# UNIT - IV

## Taylor's and Laurent's Series

### Sec: 9.1 :- Taylor's Series :- Define $f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$

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Given a function $f(z)$ and a point $z=a$, if `c' is the largest circle with centre at $z=a$ such that $f(z)$ is analytic in $C_i$, then $f(z)$ can be expanded as a series of non-negative integral powers which is convergent for all $z$ in $C_i$. This series is called Taylor's series for $f(z)$ about $z=a$.

Define Laurent's series

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ where $a_n = \frac{1}{2\pi i}\int \frac{f(t)}{(t-a)^{n+1}}\,dt$

Given a function $f(z)$ and a point $z=a$, if `$C_1$' is the largest circle and `$C_2$' is the smallest circle, both having their centres at $z=a$, such that $f(z)$ is analytic in annular regions between $C_1$ and $C_2$, then $f(z)$ can be expressed as a series of positive and negative integral powers of $z-a$ which is convergent for all $z$ in the annular region. This series is called Laurent's series of $f(z)$ about $z=a$.

### Theorem: 9.1 :- Taylor's series :-

Suppose

i) $D$ is a simply connected region

ii) $f(z)$ is analytic in $D$ and

iii) 'a' is a point in $D$

iv) $\Gamma$ is the largest circle whose interior lies in $D$.

Then, for all $z$ in $\Gamma_i$, the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$$

is absolutely convergent and its sum is $f(z)$.

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Proof:-

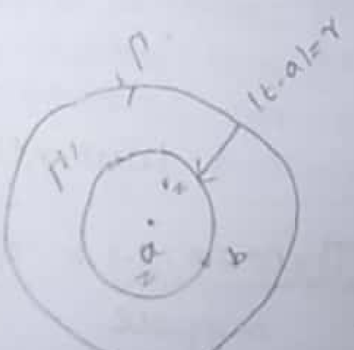
Let $\Gamma'$ is the positively oriented circle $|t-a|=r$, lying in $\Gamma$, where $z=t$ is any point on the circle and $z=a$ is a constant.

Then from Cauchy's integral formula and formula for derivatives,

$$\left.\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_{\Gamma'} \frac{f(t)}{t-a}\,dt \\
f'(a) &= \frac{1!}{2\pi i}\int_{\Gamma'} \frac{f(t)}{(t-a)^2}\,dt \\
f''(a) &= \frac{2!}{2\pi i}\int_{\Gamma'} \frac{f(t)}{(t-a)^3}\,dt \\
&\vdots \\
f^n(a) &= \frac{n!}{2\pi a}\int_{\Gamma'} \frac{f(t)}{(t-a)^{n+1}}\,dt
\end{aligned}\right\} \longrightarrow (1)$$

We know that, the geometric series,

$1+y+y^2+\cdots=(1-y)^{-1}$ with $|y|<1$ is absolutely convergent




Now, for any $z$ in $\Gamma'$, and for any $t$ on $\Gamma'$,

$|z-a| < |t-a|$

$$\Rightarrow \frac{|z-a|}{|t-a|} < 1$$

$\therefore$ The geometric series,

$$1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots=\left[1-\left(\frac{z-a}{t-a}\right)\right]^{-1} \text{ is}$$

absolutely convergent.

$$\Rightarrow 1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots = \left[\frac{t-a-z+a}{t-a}\right]^{-1}$$

$$1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots = \left[\frac{t-z}{t-a}\right]^{-1}$$

$$\Rightarrow 1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots = \frac{t-a}{t-z}$$

Multiplying both sides by $\frac{1}{2\pi i}\left(\frac{f(t)}{t-a}\right)$.

$$\left.\begin{aligned}&\frac{1}{2\pi i}\left(\frac{f(t)}{t-a}\right)+\frac{1}{2\pi i}\frac{f(t).(z-a)}{(t-a)^2}+\\ &\quad\frac{1}{2\pi i}\frac{f(t).(z-a)^2}{(t-a)^3}+\cdots\end{aligned}\right\} = \frac{1}{2\pi i}\left(\frac{f(t)}{t-a}\right)\left(\frac{t-a}{t-z}\right)$$

$$\left.\begin{aligned}&\frac{1}{2\pi i}\frac{f(t)}{t-a}+\frac{1}{2\pi i}\frac{f(t).(z-a)}{(t-a)^2}+\\ &\quad\frac{1}{2\pi i}\frac{f(t).(z-a)^2}{(t-a)^3}+\cdots\end{aligned}\right\} = \frac{1}{2\pi i}\frac{f(t)}{t-z}.$$

Integrating term by term along $\Gamma'$.

$$\left.\begin{aligned}&\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(t)}{t-a}dt+\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(t)}{(t-a)^2}dt\,(z-a)\\ &\quad+\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(t)}{(t-a)^3}(z-a)^2dt+\cdots\end{aligned}\right\} = \frac{1}{2\pi i}\int_{\Gamma'}\frac{f(t)}{t-z}dt.$$

$$f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots = f(z) \longrightarrow (2)$$

By (1)

i.e, $\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(z-a)^n = f(z)$

$\therefore$ The series on the left is absolutely convergent in $\Gamma'$.

$\therefore$ The series is called Taylors series for $f(z)$ about $z=a$.

Hence the proof.

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## Maclaurin's Series :-

If a function $f(z)$ is analytic in an open circular disc with centre at $z=0$, then for any $z$ in the disc, the series,

$$f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}(z^2)+\cdots=f(z).$$

coverge to $f(z)$. This series is called Maclaurin's series.

Proof :-

By using Taylor's series,

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots$$

Given $z=0$,

Let us take $z-a=0$.

Since $z=0$, $a=0$.

Substitude '$z$' and '$a$' in above series, we get

$$f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots=f(z).$$

Hence the thearem.

## Region of Validity : The Result :-

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots$$

is Valid for all in $\Gamma$ that is, if $z$ is replaced, on both sides, by any complex numbers in $\Gamma$, then the result holds good. that is, the series converges to the sum, if the same $f(z)$ is to be expanded in a series of powers of $z-b$, that is, if $f(z)$ is to be expanded in a series of powers of $z-b$, that is, if $f(z)$ is to be expanded in a series about $z=b$, then we should select the largest circle $c'$ with centre at $z=b$ & $f(z)$ is analytic in $c'$ in this case, we get

$$f(z)=f(b)+\frac{f'(b)}{1!}(z-a)+\frac{f''(b)}{2!}(z-a)^2+\cdots$$

which is valid for all $z$ in $c$.

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**Remark : 1**

In general, if a function $f(z)$ is analytic in an open circular disc with centre at $z=a$, then for any $z$ in that disc, the series

$$f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^n(a)}{n!}(z-a)^n+\cdots$$

converges to $f(z)$ —

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**Remark : 2 : Maclaurin's Series :-**

Maclaurin's series is the expansion of $f(z)$ about $z=0$. ie, in powers of $z=0$ if a function $f(z)$ is analytic in an open circular disc with centre at $z=0$, then for any $z$ in that disc, the series.

$$f(0)+\frac{f'(0)}{1!}(z)+\frac{f''(0)}{2!}z^2+\cdots$$

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**Series of simple Function :-**

By using Taylor's theorem, we get for all z's except $z=\infty$

$$e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots$$

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots$$

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$$

$$\cos(hz) = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots$$

$$\sin(hz) = \frac{z}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$$

The binomial series is,

$$(1+z)^n = 1+\frac{n}{1!}z+\frac{n(n-1)}{2.1}z^2+\frac{n(n-1)(n-2)}{3.2.1}z^3+\cdots$$

which is valid when $|z|<1$.

