UNIT - V.

Taylor's and Laurent's series

Section : 9.5 :-

Singular point or singularity

Definition :- (not analytic)

A singular point of a function is a point at which the function ceases to be analytic.

நிறுத்தப்படுகிறது

Eg :- 1

$f(z) = \frac{1}{z(z-1)}$, (0, 1) are singular points of the function.

Because, the function is not analytic there since the functions attains the value ∞ there. A function which is not defined at a point is not analytic there. so Such a point is a singular point of the function.

Y/N

Eg :- 2

$f(z) = \frac{e^z}{z}$ 0, ∞ are singular points.

NOTE :-

∞ is a singular point of all the trigonometric functions and hyperbalic functions because they are functions of e^z.

A singular point of a function of a point at which the function ceases to be analytic.

	$f(z)$	Singular points $z=a$	$f(a)$
$0,1$	$\frac{1}{z(1-z)}$	$0,1$	∞
$-1,2$	$\frac{z}{(z+1)(z-2)}$	$-1,2$	∞
$i,-i$	$\frac{1+z}{1+z^2}$	$i,-i$	∞
∞	$1+e^z, e^{-z}$	∞	undefined
0	$e^{1/z}, e^{-1/z}$	0	undefined
∞	$\sin z, \cos z$	∞	undefined
$\pm n\pi$	$\frac{1}{\sin z}$	$\infty; \pm n\pi, n=0,1,2\dots$	undefined; ∞
$(2n+1)\frac{\pi}{2}$	$\frac{1}{\cos z}$	$\infty; \pm(2n+1)\frac{1}{2}\pi, n=0,1,\dots$	undefined; ∞
$\pm n\pi i$	$\frac{1}{\sinh z}$	$\infty; \pm n\pi i, n=0,1,2,\dots$	undefined; ∞
$(2n+1)\frac{\pi i}{2}$	$\frac{1}{\cosh z}$	$\infty; \pm(2n+1)\frac{1}{2}\pi i, n=0,1,\dots$	undefined; ∞

Example :-

prove that $e^{1/z-a}$ is an undefined function at the singularity $z=a$.

proof:- (undefined, does not exist limit)

Given $f(z) = e^{1/z-a}$

let $z-a=h$

$f(z)$ is analytic in $\boxed{|z-a|>0}$ and it is not defined at $z=a$ because if, $a=\alpha+i\beta$ and $z=(\alpha+i\beta)+h$, then as $h\to 0$, $z\to a$ and $z-a\to 0 \Rightarrow z\to a$

$f(z) = e^{1/h} \to \infty$, if h is positive

$f(z) = e^{1/h} \to 0$, if h is negative

ie, The limit of the function as $z\to a$ does not exist.

$\therefore e^{1/z-a}$ is an undefined function at $z=a$.

Definition :-

Isolated Singularity :-

If $z=a$ is a singular point of a function $f(z)$ and if there exists a neighbourhood of $z=a$, containing no other singular point of $f(z)$, then $z=a$ is said to be an isolated singular point of $f(z)$.

① → Isolated

Definition :-

② → non Isolated

Non-Isolated Singularity :-

If $f(z)$ has infinity of singular points in every neighbourhood of $z=a$, then $z=a$ is a non-isolated singular point but it is a limit point of the set of singular points of $f(z)$.

Eg: 1

$$f(z) = \frac{1}{z(z-1)}$$ 0 and 1 are singular points

$\therefore$ They are isolated.

Eg: 2

$$f(z) = \frac{1}{(z-1)(z-2)}$$ 1 and 2 are singular points

$\therefore$ They are isolated.

Eg: 3

All singular points of $\frac{1}{\sin z}$ except ∞ are isolated singular points.

Eg: 4

The function $\frac{1}{\sin\left(\frac{1}{z}\right)}$ is not analytic at the points $z = \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \ldots \rightarrow (1)$

$\sin n\pi = 0$

Because $\sin\left(\frac{1}{z}\right)$ vanishes there, Also the function is not defined at $z=0$. Since $\sin z$ is not defined at $z=0$. Since $\sin z$ is not defined at ∞.

Thus $z=0$ also is a singular point. Further it is the limit point of the sequence of points.

Hence $z=0$ is not an isolated singular points. But the singular point ∞ is an isolated one.

Section: 9.6

Isolated Singularities :-

Isolated Singularities are classified into three different groups.

This classification is done with reference to the negative powers of the Laurent's expansion of the function about the singular point.

If $z=a$ is an isolated singular point of a function $f(z)$, then the singularities is called

types of Isolated singularities

i) a removable singularity (or) $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$

ii) a pole (or)

iii) an essential singularity. $\frac{a_{-1}}{(z-a)}, \frac{a_{-2}}{(z-a)^2}, \frac{a_{-3}}{(z-a)^3}$

according as the Laurent's Series about $z=a$ of $f(z)$, valid in a deleted neighbourhood of $z=a$, has

i) no negative powers (or)

ii) a finite number of negative powers (or) (pole)

iii) an infinite number of negative powers. (essential singularity)

Section: 9.7 :-

Removable Singularity :-

If $z=a$ in a singular point of a function and if the limit of the function as z tends to a $(z+a)$ exists and is finite, then $z=a$ is a removable singularity of the function.

Example :-

$$f(z) = \frac{\sin z}{z}, \ z \neq 0$$

Solution :-

$f(z)$ is not defined

put $z=0$, $f(z) = \frac{0}{0}$

$\therefore z=0$ is a singular point of $f(z)$

But, $\frac{\sin z}{z} = \frac{1}{z}(\sin z)$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \ 0 < |z| < \infty$$

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) = 1$$

$\therefore \lim_{z \to 0} \frac{\sin z}{z} = 1$ exists

$\therefore \frac{\sin z}{z}$ have removable singularity at $z=0$.

Example :-

$$\frac{z - \sin z}{z^3}$$

Solution :-

$$f(z) = \frac{z - \sin z}{z^3}$$

put $z=0$, $f(0) = \frac{0}{0}$.

$\therefore z=0$ is a singular point.

$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right], \ 0 < |z| < \infty$$

$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left[0 + \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots\right]$$

$$= \frac{1}{z^3}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$$

$$\lim_{z\to 0} \frac{z-\sin z}{z^3} = \lim_{z\to 0}\left(\frac{1}{6} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots\right) = \frac{1}{6}$$

$$\therefore \lim_{z\to 0} \frac{z-\sin z}{z^3} = \frac{1}{6}.$$

$\therefore \frac{z-\sin z}{z^3}$ have removable singularity at $z=0$.

Section: 9.8

Pole of order

* 2mark

If $z=a$ is a pole of $f(z)$, then the Laurent's expansion for $f(z)$ about $z=a$ is of the form

$$\frac{a_{-m}}{(z-a)^m} + \frac{a_{-(m-1)}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots$$

where $a_{-m} \neq 0$. Here $z=a$ is said to be a pole of order m or a pole of multiplicity m.

Definition:

Pole:-

* 2mark

If $z=a$ is a singular point of a function and if the limit of the function as 'z' tends to 'a' exists and equal to ∞. Then $z=a$ is called a pole.

Principal part:-

The rational function formed by the negative powers, namely

$$\frac{a_{-m}}{(z-a)^m} + \frac{a_{-(m-1)}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a}$$

is called the principal part of $f(z)$ at the pole $z=a$.

This part tends to ∞ as $z\to a$.

* 2 mark

Simple pole :-

When the order of a pole is 1, the pole is said to be a simple pole.

Double pole :-

When the order of a pole is 2, the pole is said to be a double pole.

Example : 1

$$f(z) = \frac{3}{z-1} + 4 + 5(z-1) + 6(z-1)^2.$$

The principal part of $f(z) = \frac{3}{z-1}$

Here, $z=1$ is a singular point

$\therefore$ It is a pole of order 1

It is a simple pole.

Example : 2

$z=0$ is a simple plane of the function $\frac{e^z}{z}$ because its expansion about $z=0$ is

$$\frac{e^z}{z} = \frac{1}{z}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \frac{1}{z} + \frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

principal part is $\frac{1}{z}$.

