UNIT - III.

Complex Integration

Section: 8.1 :-

Simple Rectifiable Oriented Curves :-

Region :-

A connected open set is called as region.

Definition :-

Simple arc :-

The set of all points $C = \{z(t), h \le t \le k\}$, where $z(t)$ is a continuous complex function defined in $[h, k]$ is called a simple arc.

[Jordan arc] if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$

Definition :-

Simple closed curve :-

A simple arc is called a simple closed curve [Jordan curve] if the values at end points coincide.

i.e., The set of all points $\{z(t), h \le t \le k\}$, where $z(t)$ is a continuous complex function defined in $[h, k]$ is called simple closed curve [Jordan curve].

If $z(h) = z(k)$ and $t_1 \neq t_2$.

$\Rightarrow z(t_1) \neq z(t_2)$ $(t_1 \neq h, t_2 \neq k)$.

Example : 1 :-

* $z(t) = \cos t + i \sin t$, $0 \le t \le \pi$ is a simple arc which is the upper half of the unit circle.

*. $z(t) = \cos t + i \sin t$, $0 \le t \le 2\pi$ represents a simple closed curve, which is the unit circle.

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Definition :-

Smooth :-

An arc $z(t)$ defined in $[a,b]$ is said to be smooth if $z'(t)$ exist with $z'(t) \neq 0$ and is continuous in $[a,b]$

Definition :-

Orientation of a simple closed curve :-

Let z_0 be a point inside a simple closed curve C. Then C is said to be positively oriented if $\arg\{z(t) - z_0\}$ increases by 2π when t increases from h to k. Otherwise it is said to be negatively oriented.

Example :-

The unit circles,

$z = \cos t + i\sin t \, , \; 0 \leq t \leq 2\pi.$

$z = \cos(-t) + i\sin(-t), \; 0 \leq t \leq 2\pi$

are positively oriented and negatively oriented respectively.

Definition :-

Partition ($\wp$) Norm of ($\wp$), Rectifiable, length of the arc.

Consider a simple arc C, $z = z(t)$, $h \leq t \leq k$. Suppose the closed interval $[h,k]$ is divided into 'n' subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n]$.

where,

$h = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = k.$

Denote this division or partition by $\wp$

The norm of $\wp$ denoted by $|\wp|$ is given by

$$|\wp| = \max\{[t_{i-1}, t_i]\} \text{ where } i = 1, 2, \ldots n.$$

$|\wp| = \max\{[t_{i-1}, t_i]\}$

Definition:-

Rectifiable :-

Suppose $z = z(t)$, $h \le t \le k$ be a simple arc. For each partition $\wp$, $V(\wp)$ is given by,

$$V(\wp) = \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$$

If $\sup_{\wp} \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$ is finite, then C is said to be rectifiable.

Definition:-

Length of the arc :-

If C is the simple arc, then

$\lim_{|\wp| \to 0} \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$ is called length of the arc C and is denoted by $l(C)$.

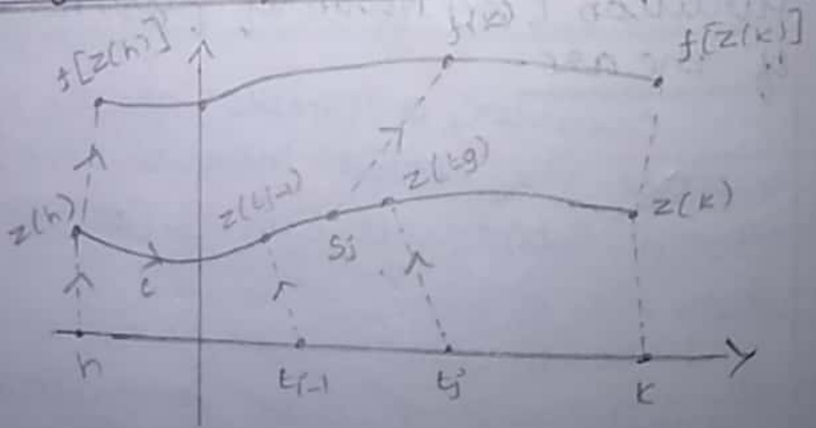
scr curves $\rightarrow$ Simple closed rectifiable curve

scro curves $\rightarrow$ simple closed rectifiable positively oriented curve.