The logrithmic series is,

$$\log(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots$$

which is valid when $|z|<1$ and when $z=1$ but not when $z=-1$.

## Steps to find a Taylor's series for $f(z)$:-

Suppase the expansion needed is about the point $z=a$, then

1. Find the singular points of $f(z)$.
2. Find the biggest circle $C$ with centre at $z=a$ and passing through the singular point nearest to $z=a$.
3. Expand $f(z)$ as a Taylor's series for all $z$ in $C_i$.

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## Theorem: 9.2:-

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### Uniqueness Theorem.

If $f(z)$ is analytic in an open circular disc with centre at $z=a$, then the Taylor's series about $z=a$ is the only power series in $z-a$ which converges to $f(z)$ in that disc.

Proof:-

The Taylor's series about $z=a$ for $f(z)$ in the disc is,

$$f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\dots=f(z) \longrightarrow (1)$$

Suppose there is another series (say)

$$a_0+a_1(z-a)+a_2(z-a)^2+\dots=f(z) \longrightarrow (2).$$

which converges in that disc for $f(z)$.

Differentiating (2) term by term and successively,

$$a_1+2\cdot a_2(z-a)+3a_3(z-a)^2+\dots=f'(z).$$

$$2\cdot 1\, a_2+2\cdot 3\, a_3(z-a)+4\cdot 3\, a_4(z-a)^2+\dots=f''(z).$$

$3!a_3 + 4.3.2.a_4(z-a) + \cdots = f'''(z)$.

Setting $(z=a)$ in these above series and in (2),

$$a_0 = f(a)$$
$$a_1 = \frac{f'(a)}{1!}$$
$$a_2 = \frac{f''(a)}{2!}$$
$$a_3 = \frac{f'''(a)}{3!}$$
$$\vdots \qquad \vdots$$

Then (2) becomes,

$$f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots = f(z).$$

which is a Taylor's series for $f(z)$.

Hence, Taylor's series is only power series in $(z-a)$ which converges to $f(z)$ in the disc.

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Remark :-

Different Taylor's series for the same function :-

If $f(z)$ is analytic in the neighbourhoods $N(a)$ and $N(b)$ of $z=a$ and $z=b$ then

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \quad \forall z \in N(a)$$

$$f(z) = f(b) + \frac{f'(b)}{2!}(z-b) + \frac{f''(b)}{2!}(z-b)^2 + \cdots \quad \forall z \in N(b).$$

## Sec: 9.2 Zeros of an analytic function

**Definition:**

**zero of f(z) of order k:-**

If a function $f(z)$ is such that !

ie $f(z) = (z-a)^k \phi(z)$.

where 'k' is a positive integer and $\phi(a) \neq 0$ then 'a' is said to be zero of $f(z)$ of order k or of multiple k.

Eg:- $f(z) = (z-1)^2 (z+1)^3 \cdot e^z$ has a zero at one is of order 2 and at zero -1 of order 3.

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**NOTE:-**

If $f(z) = (z-a)^m [a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots], a \neq 0$. then $z=a$ is zero of $f(z)$ of order m.

zeros of an analytic function are isolated.

The order of a zero can be obtained by expanding the function about the zero.

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**Example: 1**

Find the zero of $f(z) = \sin z$.

Solution:-

Taylor's series for $f(z) = \sin z$ about $z=0$ is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\sin z = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)$$

$\therefore z=0$ is a zero of $\sin z$ of order one.

Example : 2

The function $e^z-1$ has a zero $z=0$ of order 1.

Also.

$$e^z-1 = e^{2\pi i}\left[e^{z-2\pi i} - e^{-2\pi i}\right]$$

$$= e^{2\pi i}\left[\left(1+\frac{z-2\pi i}{1!}+\frac{(z-2\pi i)^2}{2!}+\cdots\right)-1\right]$$

$$= (z-2\pi i)\left[\frac{e^{2\pi i}}{1!}+\frac{e^{2\pi i}(z-2\pi i)}{2!}+\cdots\right]$$

$e^z-1$ has a zero of $z=2\pi i$ of order 1.

$-2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots, \infty$ are also its zeros.

$\infty$ is the limit point of the set of these zeros.

Theorem : 9.3:-

Suppose $f(z)$ is analytic in a region $D$ and $z_n$, $n=1,2,3,\ldots$ in $D$ are the zeros of $f(z)$, where the sequence $\{z_n\}$ converges to a limit $z=a$ in $D$. Then $f(z)$ vanishes identically in $D$.

Proof :-

Suppose that the Taylor's Series of $f(z)$ about $z=a$ is

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

which is true for all $z \in |z-a| < R$ in $D \longrightarrow (1)$

clearly $f(z)$ is continuous at $z=a$ as $f(z)$ is analytic in $D$.

Hence as $z_n \to a$, $f(z_n) \to f(a)$

$\Rightarrow f(a)=0$ $\quad$ [$\because$ given $z_n$'s zeros $f(z_n)=0 \Rightarrow f(a)=0$]

From (1)

$\Rightarrow a_0 = 0$

$$f(z) = (z-a)\left[a_1 + a_2(z-a) + \cdots\right]$$

$$= (z-a)\,\phi(z) \text{ say}$$

Since $z_n$'s are zeros, $z_n \neq a$.

$$f(z_n) = 0 \Rightarrow \phi(z_n) = 0.$$

again as $f(z)$ is continuous $\phi(z)$ is continuous and Hence $\phi(a) = 0$.

$$\therefore a_1 = 0$$

proceeding similarly, we get

$$a_2 = a_3 = \cdots = 0.$$

Thus, $f(z) = 0$ in the disc.

Suppose $z = b$ is an arbitrary point in D.

$\ni b \notin |z-a| < R$.

Join the points $a, b$ by a rectifiable curve L

$\ni$ L completely lies in D.

Let d be the minimum distance of L from the bounded.

Let L intersects $|z-a| = R$ at $a_1$ (say)

Now, consider the disc $|z-a_1| < d$.

Since, $a_1$ is the intersection points of $|z-a| < R$ and $|z-a_1| < d$ and as $a_1$ is the bounded point $\ni$ a sequence of points lying on both the discs $\ni$ the sequence converges to $a$.

As, $f(a_1) = 0$, $f(z) = 0$ in $(z-a_1) < d$.

proceeding in this manner,

It is obvious that L is covered by finite number of discs and one of such discs contains $z = b$.

Hence $f(b) = 0$.

As, $z = b$ is arbitary.

$\therefore f(z) = 0$ in the whole of D.

## Theorem: 9.4:-

### Uniqueness Theorem.

Suppose $f_1(z)$ and $f_2(z)$ are two functions in a region D such that, for all $z_n$, $f_1(z_n) = f_2(z_n)$, where $\{z_n\}$ is a sequence of points in D converging to a point in D. Then $f_1(z) \equiv f_2(z)$ in D.

Proof:- If $f(z) = f_1(z) - f_2(z)$.

Then it is obvious that $\{z_n\}$ is a converging sequence of zeros of $f(z)$ in D.

By the theorem.