Example : 3

The singularity $z=1$ of the function $\frac{1}{(z-1)^2} + \frac{1}{z-1} + 1 + (z-1) + (z-1)^2$ is a double plane

Its principal part is $\frac{1}{(z-1)^2} + \frac{1}{z-1}$

$M = 2 \rightarrow$ double pole

Example :4

$z=0$ is a double pole of the function $\frac{e^z}{z^2}$ because its expansion about $z=0$ is

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left[1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]$$

$$= \frac{1}{z^2}+\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\cdots \qquad m=2$$

Its principal part is $\frac{1}{z^2}+\frac{1}{z}$.

Example : 5

$z=1$ is a triple pole of

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2 \cdot e^{2(z-1)}}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\left[e^{2(z-1)}\right]$$

$$= \frac{e^2}{(z-1)^3}\left[1+\frac{2(z-1)}{1!}+\frac{2^2(z-1)^2}{2!}+\cdots\right]$$

$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2^2e^2}{(z-1)2!}+\frac{2^3e^2}{3!}+\cdots$$

Its principal part is

$$\frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2^2 e^2}{(z-1)2!}$$

Section; 9.9:-

Essential singularity (negative is ∞)

If $z=a$ is an essential singularity of $f(z)$, then the Laurent's expansion for $f(z)$ about $z=a$ is of the form.

$$f(z) = \sum_{n=1}^{\boxed{\infty}} \frac{a_{-n}}{(z-a)^n} + a_0 + a_1(z-a)+\cdots$$

In this, the principal part is an infinite series of negative powers of $z-a$ while the remaining part is a finite or infinite series of positive powers of $z-a$.

Example : 1

$z=0$ is an essential singularity of the function $e^{1/z}+1+z+z^2$.

$$f(z) = e^{1/z}+1+z+z^2$$

$$= \left(1+\frac{(1/z)}{1!}+\frac{(1/z)^2}{2!}+\cdots\right)+(1+z+z^2)$$

$$= \left(1+\frac{1}{z}\cdot\frac{1}{1!}+\frac{1}{z^2}\cdot\frac{1}{2!}+\cdots\right)+(1+z+z^2)$$

$$f(z) = \left(\frac{1}{z}\cdot\frac{1}{1!}+\frac{1}{z^2}\cdot\frac{1}{2!}+\cdots\right)+(2+z+z^2)$$

$$f(z) = \sum_{n=1}^{\infty}\frac{1}{n!\,z^n}+2+z+z^2$$

$\therefore \lim_{z\to 0} f(z)$ does not exist.

Example : 2

$z=-1$ is an essential singularity of the function $e^{1/z+1}+\frac{1}{2-z}$ or $e^{1/z+1}+\frac{1}{3-(z+1)}$

$$f(z) = e^{1/z+1}+\frac{1}{3-(z+1)}$$

$$= \left(1+\frac{1/z+1}{1!}+\frac{(1/z+1)^2}{2!}+\cdots\right)+\frac{1}{3\cdot\left(1-\frac{z+1}{3}\right)}$$

$$= \left(1+\frac{1}{1!}\cdot\frac{1}{z+1}+\frac{1}{2!}\left(\frac{1}{z+1}\right)^2+\cdots\right)+\frac{1}{3}\left(1-\frac{z+1}{3}\right)^{-1}$$

$(1-x)^{-1}$

$1+x+x^2+\cdots$

$$= \left(1+\frac{1}{1!}\cdot\frac{1}{z+1}+\frac{1}{2!}\left(\frac{1}{z+1}\right)^2+\cdots\right)+\frac{1}{3}\left(1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots\right)$$

$\therefore z=-1$ is an essential singularity.

Section : 9.10

Behaviour of a function at an isolated singularity :

Theorem : 9.6 :-

If $z=a$ is a removable singularity of a function $f(z)$, then there exists a deleted neighbourhood of $z=a$ in which $f(z)$ is bounded

proof :-

Given $z=a$ is a removable singularity of the function $f(z)$.

Since $z=a$ is an isalated singularity of a function $f(z)$.

If it removable singularity, the limit of $f(z)$ as $z \to a$ is finite and it is bounded in a small deleted neighbourhood of $z=a$.

i.e the deleted neighbourhood $0<|z-a|<R$ in which $f(z)$ is analytic

By the definition of removable singularity, the Laurent's series for $f(z)$ in this deleted neighbourhood contains no negative powers of $z=a$, so the expression is of the form,

$$\boxed{f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots}$$

Now, $f(z)$ is analytic in the deleted neighbourhood. if we substitute $z=a$ in $f(z)$ as a_0 (or) zero if $a_0=0$, then the new $f(z)$ is analytic at $z=a$ and hence in the whole of the neighbourhood.

$\therefore f(z)$ is bounded in some closed neighbourhood of $z=a$.

But $f(z)$ is not defined at $z=a$ (or) is discontinuous at $z=a$.

Thus, leaving out the point $z=a$, we get that $f(z)$ is bounded in some deleted neighbourhood of $z=a$.

Theorem: 9.7 [Converse of Theorem 9.6]

Riemann's Theorem:-

A function has an isolated singularity at $z=a$ but is analytic in the deleted neighbourhood $0<|z-a|<R$. If the function is bounded in the deleted neighbourhood, then then singularity is a removable singularity.

proof:

Since $f(z)$ is bounded. Suppose $|f(z)|<M$ in the deleted neighbourhood.

Let the Laurent's series of $f(z)$ about $z=a$ be $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-a)^n$, where $a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\,dt$.

where C is a circle, $|t-a|=\rho$, $\rho<R$. Then

$$|a_n| = \left|\frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\,dt\right|$$

$|i| = \sqrt{i^2}$

$$\le \frac{1}{2\pi} \max_{\text{on } C} \frac{|f(t)|}{|t-a|^{n+1}}\, l(C).$$

$|0+i|$

$\sqrt{0^2+1^2}=1$

$$< \frac{1}{2\pi}\frac{M}{\rho^{n+1}}(2\pi\rho)$$

$$|a_n| < \frac{M}{\rho^{n+1}}\rho$$

$$\Rightarrow |a_n| < \frac{M}{\rho^n}$$

let $\rho\to 0$

$|a_{-1}|<0$ but $\boxed{|a_{-1}| \text{ is a positive}}$

$\therefore |a_{-1}|=0$.

Similarly,

$n=-2,-3,\dots$

Similarly,

$$|a_{-2}|=0, \ |a_{-3}|=0 \ldots$$

$\therefore |a_n|=0$ (or) $a_n=0$ if $n=-1,-2,-3,\ldots$

$\therefore$ In Laurent's expansion of $f(z)$, all the negative powers vanish.

$$\therefore f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\ldots \rightarrow (1)$$

$\therefore$ By defn of removable singularity,

"If $z=a$ is an isolated singular point of a function $f(z)$, then the singularity is called removable singularity according as the Laurent's series about $z=a$ of $f(z)$ vaild in a deleted neighbourhood of $z=a$ has no negative powers"

From equation (1) and by the definition $z=a$ is a removable singularity.

Theorem: 9.8

If $z=a$ is a pole of a function $f(z)$, then $\lim\limits_{z\to a} f(z)=\infty$.

Proof:-

Let the order of the pole be m and the Laurent's expansion about $z=a$ in the the deleted neighbourhood $0<|z-a|<\rho$ be

$$f(z)=\frac{a_{-m}}{(z-a)^m}+\ldots+\frac{a_{-1}}{z-a}+a_0+a_1(z-a)+\ldots$$

$$=\frac{a_{-m}+\ldots+a_{-1}(z-a)^{m-1}}{(z-a)^m}+a_0+a_1(z-a)+\ldots$$

$$=\frac{\phi(z)}{(z-a)^m}+a_0+a_1(z-a)+a_2(z-a)^2+\ldots$$

where, $\phi(z)=a_{-m}+\ldots+a_{-1}(z-a)^{m-1}$

$$\lim_{z\to a} f(z) = \lim_{z\to a}\left[\frac{\phi(z)}{(z-a)^m} + a_0 + a_1(z-a) + a_2(z-a)^2 + \right] \rightarrow (1)$$

Here,

i) $\lim_{z\to a} \phi(z) = \lim_{z\to a} a_{-m} + \dots + a_{-1}(z-a)^{m-1}$

$= a_{-m}$

ii) $\lim_{z\to a} \frac{1}{(z-a)^m} = \infty$

iii) $\lim_{z\to a} a_0 + a_1(z-a) + a_2(z-a)^2 + \dots = a_0$ $\Big\} \rightarrow (2)$

Substitute (2) in (1), we get

$$\lim_{z\to a} f(z) = a_{-m}\infty + a_0 = \infty$$

$$\lim_{z\to a} f(z) = \infty$$

Remark:-

If z is a pole, $|f(z)| \rightarrow \infty$ as $z \rightarrow a$.

