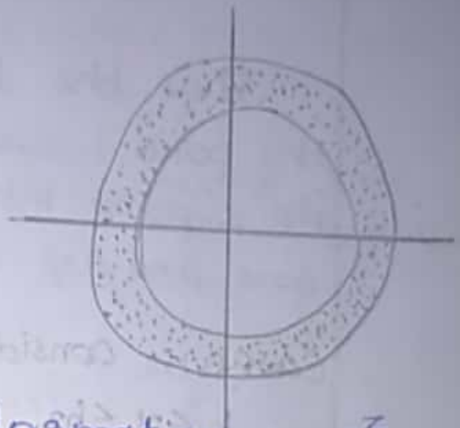


Thus the transformation $w=0$ transforms the line $x=a$ in the z-plane into the circle $|w|=e^a$ in the w-plane.

Case (ii): Consider the line $x=a$ and $x=b$ $(b>a)$.

Then we get the circles, $|w|=e^a$ and $|w|=e^b$.



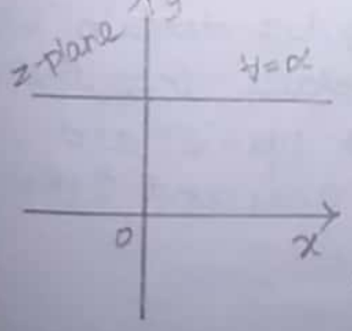


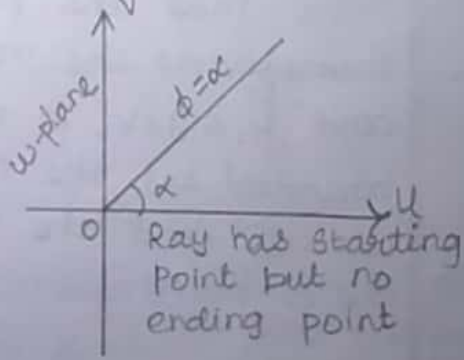
Thus the transformation $w=e^z$ transforms the points between the lines $x=a$ and $x=b$ in the z-plane into the points between the circles $|w|=e^a$ in the w-plane.

Case (iii) :- Consider the line $y=\alpha$

Then we get ~~the~~ circles

Then (3) $\Rightarrow R=e^x,\ \phi=\alpha$.

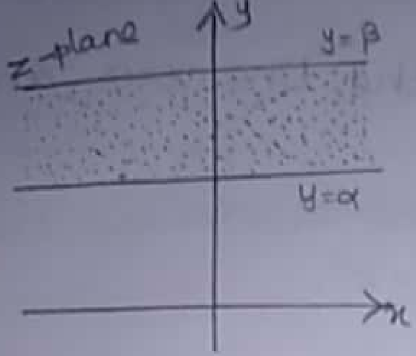


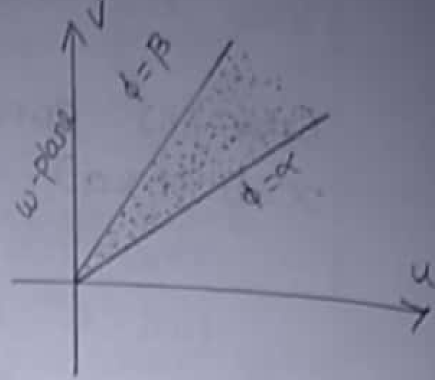


Thus the transformation $w=e^z$ transforms the line $y=\alpha$ in the z-plane into a ray $\phi=\alpha$ in the w-plane.

Case (iv) :- consider the line $y=\alpha$, $y=\beta$ $(\beta>\alpha)$

Then we get the rays, $\phi=\alpha$ and $\phi=\beta$.

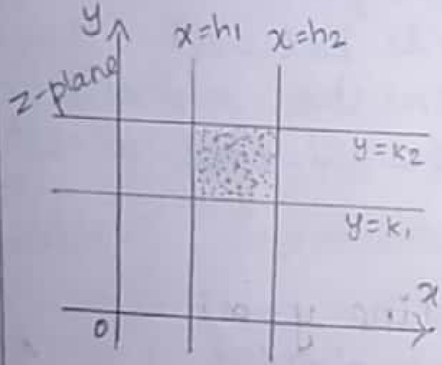


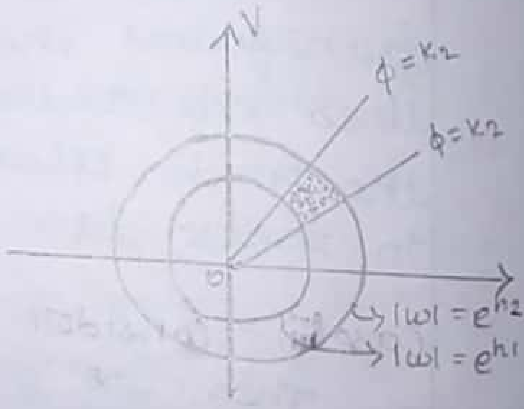


Thus the transformation $w=e^z$ transforms the point between the lines $y=\alpha$ and $y=\beta$ in the z-plane into the points between the rays $\phi=\alpha$ and $\phi=\beta$ in the w-plane.

Case (v) :- Consider the rectangular domain

$h_1 \le x \le h_2$ and $k_1 \le y \le k_2$





Thus the transformation $w=e^z$ transforms the rectangular domain $h_1 \le x \le h_2$ and $k_1 \le y \le k_2$ in the z-plane into the domain bounded by the circles $|w| = e^{h_1}$ and $|w| = e^{h_2}$ and the rays $\phi = k_1$ and $\phi = k_2$ in the w-plane.

4. Discuss the transformation $w = \cos z$

Solution :-

Consider the transformation, $w = \cos z$

Let $z = x + iy$, $w = u + iv$

Now, $w = \cos(x + iy)$

$$u + iv = \cos x \cos(iy) - \sin x \sin(iy)$$

$$= \cos x \cosh y - \sin x (i \sinh y)$$

$$u + iv = \cos x \cosh y - i \sin x \sinh y$$

$$\therefore u = \cos x \cosh y \text{ and } v = -\sin x \sinh y \longrightarrow (1)$$

case (i):-

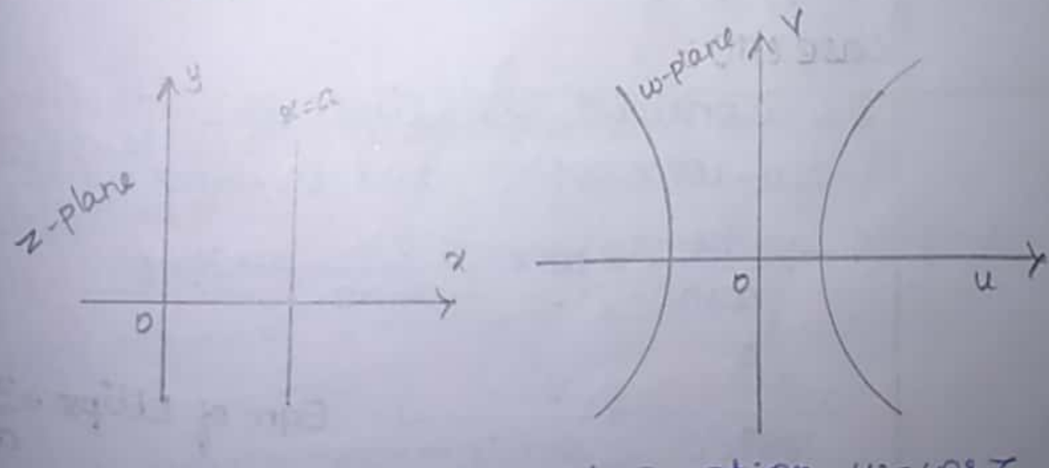
consider the line $x = a$ (straight line curve)

$(1) \Rightarrow u = \cos a \cosh y$, $v = -\sin a \sinh y$

$$\frac{u}{\cos a} = \cosh y, \quad \frac{v}{-\sin a} = \sinh y$$

$$\frac{u^2}{\cos^2 a} = \frac{v^2}{\sin^2 a} = \cosh^2 y - \sinh^2 y$$

$$\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1 \quad [\because \cosh^2 y - \sin^2 y = 1]$$



Hence the transformation $w = \cos z$ transforms the line $x = a$ in the z-plane into the hyperbola.