" Suppose $f(z)$ is analytic in a region D and $\{z_n\}$, $n = 1, 2, 3, \dots$ in D are the zeros of $f(z)$. Where the sequence $\{z_n\}$ converges to a limit $z = a$ in D, then $f(z)$ vanishes identically in D.

$$f(z) = 0$$

$$f_1(z) - f_2(z) = 0$$

$$\Rightarrow f_1(z) = f_2(z).$$

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## Sec: 9.3:- Laurent's series.

### Definition:-

### Laurent's Series:-

Suppose the radii of convergence of the power series.

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

$$b_0 + b_1(z-a) + b_2(z-a)^2 + \dots$$

are $R_1, R_2 (R_2 > R_1)$ then the series are absolutely convergent in $|z-a| < R_1$, and the series $|z-a| < R_2$ respectively and the series

$$a_0 + \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \dots$$ is absolutely convergent

in

$$\left|\frac{1}{z-a}\right| < R_1 \text{ (or) in } |z-a| > R_1,$$

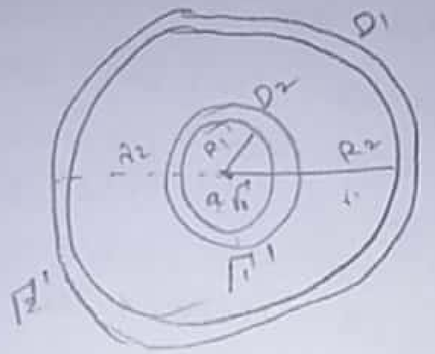
Thus the combined series,

$$\left[a_0 + \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \cdots\right] + \left[b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots\right]$$

is absolutely convergent in the annular

$$R_1 < |z-a| < R_2$$

This type of expansion with positive and negative powers of $z-a$ is called the Laurent's series about $z=a$.



## Theorem : 9.5 :-

### Laurent's series

Suppose

i) $D$ is a multiply-connected region

ii) $f(z)$ is analytic in $D$ and

iii) $\Gamma_1 : |t-a| = R_1$ and $\Gamma_2 : |t-a| = R_2$ are the smallest and biggest circles such that, the annulus region $D_1$ between $\Gamma_1$ and $\Gamma_2$ lies in $D$.

Then for all $z$ in $D_1$, the power series.

$$\sum_{-\infty}^{\infty} a_n (z-a)^n, \text{ where } a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\, dt.$$

and $C$ is any scro curve in $D_1$ (encircling $z=a$), is absolutely convergent and its sum is $f(z)$.

Proof :-

Let $\Gamma_1' : |t-a| = r_1$ and

$\Gamma_2' : |t-a| = r_2$ be two positively oriented circles in D as in the diagram.

Now, by extension to Cauchy integral formula for any $z \in D_2$.

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_2'} \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_{\Gamma_1'} \frac{f(t)}{t-z}\,dt \rightarrow (1).$$

<u>Evaluation of the integral along $\Gamma_2'$ :-</u>

Now, for any point $z \in D_2$ between $\Gamma_1'$ and $\Gamma_2'$ and for any $t$ on $\Gamma_2'$.

$$|z-a| < |t-a|$$

$$\Rightarrow \frac{|z-a|}{|t-a|} < 1$$

$$\Rightarrow \left|\frac{z-a}{t-a}\right| < 1$$

We know that, if $|y|<1$, then the geometric series $1+y+y^2+\dots = (1-y)^{-1}$ is absolutely convergent. Thus series,

$$1 + \left(\frac{z-a}{t-a}\right) + \left(\frac{z-a}{t-a}\right)^2 + \dots = \left[1 - \left(\frac{z-a}{t-a}\right)\right]^{-1}$$

is absolutely convergent.

$$\therefore 1 + \left(\frac{z-a}{t-a}\right) + \left(\frac{z-a}{t-a}\right)^2 + \dots = \left[\frac{t-a-z+a}{t-a}\right]^{-1}$$

$$= \left[\frac{t-z}{t-a}\right]^{-1}$$

$$\Rightarrow 1 + \left(\frac{z-a}{t-a}\right) + \left(\frac{z-a}{t-a}\right)^2 + \dots = \frac{t-a}{t-z}$$

Multiplying both sides by $\frac{1}{2\pi i} \cdot \frac{f(t)}{t-a}$, we get

$$\frac{1}{2\pi i}\frac{f(t)}{t-a} + \frac{1}{2\pi i}\frac{f(t)}{(t-a)^2}(z-a) + \dots = \frac{1}{2\pi i}\frac{f(t)}{(t-a)}\frac{(t-a)}{(t-z)}$$

$$\Rightarrow \frac{1}{2\pi i}\left(\frac{f(t)}{t-a}\right) + \frac{1}{2\pi i}\frac{f(t)}{(t-a)^2}(z-a) + \dots = \frac{1}{2\pi i}\frac{f(t)}{t-z}$$

By integrating term by term along $\Gamma_2'$.

$$\Rightarrow \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_{\Gamma'} \frac{f(t)}{(t-a)^{n+1}}\,dt = \frac{1}{2\pi i}\,\frac{f(t)}{t-z}\,dt \rightarrow (2)$$

The series on the left is absolutely convergent

<u>Evaluation of the integral along $\Gamma_1'$:-</u>

Now for the same $z \in D_2$ and for any $t$ on $\Gamma_1'$

$|t-a| < |z-a|$

$\Rightarrow \frac{|t-a|}{|z-a|} < 1$

$\Rightarrow \left|\frac{t-a}{z-a}\right| < 1$

Then the geometric series,

$1+\left(\frac{t-a}{z-a}\right)+\left(\frac{t-a}{z-a}\right)^2+\cdots = \left[1-\left(\frac{t-a}{z-a}\right)\right]^{-1}$ is absolutely convergent.

$$\therefore 1+\left(\frac{t-a}{z-a}\right)+\left(\frac{t-a}{z-a}\right)^2+\cdots = \left[\frac{z-a-t+a}{z-a}\right]^{-1}$$

$$= \left[\frac{z-t}{z-a}\right]^{-1}$$

$$1+\left(\frac{t-a}{z-a}\right)+\left(\frac{t-a}{z-a}\right)^2+\cdots = -\left[\frac{z-a}{t-z}\right]$$

Multiplying both sides by $\frac{1}{2\pi i}\,\frac{f(t)}{z-a}$.

$$\frac{1}{2\pi i}\left(\frac{f(t)}{z-a}\right)+\frac{1}{2\pi i}\,\frac{f(t)}{(z-a)^2}(t-a)+\cdots = \frac{-1}{2\pi i}\,\frac{f(t)}{t-z}$$

$$\Rightarrow \frac{1}{2\pi i} f(t)(z-a)^{-1}+\frac{1}{2\pi i}\,\frac{f(t)}{(t-a)^{-1}}(z-a)^{-2}+\cdots = \frac{-1}{2\pi i}\,\frac{f(t)}{t-z}$$

Integrating term by term along $\Gamma_1'$

$$\Rightarrow \frac{1}{2\pi i}\int_{\Gamma_1'} f(t)(z-a)^{-1}dt+\frac{1}{2\pi i}\int_{\Gamma_1'}\frac{f(t)}{(t-a)^{-1}}(z-a)^{-2}dt+\cdots$$

$$= \frac{-1}{2\pi i}\int_{\Gamma_1'}\frac{f(t)}{z-t}\,dt.$$

$$\Rightarrow \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{\Gamma_1'}\frac{f(t)}{(t-a)^{n+1}}\,dt = \frac{-1}{2\pi i}\int\frac{f(t)}{t-z}\,dt \rightarrow (3)$$

The series on the left is absolutely convergent

Now,

$$\sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(t)}{(t-a)^{n+1}}\,dt+\sum_{n=(-1)}^{-\infty}(z-a)^n\frac{1}{2\pi i}$$

$$\int_{\Gamma_1'}\frac{f(t)}{(t-a)^{n+1}}\,dt \rightarrow (4)$$

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Now, C is any closed contour between $\Gamma_1'$ and $\Gamma_2'$ then by extension to Cauchy's fundamental theorem,

$$\int_{\Gamma_2'} = \int_C \text{ and } \int_C = \int_{\Gamma_1'}$$

Then (#) becomes,

$$(\#) \Rightarrow f(z) = \sum_{n=0}^{\infty} (z-a)^n \cdot \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$$

$$+ \sum_{n=-1}^{-\infty} (z-a)^n \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt.$$

$$f(z) = \sum_{n=-\infty}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt.$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

where, $a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt.$

Since, $r_1$ and $r_2$ arbitrary.