Theorem: 9.9: [converse of 9.8] :-

If a function $f(z)$ is analytic in a deleted neighbourhood of $z=a$ and if

$$\lim_{z\to a} f(z) = \infty,$$

then $z=a$ is a pole of $f(z)$.

proof:-

Given $\lim_{z\to a} f(z) = \infty$

infinite limit at finite limits

ie Given a positive number A, $\exists$ a $\delta \ni$

$|f(z)| > A$, $\forall z \in |z-a| < \delta$

consider the function $g(z)$ defined by

$$g(z) = \frac{1}{f(z)}$$

Then,

$$|g(z)| = \frac{1}{|f(z)|} < \frac{1}{A}, \forall z \in |z-a| < \delta.$$

ie, in $|z-a|<\delta$, $g(z)$ is bounded and analytic.

$\Rightarrow g(z)$ has a removable singularity at $z=a$.

So, expressing $g(z)$ in Laurent's series about $z=a$,

$$g(z) = b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots$$

Since,

$$\lim_{z\to a} f(z) = \infty$$

$$\therefore \lim_{z\to a} g(z) = 0$$

$b_0 = 0$ or the first few coefficients are zero.

Suppose,

$$b_0 = b_1 = \cdots = b_{m-1} = 0 ; \; b_m \neq 0.$$

Then,

$$g(z) = (z-a)^m \left[b_m + b_{m+1}(z-a) + \cdots \right]$$

$$g(z) = (z-a)^m \cdot \phi(z) \cdot \Rightarrow \frac{g(z)}{\phi(z)} = (z-a)^m \Rightarrow \frac{g(z)}{b_m + b_{m+1}(z-a) + \cdots} = (z-a)^m$$

Here $\phi(z)$ is analytic in a neighbourhood of $z=a$, and $\phi(a) = b_m \neq 0$.

$\therefore \dfrac{1}{\phi(z)}$ is analytic in a neighbourhood of $z=a$.

Expanding $\dfrac{1}{\phi(z)}$ as a Taylor's series about $z=a$.

$$\frac{1}{\phi(z)} = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

$$\therefore f(z) = \frac{1}{g(z)} = \frac{1}{(z-a)^m \phi(z)}$$

$$f(z) = \frac{1}{(z-a)^m}\left[c_0 + c_1(z-a) + \cdots \right]$$

$$f(z) = \frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \frac{c_2}{(z-a)^{m-2}} + \cdots$$

$z=a$ is a pole of $f(z)$

Theorem :9.10:-

If $z=a$ is a zero of order m of an analytic function $f(z)$, then $z=a$ is a pole of order m of the function $\frac{1}{f(z)}$.

proof:-

Given $z=a$ is a zero of order m of an analytic function $f(z)$.

zero of order def

Suppose $\boxed{f(z)=(z-a)^m \phi(z)}$. $[\phi(a) \neq 0]$

and $\phi(z)$ is analytic in a neighbourhood of 'a'.

$\therefore \frac{1}{\phi(z)}$ is analytic in a neighbourhood of 'a'.

using Taylor's series about $z=a$,

$$\frac{1}{\phi(z)} = b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots$$

$$\frac{1}{f(z)} = \frac{1}{(z-a)^m} \cdot \frac{1}{\phi(z)}$$

$$\frac{1}{f(z)} = \frac{1}{(z-a)^m}\left[b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots\right]$$

$$\frac{1}{f(z)} = \frac{b_0}{(z-a)^m} + \frac{b_1}{(z-a)^{m-1}} + \frac{b_2}{(z-a)^{m-2}} + \cdots$$

$\therefore z=a$ is a pole of order m of the function $\frac{1}{f(z)}$

corallary

We know that zeros are isolated points also pole are isolated points.

Theorem: 9.11

If $z=a$ is a pole of order m of a function $f(z)$, then $z=a$ is a zero of order m of the function $\frac{1}{f(z)}$.

Proof:-

Given $z=a$ is a pole of order m of a function ~~f(z)~~ $f(z)$. $\exists$ a deleted neighbourhood of $z=a$ in which the

Scanned with CamScanner

Laurent's expansion for $f(z)$ about $z=a$ is

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots$$

Let $\psi(z) = (z-a)^m f(z)$

$$= a_{-m} + a_{-(m-1)}(z-a) + a_{-(m-2)}(z-a)^2 + \cdots$$

Then,

$\psi(a) = a_{-m}$ and $\frac{1}{\psi(a)} \neq 0$.

Now, $\frac{1}{f(z)} = (z-a)^m \frac{1}{\psi(z)} \left[\because \frac{1}{\psi(a)} \neq 0 \right]$

$\therefore \frac{1}{f(z)}$ has a zero of order m.

Behaviour in a neighbourhood of an essential singularity :-

The complicated behaviour of an analytic function in a neighbourhood of an essential singularity is embodied in the following theorem of weierstrass. The complicated behaviour is that an analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity, however small it may be.

Theorem : 9.12 :-

Weierstrass Theorem

If $f(z)$ has an isolated essential singularity at $z=a$ and if c is any complex constant, then for any positive ϵ, however small, there exist an z in every deleted neighbourhood of $z=a$ such that

$$|f(z) - c| < \epsilon.$$

proof :-

Suppose that $D: 0<|z-a|<\rho$ be any arbitrary chosen small deleted neighbourhood

Suppose that $\varepsilon_1>0$ is given and that

$|f(z)-C| \geq \varepsilon_1, \forall z \in D$.

$$\Rightarrow \left|\frac{1}{f(z)-C}\right| \leq \frac{1}{\varepsilon_1}$$

Hence, $\frac{1}{f(z)-C}$ is bounded and analytic in D except at $z=a$.

As $f(z)$ is not analytic at $z=a$

By the theorem,

"A function has an isolated singularity at $z=a$ but is analytic in the deleted neighbourhood $0<|z-a|<R$. If the function is bounded in the deleted neighbourhood, then the singularity is a removable singularity".

$\therefore$ $z=a$ becomes removable singularity of $\frac{1}{f(z)-C}$.

$\therefore$ $\frac{1}{f(z)-C}$ has that Laurents expansion about $z=a$ is of the form,

$a_0 + a_1(z-a) + \dots$ where $a_0 \neq 0$.

(or)

$a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots$ where $a_m \neq 0$.

i.e either $z=a$ is not a zero of $\frac{1}{f(z)-C}$ (or) $z=a$ is a zero of order m of $\frac{1}{f(z)-C}$

$\therefore$ $f(z)-C$ has a removable singularity at $z=a$

(or)

$f(z)-C$ has a pole $z=a$ of order m.

Thus $f(z)$ has a removable singularity at $z=a$ (or) $f(z)$ has a pole $z=a$ of order m.

$\therefore$ Which is a contradiction as $z=a$ is an essential singularity.

Hence the theorem.

Section : 9.11 :-

Determination of the nature of singularities.

We defined a removable singularity, a pole and an essential singularity of a function $f(z)$ at $z=a$ with reference to the number of negative powers in the Laurent's expansion of $f(z)$ about $z=a$.

If $z=a$ is a singularity of $f(z)$, then it is

i) a removable singularity if $\lim_{z\to a} f(z)$ exists and is finite.

ii) a pole if $\lim_{z\to a} f(z)$ exists and equals ∞

iii) an essential singularity if $\lim_{z\to a} f(z)$ does not exist.

In finding limits we can use L Hospital's rule which is true for complex function also.