Sec: 8.2 :-

Definition:-

Integration of complex functions:



Let $C: z = z(t)$, $h \le t \le k$, be a simple rectifiable arc. Let $f(z)$ be a continuous function defined on C.

If $\wp$ is a partition of the closed interval $[h, k]$ with subintervals.

$[t_0, t_1], [t_1, t_2], \ldots [t_{n-1}, t_n]$

where $t_0 = h$, $t_n = k$ and if δ_j is any point on the j^{th} subarc with end points $z(t_j - 1)$ and $z(t_j)$ then,

$$\lim_{|\delta| \to 0} \sum_{j=1}^{n} [z(t_j) - z(t_{j-1})] f(\delta_j)$$

is called the integral of $f(z)$ along C and is denoted by

$$\int_C f(z) dz.$$

Definition:-

Contour integral:-

Integrals along ~~scr~~ curves are called contour integrals while the paths are called contours.

Section: 8.3

Simple integrals using definition:-

Result: 1:-

$\int_C dz = b - a$, where $z(t_0) = a$ and $z(t_n) = b$.

proof:-

$$\int_C 1 \cdot dz = \lim_{|\delta| \to 0} \sum_{j=1}^{n} [z(t_j) - z(t_{j-1})](1)$$

$$= \lim_{|\delta| \to 0} z(t_1) - z(t_0) + z(t_2) - z(t_1) + \cdots + z(t_n) - z(t_{n-1})$$

$$= \lim_{|\delta| \to 0} [z(t_n) - z(t_0)]$$

$$\int_C dz = \lim_{|\delta| \to 0} (b - a) = b - a.$$

NOTE:-

1. $\int_C dz = b - a$ for any curve c.
2. $\int_C dz = 0$, if C is closed curve.

Result : 2

$\int_C z\,dz = \frac{1}{2}(b^2-a^2)$ where $z(t_0)=a$, $z(t_n)=b$.

Proof :-

$$\int_C z.dz = \lim_{|\delta|\to 0} \sum_{j=1}^{n} [z(t_j)-z(t_{j-1})]\, z(t_j) \to (1)$$

Again

$$\int_C z\,dz = \lim_{|\delta|\to 0} \sum_{j=1}^{n} [z(t_j)-z(t_{j-1})]\, z(t_{j-1}) \to (2)$$

(1)+(2)

$$2\int_C z.dz = \lim_{|\delta|\to 0} \sum_{j=1}^{n} [z(t_j)^2 - z(t_{j-1})\, z(t_j)] +$$

$$\lim_{|\delta|\to 0} \sum_{j=1}^{n} [z(t_j).z(t_{j-1}) - [z(t_{j-1})]^2]$$

$$2\int_C z.dz = \lim_{|\delta|\to 0} \sum_{j=1}^{n} (z(t_j))^2 - z(t_{j-1})\, z(t_j) + z(t_j)z(t_{j-1}) - (z(t_{j-1}))^2]$$

$$= \lim_{|\delta|\to 0} \sum_{j=1}^{n} [z(t_j)^2 - z(t_{j-1})^2]$$

$$\int_C z\,dz = \frac{1}{2} \lim_{|\delta|\to 0} (b^2-a^2)$$

$$\int_C z.dz = \frac{1}{2}(b^2-a^2)$$

** 5 mark

Theorem : 8.3 :-

If C is the positively oriented circle whose radius is r and centre is $z=a$ then:

$$\int_C \frac{1}{z-a}\,dz = 2\pi i$$

Proof :-

Given C is the positively oriented circle whose radius is r and centre is $z=a$. i.e, $|z-a| = r$.