This result is true for all z in D and for all C in $D_1$.

## Steps to find Laurent's series for $f(z)$:-

* Find the point $z=a$ about which $f(z)$ is to be expanded. Here the expansion will be in negative integral powers of $z-a$ [with or without positive integral powers of $z-a$]
* Find the singular points of $f(z)$.
* Consider all the concentric circles having their centres at $z=a$ and each passing through a singular point.
* Expand $f(z)$ for all z in each annular region in between every two consecutive circles.

problems :

## Cauchy's Formula for Derivatives.

1. If $c$ is the positively oriented circle $|z|=2$, then show that

i) $\int_C \frac{e^z}{(z-1)^2} dz = 2\pi i e$

ii) $\int_C \frac{e^z}{(z-1)^3} dz = \pi i e$

iii) $\int_C \frac{e^z}{(z-1)^4} dz = \frac{\pi i e}{3}$.

proof:-

Given $c$ is the positively oriented circle $|z|=2$.

i) let $\int_C \frac{e^z}{(z-1)^2} dz = I$

Here, $z=1$ lies inside of $c$.

$\therefore$ By Cauchy's integral formula for derivatives

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \longrightarrow (1)$$

Here $f(z)=e^z$, $n=1$, $a=1$, $f'(z)=e^z$

$$(1) \Rightarrow I = \int_C \frac{e^z}{(z-a)^{n+1}} dz$$

$$= \frac{2\pi i}{1!} f'(a)$$

$$\therefore \int_C \frac{e^z}{(z-1)^2} dz = 2\pi i \cdot e$$

ii) Let $\int_C \frac{e^z}{(z-1)^3} dz = I$

Here $z=1$, lies inside of $C$, i.e $|z|=2$.

By cauchy's integral formula for derivatives.

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \longrightarrow (1)$$

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$f'(z)=e^z \qquad , \quad a=1, n=2$

$f''(z)=e^z$

$(1) \Rightarrow \int_C \frac{e^z}{(z-1)^3}\,dz = \int_C \frac{e^z}{(z-1)^{2+1}}\,dz$

$\int_C \frac{\cos z}{z}\,dz = \int_0^{2\pi} \frac{\cos(\cos\theta + i\sin\theta)}{z}\, iz\, d\theta$

$$= \frac{2\pi i}{2!} f''(a)$$

$$= \frac{2\pi i}{2}(e^1)$$

$$= \pi i e$$

$$\therefore \int_C \frac{e^z}{(z-1)^3}\,dz = \pi i e.$$

iii) Let $\int_C \frac{e^z}{(z-1)^4}\,dz = I$.

Here, $z=1$ lies inside of C. ie $|z|=2$.

$\therefore$ By cauchy's integral formula for derivatives,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-1)^{n+1}}\,dz \longrightarrow (1)$$

Here

$f(z)=e^z, \; a=1, \; n=3$

$f'(z)=e^z$

$f''(z)=e^z$

$f'''(z)=e^z$

$(1) \Rightarrow \int_C \frac{e^z}{(z-1)^4}\,dz = \int_C \frac{e^z}{(z-1)^{3+1}}\,dz$

$$= \frac{2\pi i}{3!} f'''(a)$$

$$= \frac{2\pi i}{6} e.$$

$$= \frac{\pi i e}{3}$$

$$\therefore \int_C \frac{e^z}{(z-1)^4}\,dz = \frac{\pi i e}{3}.$$

2. Show that the integral of $\frac{e^{iz}}{z^3}$ along the positively oriented circle $|z|=2$ is $-\pi i$.

Solution:-

Given $C$ is the positively oriented circle $|z|=2$.

To find:-

$$\int_C \frac{e^{iz}}{z^3}\,dz.$$

ie To find:- $\int_C \frac{e^{iz}}{(z-0)^3}\,dz.$

Here $z=0$ lies inside of $C$, ie $|z|=2$.

By cauchy's integral formula for derivatives.

$$f^n(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \longrightarrow (1).$$

Comparing the given problem with (1).

$f(z)=e^{iz}$, $a=0$, $n=2$

$f'(z)=i.e^{iz}$

$f''(z)=-1.e^{iz}$

$$f''(0)=\frac{2!}{2\pi i}\int_C \frac{e^{iz}}{(z-0)^{2+1}}\,dz$$

$$\int_C \frac{e^{iz}}{(z-0)^3}\,dz=\frac{2\pi i}{2i}f''(0)$$

$$=\pi i(-1)$$

$$=-\pi i$$

$$\therefore \int_C \frac{e^{iz}}{(z-0)^3}\,dz=-\pi i$$

8. Show that the integral $\frac{1}{(z^2+4)^2}$ along the positively oriented circle $|z-i|=2$ is $\frac{\pi}{16}$.

proof:-

To find:-

$$\int_C \frac{1}{(z^2+4)^2}\,dz.$$

where c is the positinely oriented circle $|z-i|=2$.

circle,
$$\frac{1}{(z^2+4)^2} = \frac{1}{[(z+2i)(z-2i)]^2}$$

$$= \frac{1}{(z+2i)^2(z-2i)^2}$$

$$= \frac{A}{z+2i} + \frac{B}{(z+2i)^2} + \frac{C}{z-2i} + \frac{D}{(z-2i)^2}$$

$$\frac{1}{(z^2+4)^2} = \frac{A(z+2i)(z-2i)^2 + B(z-2i)^2 + C(z-2i)(z+2i)^2 + D(z+2i)^2}{(z^2+4)^2}.$$