Examples :-

1. $\lim_{z\to 0} \frac{\sin z}{z}$

Solution :-

$$\lim_{z\to 0} \frac{\sin z}{z} = \frac{0}{0}.$$

So by using L Hospital's rule,

$\lim_{z\to 0} \frac{\cos z}{1} = 1$ [$\because$ limit exist and it is finite]

$\therefore$ So the singularity of $\frac{\sin z}{z}$ at $z=0$ is a removable one.

Scanned with CamScanner

2. $\frac{\sin\sqrt{z}}{\sqrt{z}}$

Salution:-

$$\lim_{z\to 0} \frac{\sin\sqrt{z}}{\sqrt{z}} = \frac{0}{0}.$$

So, by using L Hospital's rule,

$$\lim_{z\to 0} \frac{\frac{\cos\sqrt{z}}{2\sqrt{z}}}{\frac{1}{2\sqrt{z}}} = \lim_{z\to 0} \frac{\cos\sqrt{z}}{1} \quad \left[\because \sin(z)^{1/2} \; \cos\sqrt{z} = \frac{1}{2\sqrt{z}}\right]$$

$\therefore$ So the Singularity of $\frac{\sin\sqrt{z}}{\sqrt{z}}$ at $z=0$ is a removable one.

3. $\operatorname{cosec} z - \frac{1}{z}$

Salution:-

$$\lim_{z\to 0}\left(\operatorname{cosec} z - \frac{1}{z}\right) = \lim_{z\to 0}\left(\frac{1}{\sin z} - \frac{1}{z}\right)$$

$$= \lim_{z\to 0}\left(\frac{0}{\cos z} - \frac{0}{1}\right) [\because \infty - \infty]$$

$$= 0.$$

$\therefore \lim_{z\to 0}\left(\operatorname{cosec} z - \frac{1}{z}\right) = 0$ [$\because$ Finite and limit exist

$\therefore z=0$ is a removable singularity of $\operatorname{cosec} z - \frac{1}{z}$.

4. $\lim_{z\to 1} \frac{e^z}{(z-1)^2} = \infty$.

Salution:-

Limit exist and it is ∞

$\therefore z=1$ is a pole of $\frac{e^z}{(z-1)^2}$

5. $\lim_{z\to 0} e^{1/z}$

Salution:-

It does not exist.

So $z=0$ is an essential singularity of $e^{1/z}$

Limit point of zeros and limit point of poles:

We know that poles are isolated singularity.

$\therefore$ The limit point cannot be a pole.

$\therefore$ The limit points of poles in a non-isolated singularities.

Examples:-

1. The zeros of $f(z) = \sin\left(\frac{1}{(z-a)}\right)$ are given by

$$\sin\left(\frac{1}{z-a}\right) = 0 \text{ as } \frac{1}{z-a} = n\pi, \; n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow z - a = \frac{1}{n\pi}, \; n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow z = a + \frac{1}{n\pi}, \; n = \pm 1, \pm 2, \ldots$$

$\therefore z = a$ is a limit point of the zeros.

$\therefore$ It is an isolated essential singularity of $f(z)$.

2. The poles of $f(z) = \frac{1}{\sin\left(\frac{1}{z-a}\right)}$ are given by

$$\sin\left(\frac{1}{z-a}\right) = 0 \text{ as } z = a + \frac{1}{n\pi}, \; n = \pm 1, \pm 2, \ldots$$

$\therefore z = a$ is the limit point of the poles.

$\therefore$ It is a non-isolated essential singularity of $f(z)$.

NOTE:

Removable singularity and poles are isolated singularity, The essential singularity is either an isolated (or) a non isolated singularity.

Section: 9.12 : Nature of singularity at ∞.

To examine the nature of singularity of $f(z)$ at $z = \infty$ we should examine the nature of singularity of $f(1/z)$ at $z = 0$.

Residues :-

Section : 10.1 : Residue :-

Definition :

If $z=a$ is an isolated singularity of $f(z)$ and if the Laurent's expansion for $f(z)$ about $z=a$ is

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

then the coefficient of $(z-a)^{-1}$, namely a_{-1}, is called the residue of $f(z)$ at $z=a$

Examples :-

1. Find the residue of $f(z)=\frac{e^z}{z}$

Solution :-

Given $f(z)=\frac{e^z}{z}$.

Laurent's expansion for $f(z)$,

$$f(z)=e^z\left(\frac{1}{z}\right).$$

$$=\frac{1}{z}\left[1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots\right]$$

$$f(z)=\frac{1}{z}+\frac{1}{1!}+\frac{z}{2!}+\cdots$$

$\therefore$ The coefficient of z^{-1} is 1.

$\therefore$ Residue of $f(z)$ at $z-0=1$

By using Laurent's series thorem,

If $a_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$

Where 'c' is a scro curve encircling $z=0$ and setting $n=-1$, then residue of $f(z)$ at $z=a$ is,

$$a_{-1}=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^0}\,dz$$

$$a_{-1}=\frac{1}{2\pi i}\int_C f(z)\,dz.$$

Scanned with CamScanner

2. If $f(z) = \frac{e^{z-a}}{(z-a)^2}$, the Laurent's expansion about the singularity $z=a$.

Solution:-

Given $f(z) = \frac{e^{z-a}}{(z-a)^2}$

$$= \frac{1}{(z-a)^2}\left(e^{z-a}\right)$$

$$= \frac{1}{(z-a)^2}\left[1 + \frac{(z-a)}{1!} + \frac{(z-a)^2}{2!} + \cdots\right]$$

$$= \frac{1}{(z-a)^2} + \frac{1}{(z-a)1!} + \frac{1}{2!} + \cdots$$

$$f(z) = (z-a)^{-2} + (z-a)^{-1} + \frac{1}{2!} + \cdots$$

$\therefore$ The coefficient of $(z-a)^{-1}$ is $\frac{1}{1!}$.

$\therefore$ The residue of $f(z)$ at the double pole $z=a$ is $\frac{1}{1!} = 1$.

Section: 10.2: Calculation of residues:

The following are the various methods of calculation of residues under different situations:

1. The residue at $z=a$ of a function $f(z)$ can be calculated directly by evaluating

$$\frac{1}{2\pi i}\int_C f(z)\,dz.$$

choosing suitably a small scro curve C around $z=a$.

Scanned with CamScanner

Example :-

$$f(z) = \frac{2z+1}{z^2 - z - 2} = \frac{2z+1}{(z+1)(z-2)}$$

Solution :-

Given function is $f(z) = \dfrac{2z+1}{(z+1)(z-2)}$

$\therefore z = -1$ and $z = 2$ are poles.

By using partial fractions,

$$\frac{2z+1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}.$$

$$\frac{2z+1}{(z+1)(z-2)} = \frac{A(z-2) + B(z+1)}{(z+1)(z-2)}.$$

$$2z+1 = A(z-2) + B(z+1)$$

put $z = 2$,

$$2(2)+1 = A(2-2) + B(2+1)$$

$$5 = 3B$$

$$B = 5/3.$$

Put $z = -1$

$$2(-1)+1 = A(-1-2) + B(-1+1)$$

$$-1 = -3A$$

$$A = 1/3$$

$$\therefore \frac{2z+1}{(z+1)(z+2)} = \frac{1/3}{z+1} + \frac{5/3}{z-2}$$

$\therefore$ So, if c_1 is the circle $|z+1| = 1$ and c_2 is the circle $|z-2| = 1$.

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=-1\end{array}\right\} = \frac{1}{2\pi i}\int_{C_1}\left[\frac{1}{3}\left(\frac{1}{z+1}\right) + \frac{5}{3}\left(\frac{1}{z-2}\right)\right]dz$$

$$= \frac{1}{2\pi i}\left[\frac{1}{3}(2\pi i) + \frac{5}{3}(0)\right]$$

$$= \frac{1}{3} \qquad \left[\because \int_C \frac{1}{z-a}\,dz = 2\pi i\right]$$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=2\end{array}\right\} = \frac{1}{2\pi i}\int_{C_2} f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \left[\frac{1}{3}\left(\frac{1}{z+1}\right)+\frac{5}{3}\left(\frac{1}{z-2}\right)\right]dz$$

$$= \frac{1}{2\pi i}\left[\frac{1}{3}(0)+\frac{5}{3}(2\pi i)\right]$$

$$= \frac{5}{3}.$$

---

2. If $z=a$ is a simple pole of $f(z)$, then

2nd method.