$\Rightarrow z-a = re^{i\theta}, \quad 0 \le \theta \le 2\pi$

$z = a + re^{i\theta}, \quad 0 \le \theta \le 2\pi$ $\quad z - a = (a + re^{i\theta}) - a$

We know that,

If C is smooth, then

$$\int_C f(z)dz = \int_C f(z(t))z'(t)dt.$$

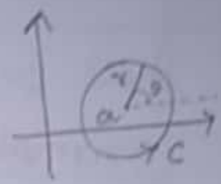
$$\therefore \int_C \frac{dz}{z-a} = \int_0 \frac{d(a+re^{i\theta})}{(a+re^{i\theta})-a}\, d\theta$$

$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{d(a+re^{i\theta})}{re^{i\theta}}\, d\theta$$

$$= \int_0^{2\pi} \frac{re^{i\theta}\, i}{re^{i\theta}}\, d\theta$$

$$= \int_0^{2\pi} i\, d\theta$$

$$= i\int_0^{2\pi} d\theta$$



$$= i\,[\theta]_0^{2\pi}$$

$$= (2\pi - 0)\, i$$

$$= 2\pi i$$

$$\int_C \frac{dz}{z-a} = 2\pi i.$$

Section : 8.5 :-

Interior and exterior of a closed curve:

Definition :-

If C is a simple closed curve then the area bounded by C is called the interior of C and denoted by C_i. The unbounded area is called exterior of C denoted by C_e.

$$C = C_i \cap C_e$$

$$C^* = C \cup C_i$$

↳ closed region.

Section : 8.6 :-

Simply - Connected Region :

Definition :-

A region D is said to be simply connected if for every closed curve C in D. C_i is contained in D.

Note :-

The exterior of any closed curve is not simply connected.

Definition :-

Multiply - connected Region :-

A region which is not simply-connected is called multiply - connected region.

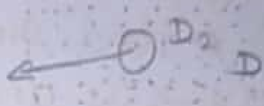
Definition :-

Annular region :-

A multiply connected region which has only one hole is called annular region.

Example :-

$|z-a|<r$, $r<|z-a|<R$.



annular region

Section : 8.7 State and prove

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Cauchy's Fundamental theorem :-

Theorem : 8.4 :-

State and prove cauchy's fundamental theorem:

Suppose that :-

i) D is a simply - connected region

ii) $f(z)$ is analytic in D and $f'(z)$ is continuous in D.

iii) C is an scr curve in D. simply closed rectifiable

Then

$$\int_C f(z)\,dz = 0.$$

$f(z)=u+iv$ $z=x+iy$ $dz=dx+idy$

proof :-

Let $f(z) = u(x,y) + iv(x,y)$

we know that

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx) \rightarrow (1)$$

By (ii), u_x, u_y, v_x, v_y exist and all are continuous.

$z=x+iy$
$dz=dx+idy$
$f(z)=u+iv$

By Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_C (Q_x - P_y)\,dx\,dy$$

$$\int_C u\,dx - v\,dy = \iint_C (-v_x - u_y)\,dx\,dy$$

$(u\,dx - v\,dy) + i(u\,dy + v\,dx)$

$$= \iint_C (-V_x + V_x)\, dx\, dy$$

$$\int_C u\, dx - v\, dy = 0.$$

Similarly,

$$\int_C V \cdot dx + u \cdot dy = \iint_C (u_x - V_y)\, dx\, dy \quad [\because u_x = V_y]$$

$$\therefore \int_C V dx + u dy = 0.$$

From (1)

$$\int_C f(z)\, dz = 0.$$

** remark

Goursat's Lemma :-

Suppose that

i) D is a simply-connected region

ii) $f(z)$ is analytic in D and

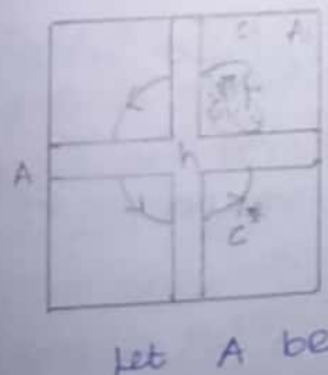
iii) C is an scr curve in D.