$$1 = A(z+2i)(z-2i)^2 + B(z-2i)^2 + C(z-2i)(z+2i)^2 + D(z+2i)^2$$

Put $z=2i$,

$$1 = D(2i+2i)^2 = D(4i)^2.$$

$$1 = -16D.$$

$$\Rightarrow D = \frac{-1}{16}.$$

Put $z=-2i$

$$1 = B(-2i-2i)^2$$

$$1 = B(-4i)^2 = (16B)(-1).$$

$$B = -1/16.$$

co-efficient of $z^3$

$$\boxed{A+C=0}.$$

Coefficient of $z^2$

$$(2i-4i)A + B + (4i-2i)C + D = 0.$$

$$(-2i)A + B + (2i)C + D = 0.$$

$A(z+2i)(z-2i)^2$

$= A(z+2i)(z^2-4zi-4)$

$= A(z^3-4z^2i-4z+2iz^2-8zi^2-8i)$

$= Az^3 - 2iz^2A$

$B(z-2i)^2 = B(z^2-4iz+4)$

$= z^2B - 4Biz + B4$

$C(z-2i)(z+2i)^2$

$= C(z-2i)(z^2+4iz-4)$

$= C[z^3+4iz^2-4z-2iz^2-8zi^2+8i]$

$= Cz^3 + C2iz^2$

$D(z+2i)^2$

$= D(z^2+4iz-4)$

$= Dz^2 + Diz4 - 4D$

$(-2i)A - \frac{1}{16} + 2iC - \frac{1}{16} = 0$

$\Rightarrow -2iA + 2iC = \frac{1}{16} + \frac{1}{16}$

$\Rightarrow 2i(-A+C) = \frac{2}{16}$

$\Rightarrow -A + C = \frac{1}{8} \times \frac{1}{2i}$

$-A + C = \frac{1}{16i}$

$A - C = -\frac{1}{16i}$

$A + C = 0 \Rightarrow A = -C$ (or) $C = -A$

$2A = -\frac{1}{16i} \Rightarrow A = -\frac{1}{32i}$

$A + C = 0 \Rightarrow \frac{-1}{32i} + C = 0$

$\Rightarrow C = \frac{1}{32i}$

$$\frac{1}{(z^2+4)^2} = \frac{-1}{32i}\left(\frac{1}{z+2i}\right) - \frac{1}{16}\left(\frac{1}{(z+2i)^2}\right) + \frac{1}{32i}\left(\frac{1}{z-2i}\right) - \frac{1}{16}\left(\frac{1}{(z-2i)^2}\right) \rightarrow (1)$$

Here $\boxed{z = 2i}$ lies inside by C.

$\therefore$ By Cauchy's integral formula for derivative,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \longrightarrow (2)$$

Comparing (1) and (2) we get

$f(z) = 1$ , $f'(z) = 0$ , $f''(z) = 0$ .

$$\int_C \frac{1}{(z^2+4)^2}\,dz = \frac{-1}{32i}\int_C \frac{1}{z+2i}\,dz - \frac{1}{16}\int_C \frac{dz}{(z+2i)^2} + \frac{1}{32}\int_C \frac{dz}{(z-2i)} - \frac{1}{16}\int_C \frac{dz}{(z-2i)^2}.$$

In the right of the integral first and second integral $z = -2i$ does not lie inside c and in third, fourth integral $z = 2i$ lies inside of C.

$$\therefore \int_C \frac{dz}{(z^2+4)^2}\,dz = 0 - 0 + \frac{1}{32i}2\pi i - \frac{1}{16}\frac{2\pi i}{1!}f'(2i)$$

$$= \frac{1}{32i} \times 2\pi i - 0.$$

$$\therefore \int_C \frac{1}{(z^2+4)^2}\,dz = \frac{\pi}{16}.$$

Hence proved.

4. If $c$ is the positively oriented circle $|z|=1$ and $k$ is a real constant, prove that $\int_C \frac{e^{kz}}{z} \cdot dz = 2\pi i$

Deduce that $\int_0^{\pi} e^{k \cdot \cos\theta} \cos(k \sin\theta)\, d\theta = \pi$.

Proof:-

Given $c$ is the positively oriented circle $|z|=1$ and $k$ is the real constant.

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz \longrightarrow (1)$$

Here,

$f(z) = e^{kz}$ and $a=0$ lies inside of $C$.

$f(0) = e^{k.0}$.

$$(1) \Rightarrow e^{k.0} = \frac{1}{2\pi i} \int_C \frac{e^{kz}}{(z-0)}\, dz.$$

$$\Rightarrow \int_C \frac{e^{k.z}}{z}\, dz = 2\pi i \longrightarrow (2).$$

(or)

$$\text{Im} \int_C \frac{e^{kz}}{z}\, dz = 2\pi.$$

Let $z = e^{i\theta} = \cos\theta + i\sin\theta$, $0 \le \theta \le 2\pi$

$dz = i \cdot e^{i\theta} \cdot d\theta = iz\,d\theta \cdot$ $e^{k(\cos\theta + i\sin\theta)}$

$$\int \frac{e^{kz}}{z}\, dz = \int_0^{2\pi} \frac{e^{k}(\cos\theta + i\sin\theta)}{z}\, i z\, d\theta.$$ $e^{k(\cos\theta+i\sin\theta)}$

$$= i \int_0^{2\pi} e^{k}(\cos\theta + i\sin\theta)\, d\theta$$

$$= i \int_0^{2\pi} e^{k\cos\theta} \cdot e^{ik\sin\theta}\, d\theta.$$

$$= i \int_0^{2\pi} e^{k\cos\theta} [\cos(k\sin\theta) + i\sin(k\sin\theta)]\, d\theta.$$

$$\int_C \frac{e^{kz}}{z}\, dz = i \int_0^{2\pi} e^{k\cos\theta} \cdot \cos(k\sin\theta)\, d\theta - \int_0^{2\pi} e^{k\cos\theta} \cdot \sin(k\sin\theta)\, d\theta.$$

From (2),

$$2\pi i = i\int_0^{2\pi} e^{k\cos\theta}\cdot\cos(k\sin\theta)d\theta - \int_0^{2\pi} e^{k\cos\theta}\sin(k\sin\theta)d\theta$$

Comparing imaginary parts,

$$\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)d\theta = 2\pi \Rightarrow \int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)d\theta = \pi$$

5. Prove that $\int_0^{2\pi}\cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi$

Proof :-

Consider $\int_C \frac{\cos z}{z}dz$ along $|z|=1$.

$f(a)=\cos z$

Here, $f(z)=\cos z \Rightarrow a=0,\ f(0)=1.$

By Cauchy's integral formula,

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \longrightarrow (1).$$

$$1=\frac{1}{2\pi i}\int_C \frac{\cos z}{z}dz.$$

$$\therefore \int_C \frac{\cos z}{z}dz = 2\pi i \longrightarrow (2)$$

Let $z=e^{i\theta}=\cos\theta+i\sin\theta,\quad 0\le\theta\le 2\pi$

$$dz = ie^{i\theta}.d\theta = iz\,d\theta.$$

$$\int_C \frac{\cos z}{z}dz = \int_0^{2\pi}\frac{\cos(\cos\theta+i\sin\theta)}{z}iz\,d\theta$$

$$= i\int_0^{2\pi}\cos(\cos\theta+i\sin\theta)d\theta$$

$$= i\int_0^{2\pi}[\cos(\cos\theta)\cos(i\sin\theta)]-[\sin(\cos\theta)\cdot\sin(i\sin\theta)]\,d\theta.$$

$$= i\int_0^{2\pi}[\cos(\cos\theta)\cosh(\sin\theta)]-[i\sin(\cos\theta)\sinh(\sin\theta)]d\theta$$

$$\int_C \frac{\cos z}{z} dz = i\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta + \int_0^{2\pi} \sin(\cos\theta)\cdot \sinh(\sin\theta)\, d\theta.$$