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=a\end{array}\right\} = \lim_{z\to a}(z-a)\,f(z).$$

$$\lim_{z\to a}(z-a)\,f(z) = \lim_{z\to a}(z-a)\left[\frac{a_{-1}}{z-a}+a_0+a_1(z-a)+\cdots\right]$$

$$= a_{-1}$$

Examples:-

1. Find the residue of $f(z) = \dfrac{2z+1}{z^2-z-2}$

★ 2m

Solution

Given $f(z) = \dfrac{2z+1}{z^2-z-2} = \dfrac{2z+1}{(z+1)(z-2)}$

$\therefore z=-1$ and $z=2$ are poles.

By using partial fractions,

$$\frac{2z+1}{(z+1)(z-2)} = \frac{A}{z+1}+\frac{B}{z-2}.$$

$$\frac{2z+1}{(z+1)(z-2)} = \frac{A(z-2)+B(z+1)}{(z+1)(z-2)}$$

$$2z+1 = A(z-2)+B(z+1)$$

put $z=2$,

$$2(2)+1 = A(2-2)+B(2+1)$$

$$5 = 3B$$

$$\Rightarrow B = 5/3.$$

put $z=-1$,

$$2(-1)+1 = A(-1-2)+B(-1+1)$$

$$-1 = -3A$$

$$\Rightarrow A = 1/3$$

$$\frac{2z+1}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{5/3}{z-2}$$

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=-1\end{array}\right\} = \lim_{z\to -1} (z+1)\cdot\frac{2z+1}{(z+1)(z-2)}$$

$$= \frac{2(-1)+1}{(-1-2)} = \frac{-1}{-3}$$

$$= \frac{1}{3}$$

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=2\end{array}\right\} = \lim_{z\to 2} (z-2)\cdot\frac{2z+1}{(z+1)(z-2)}$$

$$= \frac{2(2)+1}{2+1}$$

$$= \frac{5}{3}$$

---

2. Find the residue of $f(z) = \cot z$ at the simple pole $z=0$.

Solution:-

Given $f(z) = \cot z = \frac{\cos z}{\sin z}$, pole is $z=0$.

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=0\end{array}\right\} = \lim_{z\to 0} z\cdot\frac{\cos z}{\sin z}$$

$$= \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\cos z\right)$$

$$= \left(\lim_{z\to 0}\frac{1}{\cos z}\right)(1).$$

$$= (1)(1) \quad \text{[By L Hospital's rule]}$$

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=0\end{array}\right\} = 1.$$

Scanned with CamScanner

3. If $z=a$ is a simple pole of $f(z)$ and if $f(z)$ can be expressed as a quotient of two analytic functions $h(z)$ and $g(z)$ as

$$f(z) = \frac{h(z)}{g(z)}$$

where $h(a) \neq 0$, $g(a)=0$, $g'(a) \neq 0$.

Solution :-

$$\text{Residue at } (z=a) = \lim_{z \to a} (z-a) \frac{h(z)}{g(z)}$$

$$= \lim_{z \to a} \frac{(z-a)}{g(z)-g(a)} h(z)$$

$$= \frac{h(a)}{g'(a)}$$

4. Find the residue of $f(z) = \cot z$.

Solution :-

Given $f(z) = \cot z = \frac{\cos z}{\sin z}$

$z=0$ is a pole of $f(z)$. it is a simple pole of $f(z)$.

$h(z) = \cos z$; $g(z) = \sin z$

$h(0) = 1$; $g'(z) = \cos z$

$g'(0) = 1$

$$\therefore \left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=0\end{array}\right\} = \frac{h(a)}{g'(a)} = \frac{1}{1}$$

$= 1$.

5. If $z=a$ is a pole of order m of $f(z)$, Suppose $(z-a)^m \cdot f(z) = \phi(z)$.

Then $\phi(z)$ is analytic in a neighbourhood of $z=a$.

Solution :-

$\phi(z)$ can be expanded about $z=a$ as,

$$\phi(z) = \phi(a) + \frac{\phi'(a)}{1!}(z-a) + \frac{\phi''(a)}{2!}(z-a)^2 + \cdots + \frac{\phi^{m-1}(a)}{(m-1)!}(z-a)^{m-1} + \cdots$$

$$\therefore f(z) = \frac{\phi(z)}{(z-a)^m}$$

$$f(z) = \frac{1}{(z-a)^m}\left[\phi(a) + \frac{\phi'(a)}{1!}(z-a) + \cdots + \frac{\phi^{m-1}(a)}{(m-1)!}(z-a)^{m-1} + \cdots\right]$$

$$= \frac{\phi(a)}{(z-a)^m} + \frac{\phi'(a)}{1!}(z-a)^{-m+1} + \cdots + \frac{\phi^{m-1}(a)}{(m-1)!}\cdot\frac{1}{z-a} + \cdots$$

$$= \frac{\phi(a)}{(z-a)^m} + \frac{\phi'(a)}{1!}\cdot\frac{1}{(z-a)^{m-1}} + \cdots + \frac{\phi^{m-1}(a)}{(m-1)!}\cdot\frac{1}{(z-a)} + \cdots$$

The residue of $f(z)$ at $z=a$ is

$$\frac{\phi^{m-1}(a)}{(m-1)!} \text{ (or) } \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\phi(z)\right]_{z=a}.$$

i.e $\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]_{z=a}$

** 10 mark

Theorem: 10·1: Residue Theorem:-

Suppose

i) D is a simply-connected region

ii) $f(z)$ is analytic in D except at its singularities which are finite in number

iii) C is a scro curve in D, not passing through any singularity.

Then,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of the residues of } f(z) \text{ in } c].$$

Scanned with CamScanner

proof :-

Let the singular points of $f(z)$ in C_i be $a_1, a_2, \ldots a_n$.

Since there are only finitely many singularities points of $f(z)$ in D and they are isolated.

Since C is a scro curve in D.

Let $\Gamma_1, \Gamma_2, \ldots \Gamma_n$ are the non-intersecting positively oriented circle that lies inside C.

$|z-a_1| = r_1, |z-a_2| = r_2, \ldots, |z-a_n| = r_n$.

Then by extension to Cauchy's fundamental theorem,

$$\int_C f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \ldots + \int_{\Gamma_n} f(z)\,dz$$

$$= 2\pi i \,(\text{Residue of } f(z) \text{ at } z=a_1)$$
$$+ 2\pi i \,(\text{Residue of } f(z) \text{ at } z=a_2)$$
$$+ \ldots + 2\pi i \,(\text{Residue of } f(z) \text{ at } z=a_n).$$

$\int_C f(z)\,dz = 2\pi i$ [$\because$ Sum of the residues of $f(z)$ in c_i]

Residue at $z=\infty$:-

If c is a positively oriented circle $|z| = R$, then it is negative oriented with respect to $z=\infty$ and $-c$ is positively oriented with respect to $z=\infty$. Hence,

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=\infty\end{array}\right\} = \frac{1}{2\pi i}\int_{-C} f(z)\,dz$$

$$= \frac{-1}{2\pi i}\int_C f(z)\,dz$$

where the exterior of the circle C contains no singularity of $f(z)$ other than $z=\infty$.

Theorem 10.2:-

If a function $f(z)$ is analytic in the extended plane except at a finite number of singularities including $z=\infty$. then the sum of the residues of $f(z)$ is zero.

proof:-

Suppose the singularities are $a_1, a_2, \dots a_{n-1}$ and $a_n = \infty$.

If C is a scro curve encircling $a_1, a_2, \dots a_{n-1}$.

Then, by the residue theorem.

$$\left.\begin{matrix}\text{Sum of the residues of} \\ f(z) \text{ at } a_1, a_2, \dots a_{n-1}\end{matrix}\right\} = \frac{1}{2\pi i}\int_c f(z)\,dz \rightarrow (1)$$

$$\left.\begin{matrix}\text{Residue of } f(z) \\ \text{at } z=\infty\end{matrix}\right\} = \frac{-1}{2\pi i}\int_c f(z)\,dz \rightarrow (2)$$

Add (1) and (2) we get

The sum of all the residues is zero.