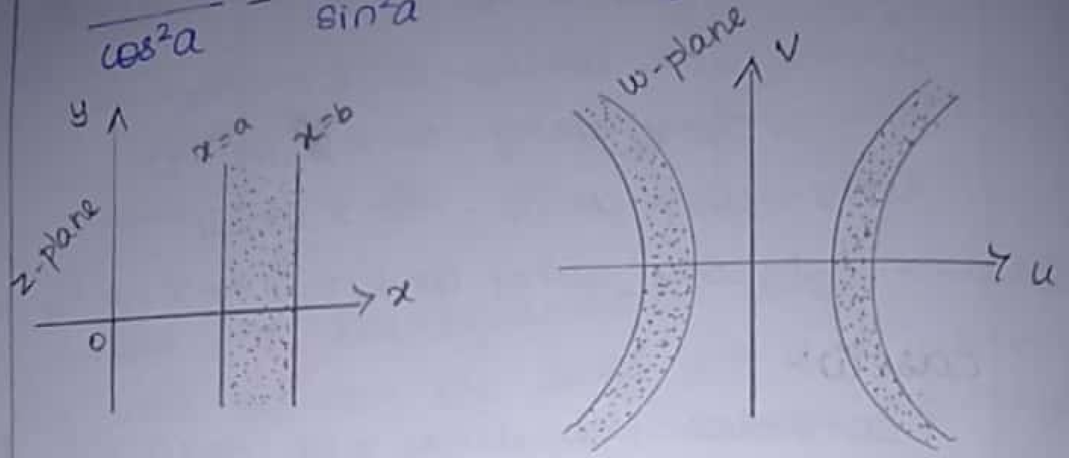
$$\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1 \text{ in the w-plane.}$$

Case (ii):-

consider the lines $x=a$ and $x=b$ $(b>a)$

Thus, we get the hyperbola.

$$\frac{u^2}{\cos^2 a}-\frac{v^2}{\sin^2 a}=1 \text{ and } \frac{u^2}{\cos^2 b}-\frac{v^2}{\sin^2 b}=1.$$



Thus the transformation $w=\cos z$ transforms the points between the lines $x=a$ and $x=b$ in the z-plane into the points between the hyperbola's,

$$\frac{u^2}{\cos^2 a}-\frac{v^2}{\sin^2 a}=1 \text{ and } \frac{u^2}{\cos^2 b}-\frac{v^2}{\sin^2 b}=1 \text{ in}$$

the w-plane.

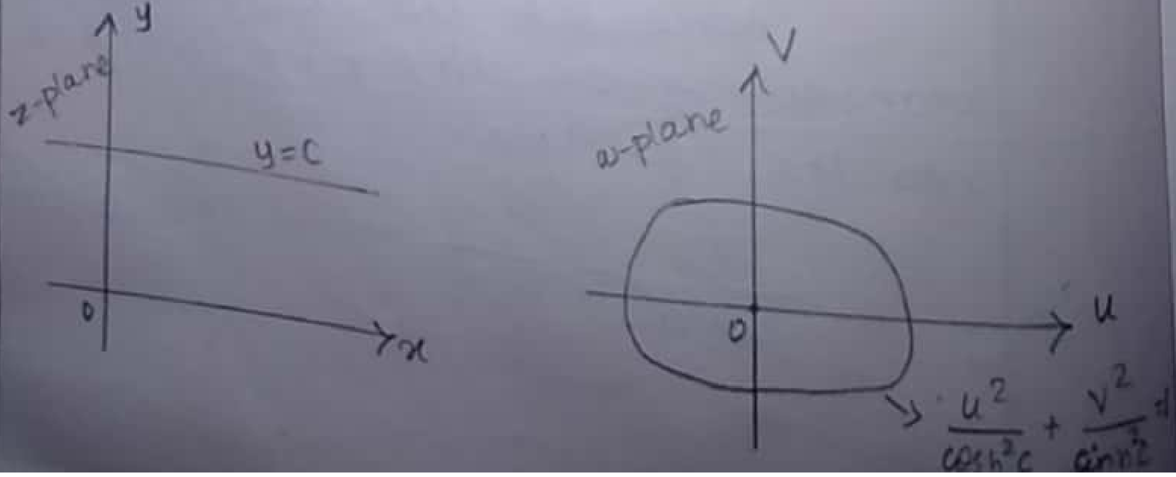
Case (iii):-

Consider the line $y=c$

$\therefore u=\cos x \cosh c$ and $v=-\sin x \sinh c$

$$\Rightarrow \frac{u}{\cosh c}=\cos x, \quad \frac{v}{\sinh c}=-\sin x$$

$$\frac{u^2}{\cosh^2 c}+\frac{v^2}{\sinh^2 c}=1. \quad \text{Eqn of Ellipse} = \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Thus the transformation $w=\cos z$ transforms the points between the lines $y=c$ and $y=d$ in the z-plane into the points between ellipses.

$$\frac{u^2}{\cosh^2 c}+\frac{v^2}{\sinh^2 c}=1 \text{ and } \frac{u^2}{\cosh^2 d}+\frac{v^2}{\sinh^2 d}=1.$$

in the w-plane.

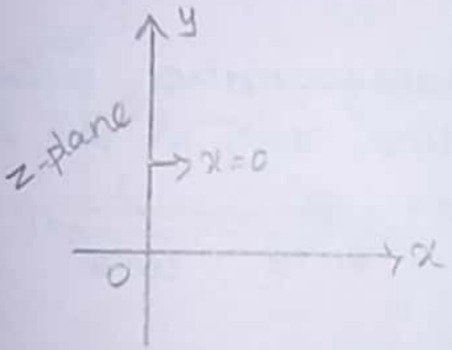
Case (v):-

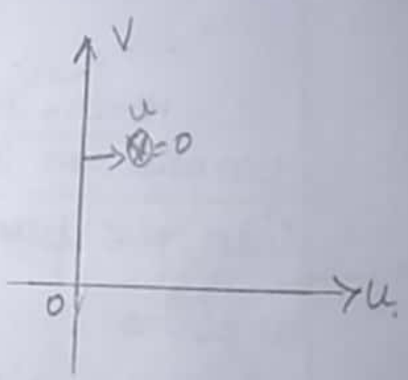
Consider the line $x=0$.

Then, $u=\cos 0 \cdot \cosh y$, $v=-\sin 0 \sinh^2 y$

$u=\cosh y$, $v=0$.

Thus, the transformation $w=\cos z$ transforms the line $x=0$ in the z-plane into the line $v=0$ in the w-plane.





5. Discuss the transformation $w=\sin z$.

** Solution:-

10mark

Consider the transformation $w=\sin z$.

let $z=x+iy$, $w=u+iv$ then

$$w=\sin z$$

$$u+iv=\sin(x+iy)$$

$$=\sin x \cos(iy)+\cos x \sin(iy)$$

$$u+iv=\sin x \cosh y + i\cos x \sinh y$$

$\therefore u=\sin x \cosh y$ and $v=\cos x \sinh y$.

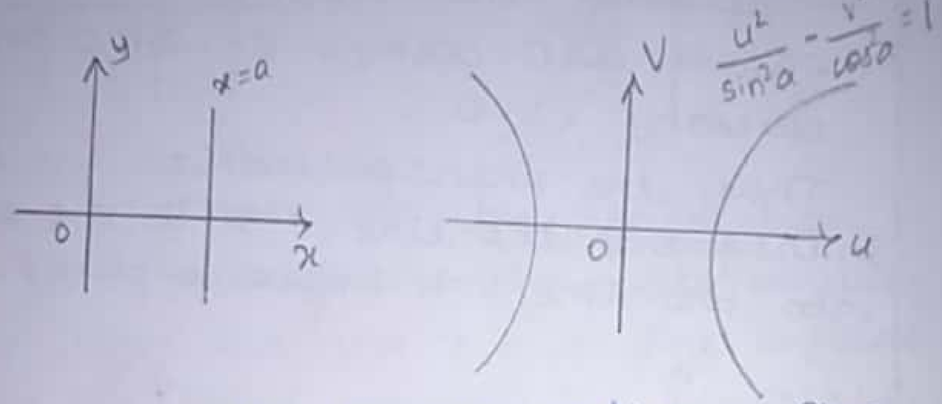
Case (i) :- consider the line $x=a$.