From (2),

$$2\pi i = i\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta + \int_0^{2\pi} \sin(\cos\theta)\cdot\sinh(\sin\theta)\, d\theta.$$

comparing imaginary parts on both sides,

$$\int_0^{2\pi} \cos(\cos\theta)\cdot\cosh(\sin\theta)\, d\theta = 2\pi.$$

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6. If C is a scro. curve enclosing the origin, show that $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{a^n\cdot(e^{az})}{n!\, z^{n+1}} dz$ and deduce that

$$\sum_{n=0}^{\infty}\left[\frac{a^n}{n!}\right]^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta.$$

proof :-

Given C is a scro curve enclosing the origin. Now, let us take $f(z) = e^{az}$, then by cauchy's formula for $n^{th}$ derivation,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}} dz \rightarrow (1).$$

$$f'(z) = a\cdot e^{az},\ f''(z) = a^2 e^{az}, \dots f^n(z) = a^n e^{az}.$$

$$f^n(a) = a^n e^0 = a^n \qquad \boxed{z=0}$$

$$(1) \Rightarrow a^n = \frac{n!}{2\pi i}\int \frac{e^{az}}{z^{n+1}} dz$$

Multiplying on both sides by the constant $\frac{a^n}{(n!)^2}$ we get,

$$\frac{a^n}{(n!)^2} a^n = \frac{n!}{2\pi i}\int_C \frac{a^n e^{az}}{(n!)^2 z^{n+1}} dz$$

$$\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{a^n e^{az}}{n!\, z^{n+1}} dz.$$

Taking summation on both sides,

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} \frac{a^n e^{az}}{n!\, z^{n+1}}\, dz.$$

$$= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} \frac{a^n e^{az}}{n!\, z^{n+1}}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{az}}{z}\left[\sum_{n=0}^{\infty} \frac{(a/z)^n}{n!}\right] dz$$

$$= \frac{1}{2\pi i}\int \frac{e^{az}}{z} e^{a/z}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{az+a/z}}{z}\, dz$$

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{e^{a(z+1/z)}}{z}\, dz.$$

where, $c$ is the $z = e^{i\theta}$ circle, $0 \le \theta \le 2\pi$

$$\Rightarrow \frac{1}{z} = e^{-i\theta}$$

$z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$.

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$z + \frac{1}{z} = 2\cos\theta.$$

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{2a\cos\theta}}{z}\, iz\, d\theta \cdot \left[\because z = e^{i\theta},\ dz = ie^{i\theta} d\theta\right]$$

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta$$

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## Taylor's Series :-

1. Find the taylor's series for $\frac{1}{z}$ about $z=1, z=2$ and $z=i$ and the corresponding neighbourhood [open circular disc] in which they are convergent - that is valid.

Solution :-

Given $f(z) = \frac{1}{z}$.

At $z=0$, the given function is not analytic and the distance between 0 and 1, 2, i. and the neighbourhood of 1, 2, i in which $\frac{1}{z}$ is analytic are

$|z-1|<1, \quad |z-2|<2, \quad |z-i|<i$

$\therefore$ By Taylor's series formula,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots \longrightarrow (1)$$

$f(z) = \frac{1}{z}; \; f'(z) = -\frac{1}{z^2}; \; f''(z) = \frac{2!}{z^3}; \; f'''(z) = \frac{-3!}{z^4}$.

$f(1) = 1; \; f'(1) = -1; \; f''(1) = 2!; \; f'''(1) = -3!$

$f(2) = \frac{1}{2}; \; f'(2) = -\frac{1}{2^2}; \; f''(2) = \frac{2!}{2^3}; \; f'''(2) = \frac{-3!}{2^4}$.

$f(i) = \frac{1}{i}; \; f'(i) = \frac{-1}{i^2}; \; f''(i) = \frac{2!}{i^3}; \; f'''(i) = \frac{-3!}{i^4}$

$(1) \Rightarrow \frac{1}{z} = 1 + (z-1)(-1) + \frac{(z-1)^2}{2!}(2!) + \frac{(z-1)^3}{3!}(-3!) + \cdots$

$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots \quad \forall z \in |z-n| < 1$.

$\frac{1}{z} = \frac{1}{2} + \frac{(z-2)}{1!}\left(\frac{-1}{2^2}\right) + \frac{(z-2)^2}{2!}\left(\frac{2!}{2^3}\right) + \frac{(z-2)^3}{3!}\left(\frac{-3!}{2^4}\right) + \cdots$

$\frac{1}{z} = \frac{1}{2} - \frac{(z-2)}{2^2} + \frac{(z-2)^2}{2^3} - \frac{(z-2)^3}{2^4} + \cdots \quad \forall z \in |z-n| < 2$.

$\frac{1}{i} = \frac{1}{i} + \frac{z-i}{1!}\left(\frac{-1}{i^2}\right) + \frac{(z-i)^2}{2!}\left(\frac{2!}{i^3}\right) + \frac{(z-i)^3}{3!}\left(\frac{-3!}{i^4}\right) + \cdots$

$\frac{1}{i} = \frac{1}{i} - \frac{(z-i)}{i^2} + \frac{(z-i)^2}{i^3} - \frac{(z-i)^3}{i^4} + \cdots \quad \forall z \in |z-n| < i$.

2. Show that, for every finite $z$,

$$e^z = e\left[1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right]$$ Find the Taylors series for $e^z$ about $z=1$. State the region for validity of expansion.

proof :-

Given $f(z) = e^z$

$f(z)$ is analytic in the finite plane and the given series in an expansion about $z=1$.

Now, $f(z) = f'(z) = f''(z) = \cdots = e^z$

$f(1) = f'(1) = f''(1) = \cdots = e^1$.

By Taylor's series for every $z$ in a circular discs with centre at $z=1$, say $|z-1| < \rho$.

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$$e^z = e + \frac{z-1}{1!} e + \frac{(z-1)^2}{2!} e + \cdots$$

$$e^z = e\left[1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \cdots\right]$$

But '$\rho$' is arbitrary. So this expansion is valid for all finite $z$.

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**Section: 8.9 :-**

**Cauchy's integral formula and formula for derivatives :-**

**Theorem: 8.8 :-**

** 10 mark

**Cauchy's integral formula.**

Suppose

i) $D$ is a simply-connected region

ii) $f(z)$ is analytic in $D$ and

iii) $C$ is a SCRC curve in $D$.

Then for any $a$ in $C_i$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz.$$

Proof:-

The function $\frac{f(z)}{z-a}$ is analytic in D except at $z=a$.

In Ci take a positively oriented circle $\Gamma$, $|z-a| = \rho$.

Now, the function $\frac{f(z)}{z-a}$ is analytic on C and $\Gamma$ and in the annulus region between C and $\Gamma$.

By extension to cauchy's fundamental theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_\Gamma \frac{f(z)}{z-a}\,dz$$

$$= \int_\Gamma \frac{f(z)-f(a)+f(a)}{z-a}\,dz$$

$$= \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + \int_\Gamma \frac{f(a)}{z-a}\,dz$$

$$= \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_\Gamma \frac{1}{z-a}\,dz$$

$$= \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + f(a)2\pi i.$$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + f(a)\,2\pi i$$

$$\int_C \frac{f(z)}{z-a}\,dz - f(a)\,2\pi i = \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz$$

$$\left|\int_C \frac{f(z)}{z-a}\,dz - f(a)2\pi i\right| = \left|\int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz\right|$$

$$\leq \left[\max\left|\frac{f(z)-f(a)}{z-a}\right| \text{on } \Gamma\right] l.(\Gamma)$$

$$\Rightarrow \left|\int_C \frac{f(z)}{z-a}\,dz - 2\pi i f(a)\right| \leq \left[\max\left|\frac{f(z)-f(a)}{z-a}\right| \text{on } \Gamma\right] l(\Gamma) \rightarrow (1)$$

Now, $f(z)$ is analytic in D.