Theorem: 10.3:-

If $f(z)$ is analytic in a deleted neighbourhood of $z=\infty$ and if its Laurent's expansion in $R < |z| < \infty$ is

$$f(z) = \dots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

Then the residue of $f(z)$ at $z=\infty$ is $-a_{-1}$

proof:-

Given $f(z)$ is analytic in a deleted neighbourhood of $z=\infty$ and if the Laurent's expansion in.

$R < |z| < \infty$ is

$$f(z) = \dots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

Scanned with CamScanner

We know that,

By definition of residue,

"The residue of $f(z)$ at $z=a$ is the coefficient of $(z-a)^{-1}$ ie, a_{-1}".

$\therefore$ Integrating $f(z)$ and the series term by term, along a circle $C: |z|=r$ which lies in $R<|z|<\infty$,

$$\int_C f(z)\,dz = \int_C \frac{a_{-1}}{z}\,dz$$

$$\therefore \int_C f(z)\,dz = \int_C a_{-1}\frac{1}{z}\,dz.$$

Since all the other integrals vanish.

$$\therefore \int_C f(z)\,dz = a_{1}(2\pi i). \quad \left[\because \int_C \frac{1}{z-a}dz = 2\pi i\right]$$

or

$$\frac{-1}{2\pi i}\int_C f(z)\,dz = -a_{-1}.$$

$$\therefore \left.\begin{matrix}\text{Residue of } f(z)\\ \text{at } z=\infty\end{matrix}\right\} = \frac{-1}{2\pi i}\int_C f(z)\,dz.$$

$$= -a_{-1}.$$

Residue of $f(z)$ at $(z=\infty)$ is $-a_{-1}$.

---

problems:

* 5 mark 1. $\displaystyle\int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a>|b|$.

Solution :-

put $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$ and

$$d\theta = \frac{dz}{iz}$$

$z=e^{i\theta}$

$dz = ie^{i\theta}.d\theta$

$\frac{dz}{ie^{i\theta}} = d\theta$

$\frac{dz}{iz} = d\theta$

$$I = \int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta.$$

$$= \int_0^{2\pi} \frac{dz/iz}{a + b(1/2i)(z - 1/z)}$$

$$= \int_0^{2\pi} \frac{1}{a + (b/2i)(z - \frac{1}{z})} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{a + \left(\frac{b}{2i}\right)\left(\frac{z^2-1}{z}\right)} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{\frac{2iza + b(z^2-1)}{2iz}} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{2iz}{\frac{2iza + b(z^2-1)}{iz}} dz$$

$$= 2\int_{|z|=1} \frac{dz}{bz^2 + 2iza - b}$$

$$I = 2\int_{|z|=1} \frac{dz}{b\left(z^2 + \frac{2ia}{b} z - 1\right)} \longrightarrow (1)$$

Take $b\left(z^2 + \frac{2ia}{b} z - 1\right) = 0.$

$$\Rightarrow z^2 + \frac{2ia}{b} z - 1 = 0.$$

$$z = \frac{-\left(\frac{2ia}{b}\right) \pm \sqrt{\left(\frac{2ia}{b}\right)^2 - 4(1)(-1)}}{2(1)}$$

$$z = \frac{-\left(\frac{2ia}{b}\right) \pm \sqrt{\frac{4i^2a^2}{b^2} - (2i)^2}}{2}$$

$$= \frac{-\left(\frac{2ia}{b}\right) \pm \frac{2i}{b}\sqrt{a^2 - b^2}}{2}$$

$$= \frac{-(a/b)\,i \pm (1/b)\,i\sqrt{a^2 - b^2}}{1}$$

$$z = -(a/b)\,i \pm \frac{i\sqrt{a^2 - b^2}}{b}$$

Scanned with CamScanner

$$z_1 = \frac{-a}{b}i + \frac{i\sqrt{a^2-b^2}}{b} \quad \text{and} \quad z_2 = \frac{-a}{b}i - \frac{i\sqrt{a^2-b^2}}{b}$$

are simple poles of integral of (1).

$$(1) \Rightarrow I = 2\int_{|z|=1} \frac{1}{b(z-z_1)(z-z_2)}\,dz \longrightarrow (2).$$

But

$$z_1 z_2 = (a/b)^2 (i)^2 - \frac{i^2(a^2-b^2)}{b^2}$$

$$= \frac{-a^2}{b^2} + \frac{a^2}{b^2} - \frac{b^2}{b^2} = -1.$$

$\therefore |z_1| = |z_2| = 1$ and $|z_1| < |z_2|$.

$\therefore z_1$ lies inside of $|z|=1$.

$\therefore$ By residue theorem, from (2).

$I = 2(2\pi i)$ [$\because$ Residue of Integral at z_1]

$$I = 4\pi i \lim_{z \to z_1} (z-z_1) \frac{1}{b(z-z_1)(z-z_2)}$$

$$= 4\pi i \frac{1}{b(z_1-z_2)}$$

$$I = \frac{4\pi i}{b(z_1-z_2)} \longrightarrow (3)$$

Now,

$$z_1 - z_2 = \frac{-ai}{b} + \frac{i\sqrt{a^2-b^2}}{b} + \frac{ai}{b} + \frac{i\sqrt{a^2-b^2}}{b}$$

$$z_1 - z_2 = \frac{2i\sqrt{a^2-b^2}}{b}$$

$$(3) \Rightarrow I = \frac{4\pi i}{b} \cdot \frac{b}{2i\sqrt{a^2-b^2}}$$

$$I = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

2. $\int_0^{2\pi} \frac{1}{1+a\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1-a^2}}, \; |a| < 1.$

Solution :-

In problem $\int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1-a^2}}, \; a > |b|$

Substitute $a=1$, $b=a$ we get the result.

3. Evaluate : $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$.

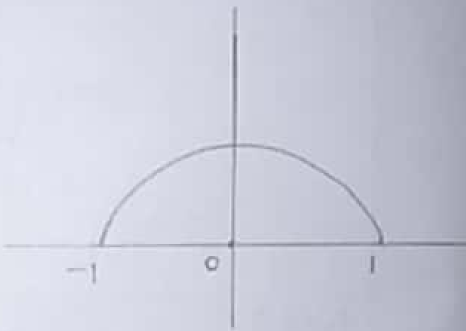
Solution :-

Consider $I = \int_C \frac{dz}{(z^2+1)^2} \rightarrow (1)$

Where 'c' is the closed curve consisting the positively oriented semicircle.

$\Gamma : |z| = R$, $\text{Im}\, z \geq 0$. and the line segment

$L : [-R, R]$.



Take, $(z^2+1)^2 = 0$.

$(z^2+1)(z^2+1) = 0$.

$z = \pm i$; $z = \pm i$

These points are double pole.

Residue of $\left.\frac{1}{(z^2+1)^2}\right\}$ at $z=i$ $= \frac{1}{1!} \frac{d}{dz}\left[(z-i)^2 \frac{1}{(z+i)^2(z-i)^2}\right]_{z=i}$.

$$= \frac{d}{dz}\left[(z+i)^{-2}\right]_{z=i}$$

$$= \left[\frac{-2}{(z+i)^3}\right]_{z=i}.$$

$$= \frac{-2}{(2i)^3}$$

$$= \frac{-2}{8i^3} = \frac{-2}{8i} = \frac{-1}{-4i} = \frac{i^2}{-4i}$$

$$= \frac{-i}{4}$$

$\therefore$ Sum of the residues is $\frac{-i}{4}$

By Cauchy's residue theorem,

$$\int_C \frac{1}{(z^2+1)^2}\, dz = 2\pi i\,[\text{Sum of residue in } C_i]$$

$$= 2\pi i\left(\frac{-i}{4}\right)$$

$$= \pi/2.$$

$$\therefore \int_{\Gamma^{-1}} \frac{1}{(z^2+1)^2}\, dz + \int_{-R}^{R} \frac{1}{(x^2+1)^2}\, dx = \frac{\pi}{2} \rightarrow (2).$$