$(1) \Rightarrow u = \sin a \cosh y$, $v = \cos a \sinh y$

$$\frac{u}{\sin a} = \cosh y \quad , \quad \frac{v}{\cos a} = \sinh y$$

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = \cosh^2 y - \sinh^2 y$$

$$\therefore \frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1.$$

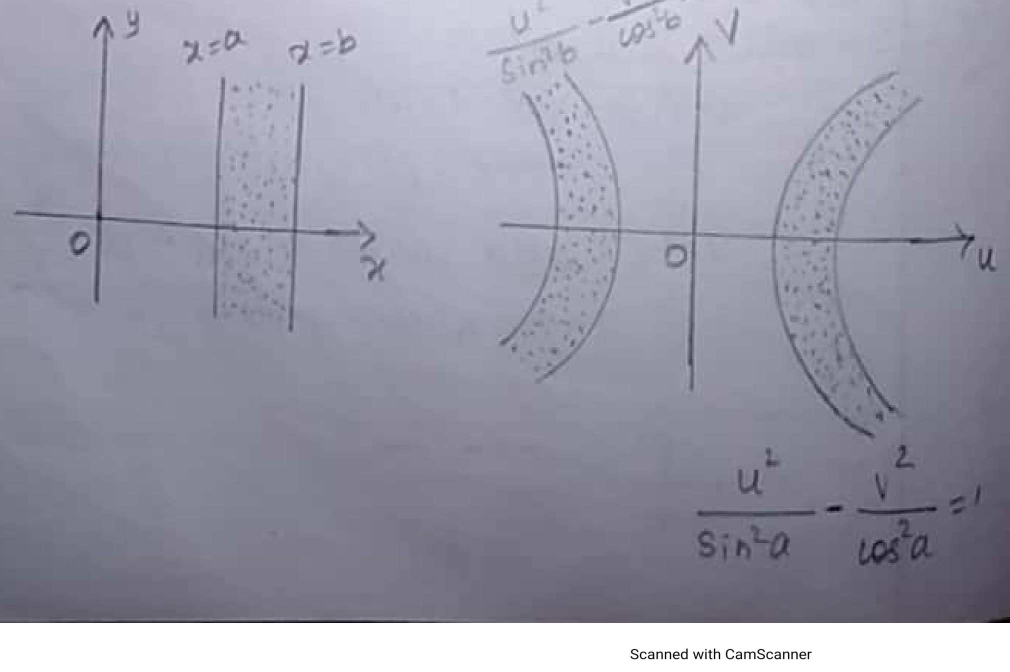


Hence, the transformation $w = \sin z$ transform the line $x=a$ in the z-plane into the hyperbola, $\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$ in the w-plane.

Case (ii) :- consider the line $x=a, x=b$ $(b>a)$.

Thus we get the hyperbola.

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1 \text{ and } \frac{u^2}{\sin^2 b} - \frac{v^2}{\cos^2 b} = 1.$$



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Thus the transformation $w = \sin z$ transforms the points between the lines $x = a$ and $x = b$ in the z-plane into the points between the hyperbola $\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$, $\frac{u^2}{\sin^2 b} - \frac{v^2}{\cos^2 b} = 1$ in the w-plane.

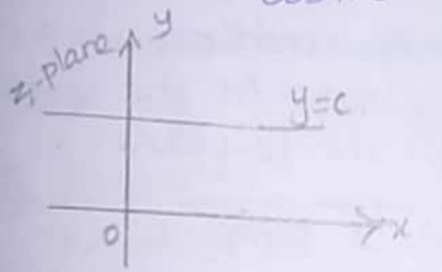
Case (iii) :- consider the line $y = c$.

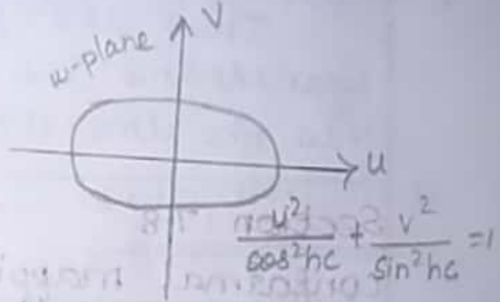
$u = \sin x \cosh c$ and $v = \cos x \sinh c$

$$\frac{u}{\cosh c} = \sin x, \quad \frac{v}{\sinh c} = \cos x$$

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = \sin^2 x + \cos^2 x = 1.$$

$$\therefore \frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1.$$



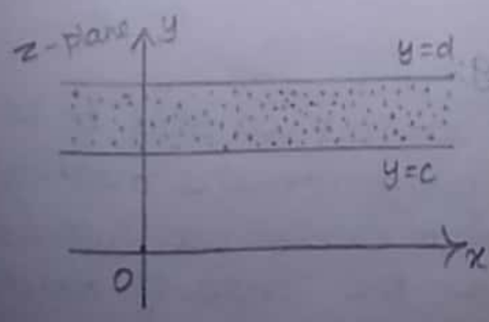


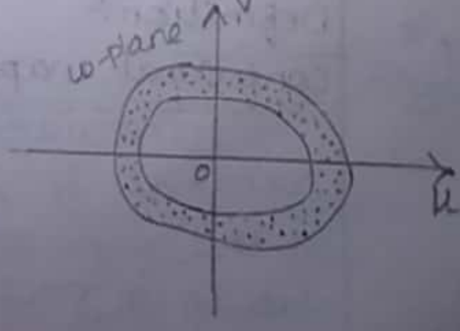
Thus the transformation $w = \sin z$ transforms the line $y = c$ in the z-plane into the ellipse $\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$ in the w-plane.

Case (iv) :- consider the line $y = c$ and $y = d$.

Then we get the ellipses,

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1 \quad \text{and} \quad \frac{u^2}{\cosh^2 d} + \frac{v^2}{\sinh^2 d} = 1.$$





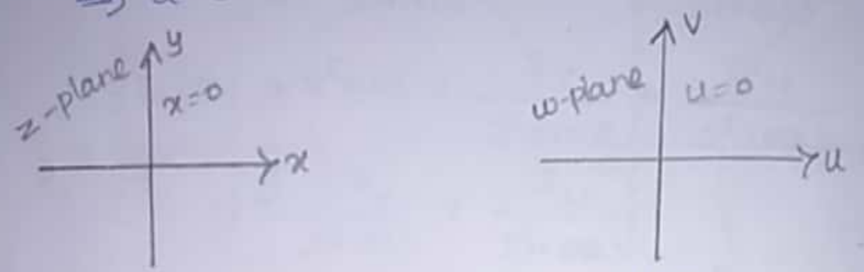
Thus, the transformation $w=\sin z$ transforms the points between the line $y=c$ and $y=d$ in z-plane into the ellipse $\frac{u^2}{\cosh^2 c}+\frac{v^2}{\sinh^2 c}=1$ and

$\frac{u^2}{\cosh^2 d}+\frac{v^2}{\sinh^2 d}=1$ in the w-plane.

case(v):- consider the line $x=0$.

Then $u=\sin 0 \cosh y$ and $v=\cos 0 \sinh y$

$\Rightarrow u=0$ and $v=0$.



Thus the transformation $w=\sin z$ transforms the line $x=0$ in the z-plane into the line $u=0$ in the w-plane.

Section : 7.8

Conformal mapping :-

** Definition :-

8 mark Isogonal mapping :-

Let $f(z)$ be a single-valued function defined in a region D. Then the mapping $w=f(z)$ is said to be isogonal if it preserves the magnitude of the angle between every two curves.

** Definition :-

8 mark Conformal mapping :-

The mapping $w=f(z)$ is said to be conformal if it preserves the magnitude of the angle between every two curves and also if it preserves the sense of orientation of the angle.

Example :-

The mapping $w=\bar{z}$ (a reflection in the real axis) is isogonal, but not conformal.

If $w=u+iv=f(z)$ is a conformal mapping, then $w=u-iv=\overline{f(\bar{z})}$ is isogonal but not conformal because $w=\overline{f(\bar{z})}$ is the mapping $w=f(z)$ followed by a reflection in the real axis.

Theorem : T-12 :-

** 10 mark

If $f(z)$ is analytic in a region D and if $f'(z)\neq 0$ in D, then the mapping $w=f(z)$ is conformal in D.

Proof :-

Given $f(z)$ is analytic in a region D and if $f'(z)\neq 0$ in D $\rightarrow$ (1)

To prove :-

The mapping $w=f(z)$ is conformal in D.

Let $z=a$ and $z\in D$.

From (1).

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a}=f'(a)$$

$$\arg\left(\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\right)=\arg f'(a)$$

i.e., $$\lim_{z\to a}\arg\frac{f(z)-f(a)}{z-a}=\arg f'(a)$$

i.e., $$\lim_{z\to a}\left[\arg(f(z)-f(a))-\arg(z-a)\right]=\arg f'(a)$$

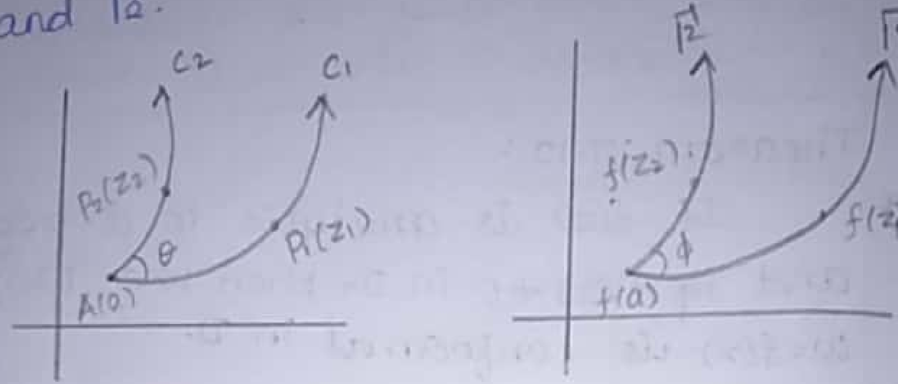
$$\lim_{z\to a}\arg[f(z)-f(a)]-\lim_{z\to a}\arg(z-a)=\arg f'(a)$$

$$\lim_{z\to a}\arg[f(z)-f(a)]=\lim_{z\to a}\arg(z-a)+\arg f'(a)\rightarrow(2)$$

where z approaches 'a' in any mode.