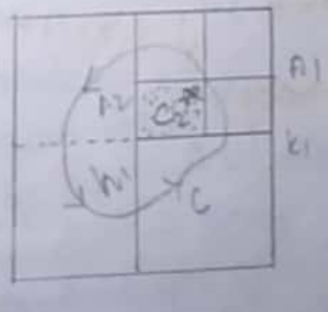
Then C^* can be partitioned into a finite number of closed sub-region, full or truncated squares, such that each subregion, satisfies the property P that, for any $\epsilon > 0$, the region has an 'a' with

$$\left| \frac{f(z) - f(a)}{z - a} - f'(a) \right| < \epsilon \text{ for all } z \text{ in that region.}$$

Proof :-

Suppose that D cannot be partitioned into a finite number of closed subregions such that each subregion satisfies P.





Let A be a square, enclosing C^* with its sides parallel to the real and imaginary axis.

Let the projection of this square on real axis be the closed interval $[h, k]$

Hence the side of the square is $k - h$.

Dividing A again into four smaller squares. Then C^* is also partitioned into 4 closed ~~xx~~ subregions and atleast one of them cannot satisfy the property P.

Let that particular subregion be C_1^*. Let C_1^* be contained in the smaller square A_1. Let the projection of A_1 on real axis be $[h_1, k_1]$.

clearly $k_1 - h_1 = \frac{k-h}{2^1} = \frac{k-h}{2}$.

If this procedure is repeated then we get a closed subregion C_n^* which does not have property P, contained in the square A_n such that

$$k_n - h_n = \frac{k-h}{2^n}$$

Clearly,

$$\lim_{n\to\infty} (k_n - h_n) = 0 \text{ and so}$$

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} h_n = \alpha \text{ (say)}$$

Similarly, if the projection is done on imaginary axis, when the process is proceeded for n times, we obtain the interval $[l_n, m_n]$ such that

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} m_n = \beta \text{ (say)}$$

Consider the complex number $\alpha + i\beta$

clearly,

$$\alpha+i\beta \in C^*, C_1^*, C_2^*, \ldots, \text{ and so on.}$$

$$\alpha+i\beta \in A, A_1, A_2, \ldots$$

As $f(z)$ is analytic in D, $a = \alpha + i\beta \in D$ and so $f'(a)$ exist

i.e, for $\varepsilon > 0$ $\exists$ a $\delta > 0$ $\ni$

$$\left| \frac{f(z)-f(a)}{z-a} - f'(a) \right| < \varepsilon, \ \forall \ |z-a| < \delta.$$

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i.e the disc $\Gamma : |z-a|$ has the property P.

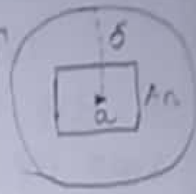
∴ It is clear from the above.

If n is large enough, then An will be contained within Γ.

Hence $c_n^* \subseteq$ An satisfies P.
which is a contradiction.

c^* can be partitioned into a finite number of closed subregions.

Hence the lemma.



Theorem: 8.5 :-

Cauchy - Goursat's lemma :-

*
Imp

Suppose

i) D is a simply - connected region

ii) f(z) is analytic in D and

iii) C is an SCT curve in D

Then $\int_C f(z)\,dz = 0$.

Extension to cauchy's fundamental theorem :-

**
Theorem : 8.6 :-

Imp
Suppose :-

i) D is an annulus region

ii) f(z) is analytic in D and

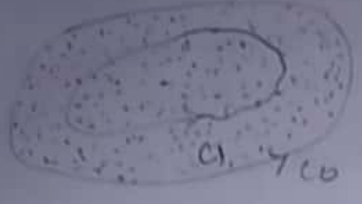
iii) C_0, C_1 are SCTO curves such that

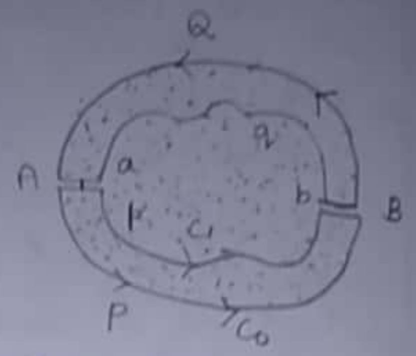
1. C_1 lies in the interior of C_0.