Its derivative exists at all points in D

$\therefore$ and hence it exists at $z=a$.

$$\therefore \lim_{z \to a} \frac{f(z)-f(a)}{z-a} = f'(a)$$

Let $\phi(z,a) = \dfrac{f(z)-f(a)}{z-a} - f'(a)$

$$\therefore \lim_{z \to a} \phi(z,a) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a} - f'(a)$$

$$= f'(a) - f'(a)$$

$$\lim_{z \to a} \phi(z,a) = 0$$

Hence for a given $\epsilon > 0$, we can find a $\delta >$

$|\phi(z,a) - 0| < \epsilon$ (or) $|\phi(z,a)| < \epsilon$, $\forall |z-a| < \delta$

$\therefore$ choosing $\rho$ to be less than $\delta$ for all $z$ on and inside the circle $\Gamma : |z-a| = \rho$.

$|\phi(z,a) - \rho| < \epsilon$ and for this $z$.

$$\left| \frac{f(z)-f(a)}{z-a} \right| = \left| f'(a) + \phi(z,a) \right|$$

$$\leq |f'(a)| + |\phi(z,a)|$$

$$< |f'(a)| + \epsilon$$

$$= M \text{ (say)}$$

where $M$ is finite because $f'(a)$ and $\epsilon$ are finite Thus on $\Gamma$,

$$\max \left| \frac{f(z)-f(a)}{z-a} \right| < M.$$

$$(1) \Rightarrow \quad \left| \int_C \frac{f(z)}{z-a} dz - 2\pi i f(a) \right| < M(2\pi\rho)$$

Let $\rho \to 0$ then

$$\left| \int_C \frac{f(z)}{z-a} dz - 2\pi i f(a) \right| = 0.$$

$$\Rightarrow \int_C \frac{f(z)}{z-a} dz - 2\pi i f(a) = 0.$$

$$\Rightarrow \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

Hence the theorem.

## Theorem: 8.9:-

### Cauchy's formula for first derivative

Suppose

i) D is a simply-connected region

(ii) $f(z)$ is analytic in D and

iii) C is a Scro curve in D.

Then for any a in C;

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz.$$

proof:-

Let a be any point in the interior of C.

Define the function $\phi(a)$ by,

$$\phi(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

Take another point $a+b$ in the interior of C.

By Cauchy's integral formula,

For a and $a+b$,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \text{ and}$$

$$f(a+b) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+b)}\,dz$$

Now,

$$\left|\frac{f(a+b)-f(a)}{a+b-a} - \phi(a)\right| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{b[z-(a+b)]}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{b(z-a)}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz\right|$$

$$\left|\frac{f(a+b)-f(a)}{b} - \phi(a)\right| = \left|\frac{1}{2\pi i b}\int_C \left[\frac{1}{z-a-b} - \frac{1}{z-a} - \frac{b}{(z-a)^2}\right]\right.$$

$$= \frac{1}{2\pi|b||i|}\left|\int_C \left[\frac{(z-a)^2-(z-a)(z-a-b)-b(z-a-b)}{(z-a-b)(z-a)^2}\right] f(z)\,dz\right|$$

$$= \frac{1}{2\pi|b|}\left|\int_C \frac{(z-a)(z-a-z+a+b)-b(z-a-b)}{(z-a-b)(z-a)^2} f(z)\,dz\right|$$

$$= \frac{1}{2\pi|b|}\left|\int_C \frac{b(z-a)-b(z-a-b)}{(z-a-b)(z-a)^2} f(z)\,dz\right|$$

$$= \frac{|b|}{2\pi} \left| \int_c \frac{f(z)}{[z-(a+b)][z-a]^2} dz \right|$$

$$\leq \frac{1}{2\pi} \left\{ \max \left| \frac{b f(z)}{[z-(a+b)](z-a)^2} \right| \text{ on } c \right\} l(c)$$

$$\left| \frac{f(a+b)-f(a)}{b} - \phi(a) \right| \leq \frac{|b|}{2\pi} \max \frac{|f(z)|\, l(c)}{|z-(a+b)||z-a|^2}$$

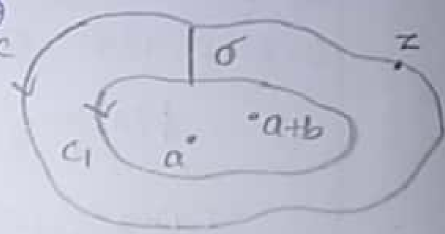
Take $\Gamma$ any simply closed curve $C_1$ in the interior of $C_i$ encircling $a$ and $a+b$.

Let $\delta$ be the minimum distance between the point of the curve $C_1$ and $C$.

If $z$ is any point on $C$,

then, $|z-a| > \delta$ and $|z-(a+b)| > \delta$

$|z-a| < \frac{1}{\delta}$ and $\frac{1}{|z-(a+b)|} < \frac{1}{\delta}$




So for any $a$ and $a+b$ in the interior of $C_1$ from,

$$\left| \frac{f(a+b)-f(a)}{b} - \phi(a) \right| < \frac{1}{2\pi} \cdot \frac{|b|}{\delta} \cdot \frac{M}{\delta^2} l(C).$$

where $M$ is $\max |f(z)|$ on $C$.

Taking $\lim_{b \to 0}$,

$$\left| \lim_{b\to 0} \frac{f(a+b)-f(a)}{b} - \phi(a) \right| = 0$$

$\Rightarrow |f'(a) - \phi(a)| = 0$.

$\Rightarrow f'(a) - \phi(a) = 0$

$\Rightarrow f'(a) = \phi(a)$

i.e, $f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$

Hence the proof.

## Theorem : 8.10

### Cauchy Integral Formula for $n^{th}$ Derivative

Suppose

i) D is a simply-connected region

ii) $f(z)$ is analytic in D and

iii) c is a sero curve in D.

Then for any a in ci

$$f^{n}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

proof :-

By cauchy's integral formula for $1^{st}$ derivative

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \rightarrow (1)$$

i.e result (1) is true for $n=1$.

Now, assume the result (1) is true for $n=m$

i.e, $$f^{m}(a) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\,dz$$

$$f^{m}(a+b) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a-b)^{m+1}}\,dz$$

Then,

$$f^{m}(a+b) - f^{m}(a) = \frac{m!}{2\pi i}\int_C \left[\frac{1}{(z-a-b)^{m+1}} - \frac{1}{(z-a)^{m+1}}\right] f(z)\,dz$$

$$= \frac{m!}{2\pi i}\int_C \left[\frac{1}{(z-a)^{m+1}\left(1-\frac{b}{z-a}\right)^{m+1}} - \frac{1}{(z-a)^{m+1}}\right] f(z)\,dz$$

$$= \frac{m!}{2\pi i}\int_C \frac{1}{(z-a)^{m+1}}\left[\frac{1}{\left(1-\frac{b}{z-a}\right)^{m+1}} - 1\right] f(z)\,dz$$

$$f^{m}(a+b) - f^{m}(a) = \frac{m!}{2\pi i}\int_C \frac{1}{(z-a)^{m+1}}\left[\left(1-\frac{b}{z-a}\right)^{-(m+1)} - 1\right] f(z)\,dz$$