Suppose that C_1, C_2 are two curves starting from the point $A(a)$: P_1, P_2 are two points respectively on C_1, C_2 and θ be the angle between C_1 and C_2.

Suppose that Γ_1, Γ_2 are the images of C_1 & C_2 let ϕ be the angle between Γ_1 and Γ_2.



Clearly $\theta = \lim (\text{angle}(AP_1, AP_2))$ as $P_1 \to A$ and $P_2 \to A$.

$$\theta = \lim_{z_2 \to a} \arg(z_2 - a) - \lim_{z_1 \to a} \arg(z_1 - a)$$

Similarly,

$$\phi = \lim_{f(z_2) \to f(a)} \arg(f(z_2) - f(a)) - \lim_{f(z_1) \to f(a)} \arg(f(z_1) - f(a))$$

$$= \left[\lim_{z_2 \to a} (\arg(z_2 - a) + \arg f'(a))\right] - \left[\lim_{z_1 \to a} \arg(z_1 - a) + \arg f'(a)\right] \quad [\because \text{by } (2)]$$

$$= \lim_{z_2 \to a} \arg(z_2 - a) - \lim_{z_1 \to a} \arg(z_1 - a)$$

$\phi = \theta$

Hence the mapping is conformal.

Remark :-

The angle of rotation at $z = a$ is $\arg f'(a)$.

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Theorem : 7.13 :-

Suppose that $f(z)$ is analytic in a region D. If l is the length of a small arc C emanating from $z=a$ and if L is the length of its image, then

$$\lim_{z \to a} \frac{L}{l} = |f'(a)|,$$ where $z \to a$ along C.

proof :-

Let z be a point on C and $z \to a$

Then $$\lim_{z \to a} \frac{L}{l} = \lim_{z \to a} \frac{|f(z) - f(a)|}{|z-a|}$$

$$= \lim_{z \to a} \left| \frac{f(z) - f(a)}{z-a} \right|$$

$$= \left| \lim_{z \to a} \frac{f(z) - f(a)}{z-a} \right|$$

$$\therefore \lim_{z \to a} \frac{L}{l} = |f'(a)|$$

Remark :-

If $|f'(a)| > 1$, then the transformation at $z=a$ has a constant magnification and the constant of magnification is $|f'(a)|$.

Definition :-

Critical point of the transformation.

A point z at which $f'(z) = 0$ is called a critical point of the transformation $w = f(z)$.

Theorem 7.14:-

If $f(z)$ is analytic in a region D and if $z=a$ is a point in it such that $f'(a)=0$, then the transformation $w=f(z)$ is not conformal at $z=a$

proof:-

Using Taylor's series for $f(z)$ about $z=a$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots \rightarrow (1)$$

As $f'(a)=0$.

$$\therefore (1) \Rightarrow f(z) = f(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

$$f(z) = f(a) + (z-a)^2 a_2 + (z-a)^3 a_3 + \cdots$$

(or) In general,

$$f(z) = f(a) + (z-a)^n a_n + (z-a)^{n+1} a_{n+1} + \cdots$$

$$\Rightarrow f(z) - f(a) = (z-a)^n \left[a_n + (z-a) a_{n+1} + \cdots\right]$$

$f(z) - f(a) = (z-a)^n \phi(z)$, where $\phi(z)$ is absolutely convergent because $f(z)$ is analytic with $\phi(a) \neq 0$. $\therefore \arg\{f(z)-f(a)\} = \arg[(z-a)^n \phi(z)]$

$$= \arg (z-a)^n + \arg \phi(z)$$

$$\arg\{f(z)-f(a)\} = n \arg (z-a) + \arg \phi(z) \rightarrow (2)$$

$\therefore$ If ϕ is the angle between the images of two curves under $w=f(z)$ then,

$$\phi = \lim_{f(z_2)\to f(a)} [\arg(f(z_2)-f(a))] - \lim_{f(z_1)\to f(a)} [\arg(f(z_1)-f(a))]$$

$$= \lim_{z_2 \to a} [n \arg(z_2-a) + \arg \phi(z_2)] -$$

$$\lim_{z_1 \to a} [n \arg(z_1-a) + \arg \phi(z_1)] \quad [\because \text{by } (2)]$$

$$= n \left[\lim_{z_2 \to a} \arg(z_2-a) - \lim_{z_1 \to a} \arg(z_1-a)\right]$$

$\therefore \phi = n\theta.$

$\therefore \phi \neq \theta.$

The mapping is not conformal.

NOTE :-

If $f(a)=0$, then $f(z) = (z-a)^n \phi(z)$, $\phi(a) \neq 0$ and hence the angle between the images of two curves at $w = f(a)$ is 'n' times the angle between the two curves.

Theorem : 7.15 :-

i) If the transformation $w=f(z)$ is conformal in a region D.

ii) If u_x, u_y, v_x, v_y exist and are continuous in D and

iii) If the magnification at each point in D is a constant $\neq 0$ then $f(z)$ is analytic in D.

proof :-

Let $ds, d\sigma$ be the lengths of an arc in z-plane at $z=a$ and its image in w-plane respectively.

by (iii), $d\sigma = k \cdot ds$, $k \neq 0$.

$$\Rightarrow d\sigma^2 = k^2 \cdot ds^2 \qquad \therefore ds^2 = dx^2 + dy^2$$

$$d\sigma^2 = k^2 (dx^2 + dy^2)$$

$$d\sigma^2 = k^2 dx^2 + k^2 dy^2 \longrightarrow (1)$$

$d\sigma^2 = du^2 + dv^2$ where $du = u_x dx + u_y dy$

$dv = v_x dx + v_y dy$

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$$\therefore d\sigma^2 = u_x^2 dx^2 + u_y^2 dy^2 + 2u_x u_y dx\,dy + v_x^2 dx^2 + v_y^2 dy^2 + 2 v_x v_y dx\,dy$$

$$= \frac{(u_x^2+v_x^2)dx^2}{k^2} + \frac{(u_y^2+v_y^2)dy^2}{k^2} + 2(u_x u_y + v_x v_y) dx\,dy.$$

Hence from(1),

$$u_x^2 + v_x^2 = k^2, \quad u_y^2 + v_y^2 = k^2$$

$$u_x u_y + v_x v_y = 0.$$

It is clear that for some α, β.

$u_x = k\cos\alpha$, $\quad v_x = k\sin\alpha$.

$u_y = k\cos\beta$, $\quad v_y = k\sin\beta$ [from (i), (ii)]

using $u_x u_y + v_x v_y = 0$, we get

$$k^2 \cos\alpha\cos\beta + k^2\sin\alpha\sin\beta = 0.$$

$$\Rightarrow \cos(\alpha-\beta) = 0.$$

$$\Rightarrow \alpha - \beta = \pm\frac{\pi}{2} \text{ then}$$

$$\alpha = \beta \pm \frac{\pi}{2}.$$

$$u_x = k\cos\left(\beta + \frac{\pi}{2}\right)$$

$$= k\left(\cos\beta \cdot \underbrace{\cos\pi/2}_{0} - \sin\beta\cdot\underbrace{\sin\pi/2}_{1}\right)$$

$$u_x = -k\sin\beta$$

$$u_x = -v_y$$

i.e. The C-R equation is not satisfied

$$\therefore \alpha = \beta - \pi/2.$$

Then $u_x = v_y$ and $u_y = -v_x$

$\therefore f(z)$ is analytic by condition (iii).

Section : 4.8 :-

C-R Equation In polar coordinates

1) problems :-

1. Show that the single-valued continuous function $f(z) = r^{1/2}(\cos\theta/2 + i\sin\theta/2) : r > 0, \alpha < \theta < 2\pi$ is analytic. find $f'(z)$.