2. The annulus region between C_0 and C_1 does not contain a hole while the interior of C_1 contains holes.

Then, $\int_{C_0} f(z)\,dz = \int_{C_1} f(z)\,dz$.

Proof:-





Taking points A, P, B, Q on C_0 and a, p, b, q on C_1 join Aa and Bb.

$f(z)$ is analytic on and in the interior of the curves APBbpaA and BQAaqbB.

Then by cauchy's theorem,

Integral along APBbpaA $=0$.

Integral along BQAaqbB $=0$.

Adding the above two integrals and using the condition.

$$\int_{Aa} + \int_{aA} = 0, \quad \int_{Bb} + \int_{bB} = 0.$$

Integral along APBQA + Integral along aqbpa $=0$.

Integral along C_0 + Integral along $(-C_1)=0$

$$\Rightarrow \int_{C_0} f(z)\,dz - \int_{C_1} f(z)\,dz = 0.$$

$$\therefore \int_{C_0} f(z)\,dz = \int_{C_1} f(z)\,dz.$$

10 mark

Section : 8.8 :-

Integral along an arc joining two points :-

Theorem : 8.7 :-

Suppose $f(z)$ is an analytic function in a simply connected region D and A(a) and B(b) are two points in D let C be any arbitrarily chosen simple rectifiable arc in D oriented from A to B.

Then the integral

$$\int_C f(z)\,dz$$

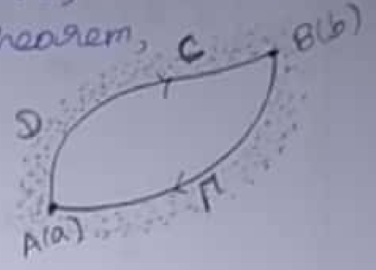
does not depend on c but depends on a and b.

proof :-

Suppose Γ is a fixed simple rectifiable arc oriented from B to A.

As $C+\Gamma$ is a SCR curve,

By Cauchy's fundamental theorem,



$$\int_{C+\Gamma} f(z)\,dz = 0.$$

$$\Rightarrow \int_C f(z)\,dz + \int_\Gamma f(z)\,dz = 0.$$

$$\Rightarrow \int_C f(z)\,dz = -\int_\Gamma f(z)\,dz \text{ is a constant}$$

The LHS independent of 'c' and depends on the points A(a) and B(a) as in definite integrals $\int_a^b f(z)\,dz$.

Note :-

$\int_C f(z)\,dz$ is written as $\int_a^b f(z)\,dz$ if a,b are the end points of C, where the orientation if C is from A to B.

Section : 8.4 :-

Definite Integrals :-

Definition :-

Let C be a simple rectifiable curve $C: z = z(t)$, $h \le t \le k$ in a region D, with end points a, b. Suppose that $f(z)$ and $F(z)$ are two functions defined in D such that

$$\frac{d}{dz}[F(z)] = f(z).$$

Further, let $z'(t)$ be continuous in $[h,k]$ then $\int_C f(z)\,dz$ is denoted by $\int_a^b f(z)\,dz$ and is given by $\int_a^b f(z)\,dz = F(b) - F(a)$.

Examples :-

As $\frac{d}{dz}\left(\frac{z^{n+1}}{n+1}\right)$, $n \neq -1$ is z^n.

$$\int_c z^n dz = \int_a^b z^n dz = \left.\frac{z^{n+1}}{n+1}\right|_a^b$$

$$= \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$$\therefore \int_c z^n dz = \frac{1}{n+1}\left[b^{n+1} - a^{n+1}\right]$$

If a and b are any two points in the plane.

$$\int_a^b e^z dz = e^b - e^a, \quad \int_a^b \cos z . dz = \sin b - \sin a$$

$$\int_a^b \sin z dz = -\cos z \Big/_a^b = -\left[\cos b - \cos a\right]$$

$$= \cos a - \cos b.$$

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