Now,

$$\left| \int_{\Gamma} \frac{1}{(z^2+1)^2}\, dz \right| = 0.$$

$$\Rightarrow \int_{\Gamma} \frac{1}{(z^2+1)^2}\, dz = 0$$

Taking $R \to \infty$, $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = \frac{\pi}{2}.$

---

4. Show that $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}.$

Solution :-

In problem $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a > |b| > 0.$

Substitute $a=5$, $b=3$ we get

$$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{2\pi}{\sqrt{5^2-3^2}}$$

$$= \frac{2\pi}{\sqrt{25-9}} = \frac{2\pi}{\sqrt{16}}$$

$$= \frac{2\pi}{4}.$$

$$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}.$$

5. Use cauchy's residue theorem, evaluate $\int_c \frac{z}{(a-z^2)(z+i)} dz$ where 'c' in the circle, $|z|=2$.

Solution :-

Let $f(z) = \frac{z}{(a-z^2)(z+i)}$

Let $I = \int_{|z|=2} \frac{z \cdot dz}{(a-z^2)(z+i)}$

$(a-z^2)(z+i) = 0.$

$\Rightarrow a-z^2 = 0$, $z=-i$

$z^2 = a \Rightarrow z = \pm\sqrt{a}$ and $z=-i$ are singular points of $|z|=2$.

$$\left.\begin{matrix}\text{Residue of } f(z) \\ \text{at } z=a\end{matrix}\right\} = \lim_{z\to a} (z-a) \cdot f(z).$$

$$\left.\begin{matrix}\text{Residue of } f(z) \\ \text{at } z=-i\end{matrix}\right\} = \lim_{z\to -i} (z+i) \cdot \frac{z}{(a-z^2)(z+i)}$$

$$= \frac{-i}{a-(-i)^2} = \frac{-i}{a+1}$$

$$= \frac{-i}{a+1}$$

$$\left.\begin{matrix}\text{Residue of } f(z) \\ \text{at } z=\sqrt{a}\end{matrix}\right\} = \lim_{z\to\sqrt{a}} (z-\sqrt{a}) \frac{z}{(a-z^2)(z+i)}$$

$$= \lim_{z\to\sqrt{a}} (z-\sqrt{a}) \cdot \frac{z}{(\sqrt{a}+z)(\sqrt{a}-z)(z+i)}$$

$a - z\sqrt{a} + z\sqrt{a} - z^2$

$$= \lim_{z\to\sqrt{a}} (z-\sqrt{a}) \frac{z}{(-(z-\sqrt{a}))(\sqrt{a}+z)(z+i)}$$

$$= \lim_{z\to\sqrt{a}} \frac{z}{-(\sqrt{a}+z)(z+i)}$$

$$= \frac{\sqrt{a}}{-2\sqrt{a}(\sqrt{a}+i)}$$

$$= \frac{-1}{2(\sqrt{a}+i)} \times \frac{\sqrt{a}-i}{\sqrt{a}-i}$$

$$= \frac{-\sqrt{a}-i}{2(a+1)}$$

Residue of $f(z)$ at $z = -\sqrt{a}$ $\Big\} = \lim_{z \to -\sqrt{a}} (z+\sqrt{a}) \cdot \frac{z}{(a-z^2)(z+i)}$

$$= \lim_{z \to -\sqrt{a}} (z+\sqrt{a}) \frac{z}{(\sqrt{a}+z)(\sqrt{a}-z)(z+i)}$$

$$= \lim_{z \to -\sqrt{a}} \frac{z}{(\sqrt{a}-z)(z+i)}$$

$$= \frac{-\sqrt{a}}{\sqrt{a}-(-\sqrt{a})(-\sqrt{a}+i)}$$

$$= \frac{-\sqrt{a}}{2\sqrt{a}(-\sqrt{a}+i)}$$

$$= \frac{-1}{2(-\sqrt{a}+i)} \times \frac{-\sqrt{a}-i}{-\sqrt{a}-i}$$

$$= \frac{\sqrt{a}+i}{2(a+1)}$$

$\therefore$ By cauchy's residue thearem.

$\int_C f(z)\,dz = 2\pi i$ [$\because$ Sum of the residues in c_i]

6. Evaluate $\int_C \frac{z^2}{(z-2)(z-3)} dz$ where c is the circle $|z|=4$.

Solution:-

Let $f(z) = \frac{z^2}{(z-2)(z-3)}$

Let $I = \int_{|z|=4} \frac{z^2}{(z-2)(z-3)} dz$

$(z-2)(z-3) = 0$

$\Rightarrow z = 2, 3$ are singular points inside of $|z|=4$.

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=a\end{array}\right\} = \lim_{z \to a} (z-a)\, f(z)$$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=2\end{array}\right\} = \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-2)(z-3)}$$

$$= \frac{2^2}{z-3}$$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=3\end{array}\right\} = \lim_{z \to 3} (z-3) \frac{z^2}{(z-2)(z-3)}$$

$$= \frac{3^2}{3-2}$$

$$= 9$$

7. Find the poles of $\frac{z+1}{z^2+2z}$

Solution:-

Let $f(z) = \frac{z+1}{z^2+2z}$

$z^2+2z = 0.$

$z(z+2) = 0$

$z=0$; $z+2=0$.

$z = 0, -2.$

$z=0, -2$ are poles of order 2.

8. Find the poles and Residues :-

i) $\frac{1}{z^2 e^z}$

Solution :-

Let $f(z) = \frac{1}{z^2 e^z}$

$$\frac{1}{z^2 e^z} = \frac{1}{z^2}(e^{-z})$$

$$= \frac{1}{z^2}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right]$$

$$\frac{1}{z^2 \cdot e^z} = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots$$

pole of the order 2 at $z=0$.

The residue there, that is, the coefficient of $\frac{1}{z}$ being -1.

ii) $\frac{z+1}{z^2-2z}$.

Solution :-

Given $f(z) = \frac{z+1}{z^2-2z}$

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$$

$$z(z-2) = 0$$

$$z = 0 \text{ (or) } z-2 = 0$$

$$z = 2$$

$z=0$ is simple pole

$z=2$ is simple pole.

Residue of $f(z)$ at $z=0$ $\Big\}$ $= \lim\limits_{z \to 0} (z-0) \cdot \frac{z+1}{z(z-2)}$

$$= \lim_{z \to 0} z \cdot \frac{z+1}{z(z-2)}$$

$$= \lim_{z \to 0} \frac{z+1}{z-2}$$

$$= \frac{0+1}{0-2} = -\frac{1}{2}$$

$$= -\frac{1}{2}.$$

Scanned with CamScanner

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=2\end{array}\right\} = \lim_{z\to 2} (z-2)\frac{z+1}{z(z-2)}$$

$$= \lim_{z\to 2} \frac{z+1}{z}.$$

$$= \frac{2+1}{2}$$

$$= \frac{3}{2}$$

iii) $\dfrac{1}{(z-1)(z-2)}$

Solution:-

Given $f(z) = \dfrac{1}{(z-1)(z-2)}$

$(z-1)(z-2) = 0.$

$z-1=0$; $z-2=0$

$z=1$ $\quad$ $z=2$

$z=1$ is a simple pole.

$z=2$ is a simple pole.

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=1\end{array}\right\} = \lim_{z\to 1} (z-1)\frac{1}{(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{1}{(z-2)}$$

$$= \frac{1}{(1-2)} = \frac{-1}{1}$$

$$= -1$$

$$\left.\begin{array}{l}\text{Residue of } f(z)\\ \text{at } z=2\end{array}\right\} = \lim_{z\to 2} (z-2)\frac{1}{(z-1)(z-2)}$$

$$= \lim_{z\to 2} \frac{1}{(z-1)}$$

$$= \frac{1}{(2-1)} = \frac{1}{1}$$

$$= 1.$$

Scanned with CamScanner

9. Find the residue of $\frac{1+e^z}{\sin z + z\cos z}$ at the pole $z=0$.

Solution :-

Given $f(z) = \frac{1+e^z}{\sin z + z\cos z}$

$z=0$ is a simple pole.