Solution :-

Given $f(z) = r^{1/2}(\cos\theta/2 + i\sin\theta/2)$ is continuous

$$f(z) = r^{1/2}\cos\theta/2 + i\, r^{1/2}\sin\theta/2 .$$

$$u = r^{1/2}\cos\theta/2 \quad ; \quad v = r^{1/2}\sin\theta/2 .$$

$$\frac{\partial u}{\partial r} = \frac{1}{2} r^{-1/2}\cos\theta/2 \quad ; \quad \frac{\partial v}{\partial r} = \frac{1}{2} r^{-1/2}\sin\theta/2$$

$$\frac{\partial u}{\partial r} = \frac{1}{2r^{1/2}}\cos\theta/2 \quad ; \quad \frac{\partial v}{\partial r} = \frac{1}{2r^{1/2}}\sin\theta/2 .$$

$$\frac{\partial u}{\partial \theta} = -r^{1/2}\sin\theta/2\left(\frac{1}{2}\right) \quad ; \quad \frac{\partial v}{\partial \theta} = r^{1/2}\cos\theta/2\,(1/2) .$$

$$\frac{\partial u}{\partial \theta} = -\frac{1}{2} r^{1/2}\sin\theta/2 \quad ; \quad \frac{\partial v}{\partial \theta} = \frac{1}{2} r^{1/2}\cos\theta/2 .$$

$$r \cdot \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad ; \quad -r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} .$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} .$$

$\therefore$ The C-R equations are satisfied. Further the first partials are continuous. So $f'(z)$ exist.

$$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= \frac{r}{z}\left(\frac{1}{2r^{1/2}}\cos\theta/2 + i\frac{1}{2r^{1/2}}\sin\theta/2\right)$$

$$= \frac{r}{z}\cdot\frac{1}{2r^{1/2}}(\cos\theta/2 + i\sin\theta/2) .$$

$$= \frac{1}{2z} r^{1/2}(\cos\theta/2 + i\sin\theta/2) .$$

$\Rightarrow (\cos\theta + i\sin\theta)^{1/2}$

$$f'(z) = \frac{1}{2z}\sqrt{z} \qquad z = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow f'(z) = \frac{1}{2\sqrt{z}} \qquad z^{1/2} = r^{1/2}(\cos\theta/2 + i\sin\theta/2) .$$

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2. If $f(z) = \log r + i\theta$ is single-valued and continuous in a region D. Show that
(5) $f'(z) = \frac{1}{z}$.

Solution:-

Given $f(z) = \log r + i\theta$ is continuous.

To prove:-

$$f'(z) = \frac{1}{z}.$$

$$f(z) = \log r + i\theta.$$ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial \theta}$

$u(x,y) = \log r$; $v(x,y) = \theta$. $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$u_r = \frac{1}{r}$; $v_r = 0$.

$u_\theta = 0$; $v_\theta = 1$.

$$r \cdot \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \; ; \; \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} r.$$

∴ The C-R equations are satisfied.

∴ The first partials of u and v are continuous

$$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= \frac{r}{z}\left(\frac{1}{r} + i(0)\right)$$

$$\therefore f'(z) = \frac{1}{z}.$$

3. Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic
(7) in a region D. Show that, in D, the level curves $u(x,y) = \lambda$ are orthogonal to the level curves $v(x,y) = \mu$, where λ and μ are parameters.

Solution:-

$\nabla u(x,y)$ is a vector normal to the curve. $u(x,y) = \lambda$ at (x,y) and

$\nabla v(x,y)$ is a vector normal to the curve $v(x,y) = \mu$ at (x,y). But the Scalar product of these vectors.

$$\{\nabla u(x,y)\} [\nabla v(x,y)] = \left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y}\right)\left(i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y}\right)$$

$= u_x v_x + u_y v_y$

$= v_y v_x - v_x v_y$ [$\because$ C-R Equations

$= 0.$ $u_x = v_y$

$u_y = -v_x$]

So the normals to the curves at (x, y) are perpendicular.

$\Rightarrow$ The tangents to the curve at (x, y) are perpendicular.

$\therefore$ The curves are orthogonal.

Remark :-

$z = x + iy$, $\bar{z} = x - iy$; $z + \bar{z} = 2x \Rightarrow x = \frac{z + \bar{z}}{2}$

$z - \bar{z} = 2iy \Rightarrow y = \frac{z - \bar{z}}{2i} \Rightarrow y = \frac{-i}{2}(z - \bar{z}).$

Theorem :-

For a analytic function $f(z)$ the C-R equations can be put in the condition for $\frac{\partial f}{\partial \bar{z}} = 0.$

proof :-

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \right) + i \left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \right).$$

$$= u_x(1/2) + u_y(i/2) + i\left[v_x(1/2) + v_y(i/2)\right]$$

$$= 1/2 \{ (u_x - v_y) + i(u_y + v_x) \}$$

$$= \frac{1}{2} \{ (u_x - u_x) + i(u_y - u_y) \}$$

$$= 1/2 \, (0)$$

$$= 0.$$

$$\therefore \frac{\partial f}{\partial \bar{z}} = 0.$$

(4). Let $f(z)$ is a function of (x,y) and x,y has a function of $z, \bar{z}$ we have shows that

** 10 mark

i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0.$

Solution:-

$z = x+iy \quad , \bar{z} = x-iy$

$x = \frac{z+\bar{z}}{2} \quad , y = \frac{z-\bar{z}}{2i}$

$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}.$

$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}(1/2) + \frac{\partial f}{\partial y}(-1/2).$

$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f) \longrightarrow (1)$

$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}.$

$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{i}{2}\right).$

$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f) \longrightarrow (2)$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$

$= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$= 2\frac{\partial}{\partial z}\cdot 2\frac{\partial}{\partial \bar{z}}.$

$\nabla^2 = 4\cdot\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}}$

$\nabla^2|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 \qquad |z|^2 = z.\bar{z}$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[f(z).f(\bar{z})]$

$= 4\cdot\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}}(f(z).f(\bar{z})).$

$= 4\left(\frac{\partial}{\partial z}f(z)\right)\left(\frac{\partial}{\partial \bar{z}}f(\bar{z})\right)$

$$= 4 f'(z) \left(\overline{\frac{\partial}{\partial z} f(z)} \right) \quad [\because |z|^2 = z \cdot \bar{z}]$$

$$= 4 f'(z) \left(\overline{f'(z)} \right)$$

$$\nabla^2 |f(z)|^2 = 4 |f'(z)|^2$$

(ii) $$\nabla^2 \log |f'(z)| = \nabla^2 \frac{1}{2} \log |f'(z)|^2$$

$$= \frac{1}{2} \nabla^2 \log \left[f'(z) \, \overline{f'(z)} \right]$$

$$= \frac{1}{2} \nabla^2 \left[\log f'(z) + \log \overline{f'(z)} \right]$$

$$= \frac{1}{2} \, 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \left[\log f'(z) + \log \overline{f'(z)} \right]$$

$$= 2 \frac{\partial}{\partial z} \left[\frac{\partial}{\partial \bar{z}} (\log f'(z)) + \frac{\partial}{\partial \bar{z}} (\log \overline{f'(z)}) \right]$$

$$= 2 \frac{\partial}{\partial z} \left(0 + \frac{\partial}{\partial \bar{z}} \log \overline{f'(z)} \right)$$

$$= 2 \left(\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} \log \overline{f'(z)} \right)$$

$$= 2(0)$$

$$= 0$$

$$\therefore \nabla^2 \log |f'(z)| = 0.$$

Section 6:12 - Harmonic Functions :-

Problems : Method: 1 :-

Exact Differential Method.

1. Check whether $xy(x+y)$ is a harmonic function. Can it be the real part of an analytic function.

(6)

Solution :-

Given $u(x,y) = xy(x+y)$

Then,

$$u_x = xy + (x+y)y$$

$$u_{xx} = y + y(1) = 2y$$

$$u_y = xy + (x+y)(x)$$

$$u_{yy} = x(1) + x(1) = 2x$$

consider,

$$u_{xx}+u_{yy}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2(x+y)\neq 0.$$

$\therefore u$ is not harmonic.

$\therefore xy(x+y)$ cannot be the real part of an analytic function.

Q. (b) Show that $u(x,y)$ is a harmonic function in the following cases and find the analytic function $f(z)$ having it as its real part. $u(x,y)=x^2-y^2$ find V and $f(z)$.

Solution:-

Consider the harmonic function $u=x^2-y^2$. Then the C-R equations are

$$u_x=V_y \quad , \quad u_y=-V_x$$

Here,

$$u_x=2x=V_y \quad , \quad u_y=-2y=-V_x$$

$$\Rightarrow V_y=2x \quad , \quad \Rightarrow V_x=2y$$

$\therefore V=\int V_x\,dx$ (treating 'y' as constant) $+$ $\int V_y\,dy$ (terms of V_y not containing x)

$$=\int 2y\,dx+\int (0)\,dy$$

$$=2y\int dx$$

$$=2xy$$

$$\therefore V(x,y)=2xy$$

$$f(z)=u(x,y)+iV(x,y)+c$$

$$=(x^2-y^2)+i\,2xy+c$$

$$f(z)=(x+iy)^2+c$$

$$\therefore f(z)=z^2+c$$

Q. (i) $u = 2x(1-y)$

Solution:-

$u = 2x(1-y) = 2x - 2xy$

$u_x = 2-2y$, $u_y = -2x$

$u_{xx} = 0$, $u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0.$

$\therefore u(x,y)$ is a harmonic function

$f(z) = u + iv$

$u = 2x(1-y)$

$u = 2x - 2xy$

$u_x = 2 - 2y = v_y$

$u_y = -2x$

$v_x = 2x$

$v = \int v_x dx + \int v_y dy$

$\int 2x dx + \int 2 - 2y dy$

$v = x^2 + 2y - y^2$

$v = x^2 - y^2 + 2y$

To find:-

The analytic function $f(z)$.