$$\left[\because (1-y)^{-n} = 1 + ny + \frac{n(n-1)}{2!}y^2 + \cdots\right]$$

$$f^{m}(a+b) - f^{m}(a) = \frac{m!}{2\pi i}\int \frac{1}{(z-a)^{m+1}}$$

$$\left[1+(m+1)\left[\frac{b}{z-a}\right] + \frac{(m+1)m}{2!}\left(\frac{b}{z-a}\right)^2 + \cdots\right] f(z)\,dz$$

$$= \frac{m!}{2\pi i}\int_C \frac{1}{(z-a)^{m+1}}\left[\frac{(m+1)b}{(z-a)}\left(1+\frac{m}{2!}\left(\frac{b}{z-a}\right)+\cdots\right) f(z)\,dz\right]$$

$$= \frac{m!}{2\pi i} \int_C \frac{(m+1)b}{(z-a)^{m+2}} \left[1 + \frac{m}{2!}\left(\frac{b}{z-a}\right) + \cdots\right] f(z)\,dz$$

$$\frac{f^m(a+b) - f^m(a)}{b} = \frac{(m+1)!}{2\pi i} \int_C \frac{1}{(z-a)^{m+2}} \left[1 + \frac{m}{2!}\left(\frac{b}{z-a}\right) + \cdots\right] f(z)\,dz$$

Taking, $\lim_{b\to 0}$ on both sides,

$$\lim_{b\to 0} \frac{f^m(a+b) - f^m(a)}{b} = \lim_{b\to 0} \frac{(m+1)!}{2\pi i} \int_C \frac{1}{(z-a)^{m+2}} \left[1 + \frac{m}{2!}\left(\frac{b}{z-a}\right) + \cdots\right] f(z)\,dz$$

$$\lim_{b\to 0} \frac{f^m(a+b) - f^m(a)}{b} = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+2}}\,dz \; [1+0+0+\cdots]$$

$$\therefore f^{m+1}(a) = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+2}}\,dz$$

$\therefore$ The results (1) is true for $n = m+1$.

Hence by mathematical induction, the result (1) is true for any positive integer $n$.

Hence the result.

---

Theorem : 8.10 A :-

A function analytic in D has derivatives of all orders in D.

proof :-

[First we prove cauchy's integral formula for nth derivative].

Clearly, if $f^m(z)$ exists then $f^{m+1}(z)$ is also exist.

Thus a function analytic in D has derivative of all or derivatives of all orders in D.

## Theorem : 8.10 B

The derivative of an analytic function is also analytic.

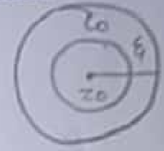
proof :-

[First we prove cauchy integral formula for $n^{th}$ derivative]

Suppose $f(z)$ is analytic at $z_0$ then ∃ a neighbourhood $|z-z_0|<\epsilon$ of $z_0$ throughout which $f$ is differentiable.

Then ∃ a circle $C : |z-z_0| < \epsilon/2$ ∋ $f$ is analytic and on $C_0$.

Then cauchy's integral formula for derivatives




$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$ at each point 'a' in the interior of $C_0$.

∴ $f''(a)$ = exists neighbourhood $|z-z_0| < \epsilon/2$

∴ $f$ is analytic at $z_0$.

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## Theorem: 8.11 :- Morera's Theorem :-

Converse of Cauchy's fundamental Theorem.

Suppose

i) D is a region

ii) $f(z)$ is continuous in D

iii) C is an SCR curve in D and

iv) $\int_C f(z) dz = 0$ for all C's

Then $f(z)$ is analytic in D.

proof :-

Let $z_0$ be a fixed point in D.

Since the integral of $f(z)$ along every closed curve is zero, the integral

$$\left[ F(z) = \int_{z_0}^{z} f(z)\, dz \right]$$

is path independent and defines a single-valued function in D.

Then for any two points $a$, $a+b$ in D.

$$F(a+b)-F(a) = \int_{z_0}^{a+b} f(z)\,dz - \int_{z_0}^{a} f(z)\,dz \qquad \int_{a}^{z_0} + \int_{z_0}^{a+b}$$

$$= \int_{a}^{a+b} f(z)\,dz$$

$$= \int_{a}^{a+b} [f(z)-f(a)+f(a)]\,dz$$

$$= \int_{a}^{a+b} [f(z)-f(a)]\,dz + \int_{a}^{a+b} f(a)\,dz$$

$$= \int_{a}^{a+b} [f(z)-f(a)]\,dz + f(a)\int_{a}^{a+b} dz$$

$$= \int_{a}^{a+b} [f(z)-f(a)]\,dz + f(a)\,(z)\Big|_{a}^{a+b}$$

$$= \int_{a}^{a+b} [f(z)-f(a)]\,dz + f(a)[(a+b)-a]$$

$$= \int_{a}^{a+b} [f(z)-f(a)]\,dz + f(a)[(\cancel{a}+b-\cancel{a})]$$

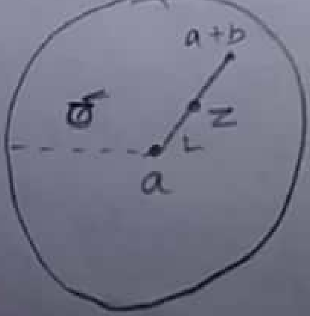
$$= \int_{a}^{a+b} [f(z)-f(a)]\,dz + b\,f(a) = \int_{a}^{a+b} b\left\{\frac{f(z)-f(a)}{b}\right\}$$

$$\frac{F(a+b)-F(a)}{b} = \frac{1}{b}\int_{a}^{a+b} [f(z)-f(a)]\,dz + f(a)$$

$$\left\{\frac{F(a+b)-F(a)}{b}\right\} - f(a) = \frac{1}{b}\int_{a}^{a+b} [f(z)-f(a)]\,dz \longrightarrow (1)$$

Since $f(z)$ is continuous in D, for any $\varepsilon>0$ $\exists$ $\delta>0$ such that

$|f(z)-f(a)|<\varepsilon$, whenever $|z-a|<\delta$.



Restrict $a+b$ to lie in the disc $|z-a| < \delta$ and denote the segment joining $a$ and $a+b$ by $L$.

Then from (1),

$$\left|\frac{F(a+b)-F(a)}{b} - f(a)\right| = \frac{1}{|b|}\left|\int_L [f(z)-f(a)]\,dz\right|$$

$$\left[\because \left|\int_C f(z)\,dz\right| \le \{\max|f(z)| \text{ on } C\}\, l(C)\right]$$

$$\le \frac{1}{|b|}[\max|f(z)-f(a)| \cdot \text{on } L]\, l(C).$$

$$< \frac{1}{|b|}\epsilon|b|.$$

$$\therefore \left|\frac{F(a+b)-F(a)}{b} - f(a)\right| < \epsilon$$

$$\Rightarrow \left|\lim_{b\to 0}\frac{F(a+b)-F(a)}{b} - f(a)\right| < \epsilon$$

$\Rightarrow |F'(a) - f(a)| < \epsilon$

$\Rightarrow |F'(a) - f(a)| = 0$ $\quad\because \epsilon$ is arbitrary

$F'(a) - f(a) = 0$

$F'(a) = f(a)$

$\therefore F'(z) = f(z), \ \forall z \in D$ [$\because a$ is arbitrary]

Thus the derivative of $F(z)$ exists throughout $D$.

$\therefore F(z)$ is analytic in $D$.

Since the derivative of an analytic function is analytic.

$F'(z)$ is analytic in $D$.

$f(z)$ is analytic in $D$.