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=0\end{array}\right\} = \lim_{z\to 0} (z-0)\frac{1+e^z}{z\left(\frac{\sin z}{z}+\cos z\right)}$$

$$= \lim_{z\to 0} z \frac{1+e^z}{z\left(\frac{\sin z}{z}+\cos z\right)}$$

$$= \lim_{z\to 0} \frac{1+e^z}{\frac{\sin z}{z}+\cos z}$$

$$= \frac{1+e^0}{\frac{\sin 0}{0}+\cos 0} = \frac{1+1}{0+1} = \frac{2}{1}.$$

$= 2.$

10. If C is the positively oriented circle $|z|=2$, evaluate $I = \int_C \frac{5z-2}{z(z-1)} dz$.

Solution :-

Given $f(z) = \frac{5z-2}{z(z-1)}$

$z(z-1) = 0$

$z=0$; $z-1=0$

$z=1$

$z=0$ is a simple pole

$z=1$ is a simple pole.

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=0\end{array}\right\} = \lim_{z\to} (z-$$

$$= \lim_{z \to 0} \frac{5z-2}{z-1}$$

$$= \frac{5(0)-2}{0-1} = \frac{-2}{-1}$$

$$= 2$$

Residue of $f(z)$ at $z=1$ $\Big\} = \lim_{z \to 1} (z-1) \cdot \frac{5z-2}{z(z-1)}$

$$= \lim_{z \to 1} \frac{5z-2}{z}$$

$$= \frac{5(1)-2}{1} = 5-2$$

$$= 3$$

By the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i \,[\text{Sum of Residues}]$$

$$\int_C \frac{5z-1}{z(z-1)}\,dz = 2\pi i\,(2+3)$$

$$= 10\pi i$$

$$\int_C \frac{5z-1}{z(z-1)}\,dz = 10\pi i.$$

11. If C be a scro curve encircling origin. Show that

i) $\int_C \frac{2e^z}{z}\,dz = 4\pi i.$

proof :-

$z=0$ is a simple curve.

Residue of $f(z)$ at $z=0$ $\Big\} = \lim_{z \to 0} (z-0) \cdot \frac{2e^z}{z}$

$$= \lim_{z \to 0} z \,\frac{2e^z}{z}$$

$$= \lim_{z \to 0} 2e^z$$

$$= 2e^0$$

$$= 2$$

By the Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \,[\text{sum of residues}]$$

$$= 2\pi i\,[2]$$

$$= 4\pi i$$

$$\int_C \frac{2e^z}{z}\,dz = 4\pi i.$$

---

ii) $\displaystyle\int_C \frac{3+z}{z}\,dz = 6\pi i$

* Smart solution: $z=0$ is a simple pole

Residue of $f(z)$ at $z=0$ $\displaystyle = \lim_{z\to 0} (z-0)\cdot\frac{3+z}{z}$

$$= \lim_{z\to 0} z\cdot\frac{3+z}{z}$$

$$= \lim_{z\to 0}(3+z)$$

$$= 3+0$$

$$= 3$$

By the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{sum of the Residue}]$$

$$= 2\pi i\,(3)$$

$$= 6\pi i$$

$$\int_C \frac{3+z}{z}\,dz = 6\pi i$$

---

12. If C is the positively oriented circle $|z|=2$ show that

i) $\displaystyle\int_C \frac{z-2}{z(z-1)}\,dz = 2\pi i.$

proof:

let $f(z) = \dfrac{z-2}{z(z-1)}$

$z(z-1)=0$

$z=0$, $z-1=0$

$z=1$

$z=0$ and $z=1$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=0\end{array}\right\} = \lim_{z\to 0} (z-0)\,\frac{z-2}{z(z-1)}$$

$$= \lim_{z\to 0} z \cdot \frac{z-2}{z(z-1)}$$

$$= \lim_{z\to 0} \frac{z-2}{z-1}$$

$$= \frac{0-2}{0-1} = \frac{-2}{-1}$$

$$= 2$$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=1\end{array}\right\} = \lim_{z\to 1} (z-1) \cdot \frac{z-2}{z(z-1)}$$

$$= \lim_{z\to 1} \frac{z-2}{z}$$

$$= \frac{1-2}{1} = -1$$

$$= -1$$

Using Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \left[\text{Sum of Residue}\right]$$

$$= 2\pi i\,[2-1]$$

$$= 2\pi i\,[1]$$

$$= 2\pi i$$

$$\therefore \int_C \frac{z-2}{z(z-1)}\,dz = 2\pi i.$$

(ii) $\dfrac{e^{2z}}{(z+1)^3}\,dz = \dfrac{4\pi i}{e^2}$.

Proof:-

Let $f(z) = \dfrac{e^{2z}}{(z+1)^3}$

$(z+1)^3 = 0$

$z+1 = 0$

$z = -1$, $m = 3$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=-1\end{array}\right\} = \lim_{z\to -1} \frac{1}{2!}\frac{d^2}{dz^2}(z+1)^3 \cdot \frac{e^{2z}}{(z+1)^3}$$

$$= \frac{1}{2}\lim_{z\to -1} \frac{d^2}{dz^2} e^{2z} \longrightarrow \left[\begin{array}{l} 2 \cdot e^{2z} \\ 2 \cdot 2 \cdot e^{2z}\end{array}\right]$$

$$= \frac{1}{2}\lim_{z\to -1} e^{2z} = \frac{4}{2} e^{-2} = 2e^{-2}$$

Scanned with CamScanner

using residue Theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of the residue}]$$

$$= 2\pi i\left(\frac{2}{e^2}\right)$$

$$= \frac{4\pi i}{e^2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^3}\,dz = \frac{4\pi i}{e^2}.$$

13. If C is the positively oriented circle $|z|=3$. Show that

i) $\int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i.$

proof:-

Given $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$

$z(z-2)=0$

$z-2=0$

$z=2$

$z=0$ is a simple pole

$z=2$ is a simple pole.

Residue of $f(z)$ at $z=0$ $\Big\} = \lim_{z\to 0}(z-0)\cdot\frac{z+1}{z(z-2)}$

$$= \lim_{z\to 0} z\cdot\frac{z+1}{z(z-2)}$$

$$= \lim_{z\to 0}\frac{z+1}{z-2}$$

$$= \frac{0+1}{0-2} = \frac{1}{-2}$$

$$= -1/2.$$

Residue of $f(z)$ at $z=2$ $\Big\} = \lim_{z\to 2}(z-2)\cdot\frac{z+1}{z(z-2)}$

$$= \lim_{z\to 2}\frac{z+1}{z}$$

$$= \frac{2+1}{2} = \frac{3}{2}$$

$$= \frac{3}{2}$$

By using residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ [Sum of Residue]}$$
$$= 2\pi i \left[-\frac{1}{2}+\frac{3}{2}\right]$$
$$= 2\pi i \left[\frac{2}{2}\right]$$
$$= 2\pi i.$$
$$\therefore \int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i.$$

14. Show that $\frac{z^2}{z^2+a^2}$.

Solution:-

Let $z^2+a^2=0$

$z^2=-a^2$

$z=\pm ia$

$z=ia,\ z=-ia$

$z=ia$ is a simple pole.

$z=-ia$ is a simple pole.

Residue of $f(z)$ at $z=ia$ $= \lim_{z\to ia} (z-ia)\cdot\frac{z^2}{a^2+z^2}$

$$= \lim_{z\to ia} (z-ia)\frac{z^2}{(z+ia)(z-ia)}$$
$$= \lim_{z\to ia} \frac{z^2}{z+ia}$$
$$= \frac{(ia)^2}{ia+ia} = \frac{i^2a^2}{2ia}$$
$$= \frac{a i}{2}$$
$$= \frac{a}{2}i.$$

Residue of $f(z)$ at $z=-ia$ $= \lim_{z\to -ia} (z+ia)\frac{z^2}{(z+ia)(z-ia)}$

$$= \lim_{z\to -ia}\frac{z^2}{(z-ia)}$$

$$= \frac{(-ia)^2}{-ia-ia} = \frac{(-i)^2 \cdot a^2}{-2ia}$$

$$= \frac{i^2 \cancel{a^2}}{-2ia}$$

$$= \frac{a}{2}$$