$f(z) = 2u\left(\frac{1}{2}z, \frac{-1}{2}iz\right) \longrightarrow (1)$

Here, $x = \frac{1}{2}z$, $y = \frac{-iz}{2}$ in u.

$\therefore u = \left(\frac{z}{2}, \frac{-iz}{2}\right) = 2\left(\frac{z}{2}\right)\left(1 + \frac{iz}{2}\right)$

$u\left(\frac{z}{2}, \frac{-iz}{2}\right) = z\left(\frac{2+iz}{2}\right) = \frac{2z + iz^2}{2}$

$(1) \Rightarrow f(z) = 2\left(\frac{2z + iz^2}{2}\right)$

$\therefore f(z) = 2z + iz^2.$

(ii) $u = x^3 - 3xy^2.$

Solution:-

Given $u = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2$, $u_y = -6xy$

$u_{xx} = 6x$, $u_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 6x - 6x = 0.$

$\therefore u(x,y)$ is a harmonic function.

To find:- $f(z)$

ie, $f(z) = 2u\left(\frac{1}{2}z, -\frac{1}{2}iz\right)$

Here, $x = \frac{z}{2}$, $y = -iz/2$

$\therefore u\left(\frac{z}{2}, \frac{iz}{2}\right) = \left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(-\frac{iz}{2}\right)^2$

$= \frac{z^3}{8} + \frac{3z}{2} \cdot \frac{z^2}{4}$

$= \frac{z^3}{8} + \frac{3z^3}{8}$

$$= \frac{4z^3}{8}$$

$$u\left(\frac{z}{2}, \frac{-iz}{2}\right) = \frac{z^3}{2}.$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$= 2 \cdot z^3/2 .$$

$$f(z) = z^3.$$

(iii) $u = x^3 - 3xy^2 + x$.

Solution:-

Given $u = x^3 - 3xy^2 + x$

$u_x = 3x^2 - 3y^2 + 1$, $u_y = -6xy$

$u_{xx} = 6x$, $u_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 6x - 6x = 0$.

$\therefore u(x, y)$ is a harmonic function.

To find :- $f(z)$

ie, $f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$

$= \cancel{2} \cdot z^3/\cancel{2}$.

$\therefore \cancel{f(z) = z^3}$

$$u\left(\frac{z}{2}, \frac{-iz}{2}\right) = \left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{-iz}{2}\right)^2 + \left(\frac{z}{2}\right)$$

$$= \frac{z^3}{8} - \left(\frac{3z}{2}\right)\left(\frac{-z^2}{4}\right) + \frac{z}{2}$$

$$= \frac{z^3}{8} + \frac{3z^3}{8} + \frac{z}{2}$$

$$= \frac{4z^3}{8} + \frac{z}{2}$$

$$= \frac{z^3}{2} + \frac{z}{2}$$

$$u\left(\frac{z}{2}, \frac{-iz}{2}\right) = \frac{1}{2}(z^3 + z)$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$= 2\left[\frac{1}{2}[z^3 + z]\right]$$

$$\therefore f(z) = z(z^2 + 1)$$

iv) $u = y^3 - 3x^2y$.

Solution:-

Given $u = y^3 - 3x^2y$

$u_x = -6xy$, $u_y = 3y^2 - 3x^2$

$u_{xx} = -6y$, $u_{yy} = 6y$

$\therefore u_{xx} + u_{yy} = -6y + 6y = 0$.

$\therefore u(x,y)$ is a harmonic function.

To find: $f(z)$:-

ie, $f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$.

$$u\left(\frac{z}{2}, -\frac{iz}{2}\right) = \left(\frac{-iz}{2}\right)^3 - 3\left(\frac{z}{2}\right)^2\left(\frac{-iz}{2}\right)$$

$$= \frac{iz^3}{8} + \frac{3iz^3}{8}$$

$$= \frac{i}{8}(z^3 + 3z^3)$$

$$= \frac{i\,4z^3}{8}$$

$$= \frac{iz^3}{2}$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$= 2\left(\frac{iz^3}{2}\right)$$

$$\therefore f(z) = iz^3.$$

v) $u = y^3 - 3x^2y + 2y$.

Solution:-

Given $u = y^3 - 3x^2y + 2y$

$u_x = -6xy$; $u_y = 3y^2 - 3x^2 + 2$

$u_{xx} = -6y$, $u_{yy} = 6y$

$u_{xx} + u_{yy} = -6y + 6y = 0$

$\therefore u(x,y)$ is a harmonic function.

To find :- $f(z)$

$$f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$$

$$u\left(\frac{z}{2}, -\frac{iz}{2}\right) = \left(-\frac{iz}{2}\right)^3 - 3\left(\frac{z}{2}\right)^2\left(-\frac{iz}{2}\right) + 2\left(-\frac{iz}{2}\right)$$

$$= \frac{iz^3}{8} + \frac{3iz^3}{8} - iz$$

$$= \frac{i4z^3}{8} - iz$$

$$= \frac{iz^3}{2} - iz$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$= 2\left(\frac{iz^3}{2} - iz\right)$$

$$\therefore f(z) = iz^3 - 2iz.$$

---

vi) $u = -x^3 + 3xy^2 + 2x$

Solution :-

Given $u = -x^3 + 3xy^2 + 2x$

$u_x = -3x^2 + 3y^2 + 2$, $u_y = 6xy$

$u_{xx} = -6x$, $u_{yy} = 6x$

$\therefore u_{xx} + u_{yy} = -6x + 6x = 0$

$\therefore u(x,y)$ is a harmonic function.

To find : $f(z)$

$$f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$$

$$u\left(\frac{z}{2}, -\frac{iz}{2}\right) = \left(-\frac{z}{2}\right)^3 + 3\left(\frac{z}{2}\right)\left(-\frac{iz}{2}\right)^2 + 2\left(\frac{z}{2}\right)$$

$$= -\frac{z^3}{8} - \frac{3z^3}{8} + z$$

$$= -\frac{4z^3}{8} + z$$

$$= -\frac{z^3}{2} + z$$

$$f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$$

$$= 2\left(-\frac{z^3}{2} + z\right)$$

$$\therefore f(z) = -z^3 + 2z.$$

vii) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Solution:-

$u_x = 3x^2 - 3y^2 + 6x$, $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$, $u_{yy} = -6x + (-6)$

$\therefore u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$

$\therefore u(x,y)$ is a harmonic function.

To find:- $f(z)$

$$f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$\therefore u\left(\frac{z}{2}, \frac{-iz}{2}\right) = \left(\frac{z}{2}\right)^3 - \frac{3z}{2}\left(\frac{-iz}{2}\right)^2 + 3\left(\frac{z}{2}\right)^2 - 3\left(\frac{-iz}{2}\right)^2$$

$$= \frac{z^3}{8} + \frac{3z^2}{8} + \frac{3z^2}{4} + \frac{3z^2}{4}$$

$$= \frac{4z^3}{8} + \frac{6z^2}{4}$$

$$= \frac{z^3}{2} + \frac{3z^2}{2}$$

$$f(z) = 2\left(\frac{z^3}{2} + \frac{3z^2}{2}\right) + 1$$

$$\therefore f(z) = z^3 + 3z^2 + 1.$$

viii) $u = x^2 - y^2 - 2xy - 2x + 3y.$

Solution:-

Given $u = x^2 - y^2 - 2xy - 2x + 3y$

$u_x = 2x - 2y - 2$, $u_y = -2y - 2x + 3$

$u_{xx} = 2$, $u_{yy} = -2$

$\therefore u_{xx} + u_{yy} = 2 - 2 = 0.$

$\therefore u(x,y)$ is a harmonic function.

To find:- $f(z)$

$$f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$u\left(\frac{z}{2}, \frac{-iz}{2}\right) = \left(\frac{z}{2}\right)^2 - \left(\frac{-iz}{2}\right)^2 - 2\left(\frac{z}{2}\right)\left(\frac{-iz}{2}\right) - 2\left(\frac{z}{2}\right) + 3\left(\frac{-iz}{2}\right)$$

$$= \frac{z^2}{4} + \frac{z^2}{4} + \frac{iz^2}{2} - z - \frac{3iz}{2}$$

$$= \frac{2z^2}{4} + \frac{iz^2}{2} - z\left(1 + \frac{3i}{2}\right)$$

$$= \frac{z^2}{2} + \frac{iz^2}{2} - z\left(\frac{2+3i}{2}\right)$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

$$= 2\left[\frac{z^2}{2} + \frac{iz^2}{2} - z\left(\frac{2+3i}{2}\right)\right]$$

$$\therefore f(z) = z^2 + iz^2 - z(2+3i)$$

$$\therefore f(z) = z^2 + iz^2 - 2z - 3iz.$$

ix) $u = 2x^3 - 5x^2 - 6xy^2 + 3x + 5y^2 - 7.$

Solution:-

Given $u = 2x^3 - 5x^2 - 6xy^2 + 3x + 5y^2 - 7.$

$u_x = 6x^2 - 10x - 6y^2 + 3$, $u_y = -12xy + 10y$

$u_{xx} = 12x - 10$, $u_{yy} = -12x + 10.$

$\therefore u_{xx} + u_{yy} = 12x - 10 - 12x + 10 = 0.$

$\therefore u(x,y)$ is a harmonic function.

To find :- $f(z)$

$$f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$$

$$u\left(\frac{z}{2}, -\frac{iz}{2}\right) = 2\left(\frac{z}{2}\right)^3 - 5\left(\frac{z}{2}\right)^2 - 6\left(\frac{z}{2}\right)\left(-\frac{iz}{2}\right)^2 + 3\left(\frac{z}{2}\right) + 5\left(-\frac{iz}{2}\right)^2 -$$

$$= 2\frac{z^3}{8} - 5\frac{z^2}{4} + 6\frac{z^3}{8} + \frac{3z}{2} - \frac{5z^2}{4} - 7$$

$$= \frac{z^3}{4} - \frac{5z^2}{4} + \frac{3z^3}{4} + \frac{3z}{2} - \frac{5z^2}{4} - 7$$

$$= \frac{4z^3}{4} - \frac{10z^2}{4} + \frac{3z}{2} - 7.$$

$$= z^3 - \frac{5z^2}{2} + \frac{3z}{2} - 7$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$$

$$= 2\left(z^3 - \frac{5z^2}{2} + \frac{3z}{2}\right) - 7$$

$$\therefore f(z) = 2z^3 - 5z^2 + 3z - 7$$

8. Find the analytic function whose real part is $\frac{x}{x^2+y^2}$.

(3)

Solution:-

Given $u(x,y) = \frac{x}{x^2+y^2}$

At $u(0,0) = \frac{0}{0}$ = undefined.

$\therefore$ we use the formula.

$$f(z) = 2u\left(\frac{z+a}{2}, \frac{z-a}{2i}\right)$$

$$= 2\left[\frac{\left(\frac{z+a}{2}\right)}{\left(\frac{z+a}{2}\right)^2 + \left(\frac{z-a}{2}\right)^2}\right]$$

$$= 2\left[\frac{\left(\frac{z+a}{2}\right)}{\frac{(z+a)^2}{4} + \frac{(z-a)^2}{4}}\right]$$

$$= 2\left[\frac{4(z-a)}{2[(z+a)^2 - (z-a)^2]}\right]$$

$$= \frac{4(z+a)}{(z^2+a^2+2za) - (z^2+a^2-2za)}$$

$$= \frac{4(z+a)}{z^2+a^2+2za-z^2-a^2+2za}$$

$$= \frac{4(z+a)}{4za}$$

$$= \frac{z+a}{za} = \frac{z}{za} + \frac{a}{za}$$

$$= \frac{1}{a} + \frac{1}{z}$$

Put $a = 1$.

$\therefore f(z) = 1 + \frac{1}{z}$.

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4\. Show that the function $u(x,y) = \sin x \cosh y$ is harmonic. Find its harmonic conjugate $V(x,y)$ and the analytic function $f(z) = u+iv$.

Solution:-

Given $u(x,y) = \sin x \cosh y$

Using method-1:-

$u_x = \cos x \cosh y$, $u_y = \sin x \sinh y$

$u_{xx} = -\sin x \cosh y$, $u_{yy} = \sin x \cosh y$

$\therefore u_{xx} + u_{yy} = -\sin x \cosh y + \sin x \cosh y = 0$

$\therefore u(x,y)$ is a harmonic function.

To find:- $V(x,y)$.

$V(x,y) = \int V_x \, dx$ (y as constant)

$+ \int V_y \, dy$ (terms of V_y not containing x)

$\therefore$ By C-R equations.

$u_x = V_y \Rightarrow V_y = \cos x \cosh y$

$u_y = -V_x \Rightarrow V_x = -\sin x \sinh y$

$V(x,y) = \int -\sin x \sinh y \, dx + \int (0) dy$

$= -\sinh y \int \sin x \, dx$

$V(x,y) = \cos x \sinh y$

$\therefore f(z) = u+iV = \sin x \cosh y + i \cos x \sinh y$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin(x+iy)$

$\therefore f(z) = \sin z$.

Using method-2:-

$u(x,y) = \sin x \cosh y$

$f(z) = 2\left[u\left(\frac{z}{2}, \frac{-iz}{2}\right)\right]$ [$\cos iz = \cos z$, $\sin iz = i \sin z$]

$= 2 \sin\left(\frac{z}{2}\right) \cosh\left(\frac{-iz}{2}\right)$

$= 2\sin\left(\frac{z}{2}\right)\cos i\left(-\frac{iz}{2}\right)$

$= 2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)$

$f(z) = \sin z = \sin(x+iy)$

$f(z) = \sin x \cos iy + \cos x \sin iy$

$f(z) = \sin x \cosh y + i\cos x \sinh y$

$\therefore V(x,y) = \cos x \sinh y.$

---

5. (10) Find the analytic function $f(z)$ if its real part is $u(x,y) = e^x(x\cos y - y\sin y)$.

Solution:-

Given $u(x,y) = e^x(x\cos y - y\sin y) \longrightarrow (1)$

$u(0,0) = e^0(0\cos 0 - 0\sin 0) = 0.$

$\therefore$ It is defined

using the formula.

$f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$

put $x = \frac{z}{2}$, $y = -\frac{iz}{2}$ in (1).

$f(z) = 2\left\{e^{(z/2)}\left[\left(\frac{z}{2}\right)\cos\left(-\frac{iz}{2}\right) - \left(-\frac{iz}{2}\right)\sin\left(-\frac{iz}{2}\right)\right]\right\}$

$= 2\left\{e^{(z/2)}\left[\left(\frac{z}{2}\right)\cos\left(\frac{hz}{2}\right) + i\left(\frac{z}{2}\right)(-i)\sin\left(\frac{hz}{2}\right)\right]\right\}$

$= 2\left\{e^{(z/2)}\left[\left(\frac{z}{2}\right)\cos\left(\frac{hz}{2}\right) + \left(\frac{z}{2}\right)\sin\left(\frac{hz}{2}\right)\right]\right\}$

$= e^{(z/2)}(z)\left[\frac{e^{z/2}+e^{-z/2}}{2} + \frac{e^{z/2}-e^{-z/2}}{2}\right]$

$= \frac{e^{(z/2)}(z)}{2}\left[e^{z/2} + e^{-z/2} + e^{z/2} - e^{-z/2}\right]$

$= \frac{2e^{z/2}}{2} e^{z/2}(z)$

$\therefore f(z) = ze^z.$

6. Find the analytic function $f(z)$ if its real part is

(11) $u(x,y) = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]$.

Solution :-

Given $u(x,y) = e^{-x}[(x^2-y^2)\cos y + 2xy \sin y] \rightarrow (1)$

using the formula,

$$f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

Substitute $x = z/2$, $y = -iz/2$ in (1).

we get,

$$u(x,y) = e^{-z/2}\left[\left(\frac{z^2}{4} - \left(\frac{-iz}{2}\right)^2\right)\cos\left(\frac{-iz}{2}\right) + 2\left(\frac{z}{2}\right)\left(\frac{-iz}{2}\right)\sin\left(\frac{-iz}{2}\right)\right]$$

$$= e^{-z/2}\left[\left(\frac{z^2}{4} + \frac{z^2}{4}\right)(\cosh z/2) - \frac{iz^2}{2}(-i)\left(\sin h\frac{z}{2}\right)\right]$$

$$= e^{-z/2}\left[\frac{z^2}{2}\cosh(z/2) - \frac{z^2}{2}\sinh\left(\frac{z}{2}\right)\right]$$

$$= \frac{z^2}{2} e^{-z/2}\left[\cosh(z/2) - \sinh(z/2)\right]$$

$$= \frac{z^2}{2} e^{-z/2}\left[\left(\frac{e^{z/2}+e^{-z/2}}{2}\right) - \left(\frac{e^{z/2}-e^{-z/2}}{2}\right)\right]$$

$$= \frac{z^2}{2} e^{-z/2}\left(e^{-z/2}\right).$$

$$u(x,y) = \frac{z^2}{2} e^{-z}$$

$$\therefore f(z) = 2u\left[z/2, -iz/2\right]$$

$$= 2\left[z^2/2\, e^{-z}\right]$$

$$\therefore f(z) = z^2 e^{-z}.$$

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1. Find the analytic function if $e^{-x}(x\sin y - y\cos y)$ is its real part.

Solution :-

Given $u(x,y) = e^{-x}(x\sin y - y\cos y) \longrightarrow (1)$

using the formula,

$$f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right).$$

Substitute $x = \frac{z}{2}$, $y = -\frac{iz}{2}$ in (1).

$$u\left(\frac{z}{2}, -\frac{iz}{2}\right) = e^{-z/2}\left[\frac{z}{2}\sin\left(-\frac{iz}{2}\right) - \left(-\frac{iz}{2}\right)\cos\left(-\frac{iz}{2}\right)\right]$$

$$= e^{-z/2}\left[\frac{z}{2}(-i)\sinh\left(\frac{z}{2}\right) + \left(\frac{iz}{2}\right)\cos\left(\frac{hz}{2}\right)\right]$$

$$= \frac{z}{2}e^{-z/2}\left[-i\sin\left(\frac{hz}{2}\right) + i\cos\left(\frac{hz}{2}\right)\right]$$

$$= \left(\frac{iz}{2}\right)e^{-z/2}\left[-\left(\frac{e^{z/2} - e^{-z/2}}{2}\right) + \left(\frac{e^{z/2} + e^{-z/2}}{2}\right)\right]$$

$$= \left(\frac{iz}{2}\right)e^{-z/2}\left[\frac{-e^{z/2}}{2} + \frac{e^{-z/2}}{2} + \frac{e^{z/2}}{2} + \frac{e^{-z/2}}{2}\right]$$

$$= \left(\frac{iz}{2}\right)e^{-z/2}\left[\frac{2e^{-z/2}}{2}\right]$$

$$= \left(\frac{iz}{2}\right)e^{-z/2}\left[e^{-z/2}\right]$$

$$u\left(\frac{z}{2}, -\frac{iz}{2}\right) = \left(\frac{iz}{2}\right)e^{-z}$$

$$\therefore f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right).$$

$$= 2\left(\frac{iz}{2}e^{-z}\right)$$

$$\therefore f(z) = iz(e^{-z}).$$

8. Find the analytic function. $e^x(x\sin y + y\cos y)$ its imaginary part.

** (13)

Solution :-

Given $V(x,y) = e^x(x\sin y + y\cos y) \rightarrow (1)$

using the formula.

$$f(z) = 2iV\left(\frac{z}{2}, \frac{-iz}{2}\right)$$

put $x = \frac{z}{2}$, $y = -\frac{iz}{2}$ in (1).

$$V(x,y) = V\left(\frac{z}{2}, -\frac{iz}{2}\right)$$

$$= e^{z/2}\left[\left(\frac{z}{2}\right)\sin\left(-\frac{iz}{2}\right) + \left(-\frac{iz}{2}\right)\cos\left(-\frac{iz}{2}\right)\right]$$

$$= e^{z/2}\left[\left(\frac{z}{2}\right)(-i)\sinh\left(\frac{z}{2}\right) - \frac{iz}{2}\cosh\left(\frac{z}{2}\right)\right]$$

$$= e^{z/2}\left(-\frac{iz}{2}\right)\left[\sinh\left(\frac{z}{2}\right) + \cosh\left(\frac{z}{2}\right)\right]$$

$$= \left(-\frac{iz}{2}\right)e^{z/2}\left[\left(\frac{e^{z/2} - e^{-z/2}}{2}\right) + \left(\frac{e^{z/2} + e^{-z/2}}{2}\right)\right]$$

$$= \left(-\frac{iz}{2}\right)e^{z/2}\left[\frac{e^{z/2}}{2} - \frac{e^{-z/2}}{2} + \frac{e^{z/2}}{2} + \frac{e^{-z/2}}{2}\right]$$

$$= \left(-\frac{iz}{2}\right)e^{z/2}\left[\frac{2e^{z/2}}{2}\right]$$

$$= \left(-\frac{iz}{2}\right)e^{z/2} \cdot e^{z/2}$$

$$V(x,y) = \left(-\frac{iz}{2}\right)e^{z}$$

$$f(z) = 2iV\left(\frac{z}{2}, -\frac{iz}{2}\right).$$

$$= 2i\left(-\frac{iz}{2} \cdot e^{z}\right)$$

$$\therefore f(z) = ze^{z}.$$

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Method 2:-

9. If $u(x,y) = x^2 - y^2$ then find $f(z)$.

Solution:-

Given $u(x,y) = x^2 - y^2 \rightarrow (1)$

we know that,

If u is given then we can find $f(z)$.

i.e $f(z) = 2u\left(\frac{z+a}{2}, \frac{z-a}{2i}\right)$

$u(0,0) = 0 - 0 = 0$.

$\therefore u(x,y)$ exist.

put $a = 0$.

$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right)$

put $x = \frac{z}{2}$, $y = \frac{z}{2i}$ in (1).

$f(z) = 2\left[\left(\frac{z}{2}\right)^2 - \left(\frac{z}{2i}\right)^2\right]$

$= 2\left[\frac{z^2}{4} + \frac{z^2}{4}\right]$

$= 2\left[\frac{2z^2}{4}\right] = \frac{4z^2}{4} = z^2$.

$\therefore f(z) = z^2$

10. $u = x^3 - 3xy^2$

Solution:-

Given $u(x,y) = x^3 - 3xy^2$

$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right)$

$f(z) = 2\left[\left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2\right]$

$= 2\left[\frac{z^3}{8} - 3\left(\frac{z}{2}\right)\left(\frac{z^2}{-4}\right)\right]$

$= 2\left[\frac{z^3}{8} + \frac{3z^3}{8}\right]$

$= 2\left[\frac{4z^3}{8}\right] = \frac{8z^3}{8}$

$\therefore f(z) = z^3$

11. $u = 2x(1-y) = 2x - 2xy$.

Solution :-

Given $u(x,y) = 2x - 2xy$

we know that,

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right)$$

$$f(z) = 2\left[2\left(\frac{z}{2}\right) - 2\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)\right]$$

$$= 2\left[z - 2\left(\frac{z^2}{4i}\right)\right] \qquad = 2\left[z - \left(\frac{z^2}{2}\right)\left(\frac{1}{i}\right)\right]$$

$$= 2\left[z - \frac{z^2}{2i}\right] \qquad = 2\left[z - \left(\frac{z^2}{2}\right)(-i)\right]$$

$$= 2\left[z + \frac{iz^2}{2}\right] \qquad = \left[\because \frac{1}{i} = -i\right]$$

$$f(z) = 2z + iz^2.$$

---

12. find the analytic function whose real part is $\frac{x}{x^2+y^2}$.

Solution :-

Given $u(x,y) = \frac{x}{x^2+y^2}$

To find : $f(z)$

If u is given

$$f(z) = 2u\left[\frac{z+a}{2}, \frac{z-a}{2i}\right]$$

$u(0,0) = \frac{0}{0}$ = undefined.

we put $a=1$,

$$f(z) = 2u\left(\frac{z+1}{2}, \frac{z-1}{2i}\right)$$

$$= 2\left[\frac{\frac{z+1}{2}}{\left(\frac{z+1}{2}\right)^2 + \left(\frac{z-1}{2i}\right)^2}\right]$$

$$= 2\left[\frac{z+1/2}{\frac{(z+1)^2}{4} + \frac{(z-1)^2}{(-4)}}\right]$$

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$$= 2\left[\frac{4(z+1)}{2(z+1)^2-(z-1)^2}\right]$$

$$= \frac{4(z+1)}{z^2+1+2z-(z^2+1-2z)}$$

$$= \frac{4(z+1)}{z^2+1+2z-z^2-1+2z}$$

$$= \frac{4(z+1)}{4z}$$

$$= \frac{z+1}{z} = \frac{z}{z} + \frac{1}{z}$$

$$\therefore f(z) = 1+\frac{1}{z}.$$

---

13. Find the analytic function whose real part is $u = \frac{1}{2}\log(x^2+y^2)$.

Solution:-

Given $u = \frac{1}{2}\log(x^2+y^2)$

To find :- $f(z)$

If u is given.

$u(0,0) = -\infty$

we want to substitute $a=1$.

$$f(z) = 2u\left(\frac{z+a}{2}, \frac{z-a}{2i}\right)$$

$$f(z) = 2u\left(\frac{z+1}{2}, \frac{z-1}{2i}\right)$$

$$= 2\left\{\frac{1}{2}\log\left[\left(\frac{z+1}{2}\right)^2+\left(\frac{z-1}{2i}\right)^2\right]\right\}$$

$$= \log\left[\frac{z^2+1+2z}{4} + \frac{z^2+1-2z}{(-4)}\right]$$

$$= \log\left[\frac{z^2+1+2z-z^2-1+2z}{4}\right]$$

$$= \log\left(\frac{4z}{4}\right)$$

$$\therefore f(z) = \log z.$$