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CHAPTER 1

Statistical Quality Control

CHAPTER OUTLINE

DISCUSSION AND REVIEW QUESTIONS ;
ASSORTED REVIEW PROBLEMS

1·1. INTRODUCTION

Statistical Quality Control, abbreviated as S.Q.C., is one of the most important applications of the statistical techniques in industry. These techniques are based on the theory of probability and sampling, and are being extensively used in almost all industries such as armament, aircraft, automobile, textile, electrical equipment, plastic, rubber, electronics, chemicals, petroleum, transportation, medicine, and so on. In fact, it is impossible to think of any industrial field where S.Q.C. is not used.

The most important word in the term 'Statistical Quality Control' is *quality*. By quality we mean an attribute of the product that determines its fitness for use. The range of these attributes is pretty wide—physical, chemical, aesthetic, etc. A product may have several aspects of quality as well as an overall quality which is something more than the sum of its individual quality aspects—a property technically known as *Synergy*.

Quality here means a level/standard of the product which, in turn, depends on four *Ms* besides many other factors—materials, manpower, machines and management.

Quality control is a powerful productivity technique for effective diagnosis of lack of quality (or conformity to settled standards) in any of the materials, processes, machines or end-products. It is essential that the end products possess the qualities that the consumer expects of them, for the progress of industry depends on the successful marketing of products. Quality control ensures this by insisting on quality specifications all along the line from the arrival of materials through each of their processing to the final delivery of goods.

Quality control, therefore, covers all the factors and processes of production which may be broadly classified as follows :

(i) *Quality of materials*. Material of good quality will result in smooth processing thereby reducing the waste and increasing the output. It will also give better finish to the end products.

(ii) *Quality of manpower*. Trained and qualified personnel will give increased efficiency due to the better quality production through the application of skill and also reduce production cost and waste.

(iii) *Quality of machines*. Better quality equipment will result in efficient work due to lack or scarcity of breakdowns and thus reduce the cost of defectives.

(iv) *Quality of management*. A good management is imperative for increase in efficiency, harmony in relations, and growth of business and markets.

Remarks 1. The course of quality control was set by the work of Walter A. Shewhart of Bell Telephone Laboratories in 1924. He first applied a statistical control chart to manufactured products and later used statistical refinements for process control. Two other men from the Bell system, H.F. Dodge and H.C. Romig, applied statistical theory to sampling inspection to produce their widely used *Sampling Inspection Tables*.

The advent of World War II awakened an otherwise lethargic interest in statistical techniques for quality control. The armed forces adopted scientifically designed sampling inspection plans which finally culminated in the publication of Military Standard 105 for acceptance sampling of attributes and put pressure on suppliers to adopt equivalent inspection procedures for their input to keep it regularly rejected by the military services. The training and research that accompanied the original and provide governmental applications spawned an enthusiastic following and aroused interest in related quality control techniques. Today many organisations internationally promote quality control, and publish books, and journals regularly report new developments.

2. The discussion on quality control explains the three basic aspects involved in any quality control

Engineering. The creation and development of a product is basically engineering; the development of quality evaluation through improved inspection procedures is also engineering ; again, the knowledge of causes of defects and sub-standard products and their rectification is engineering.

Statistical. The concept of the behaviour of a process, which has brought in the idea of 'prevention' and 'control', is statistical ; building an information system to satisfy the concept of 'prevention' and 'control' and improving upon product quality, requires statistical thinking.

Managerial. The competent use of the engineering and statistical technology is managerial ; the creation of a climate for quality consciousness in the organisation depends upon the policies and practices of the management ; the effective coordination of the quality control function with those of others is managerial.

1·2. BASIS OF STATISTICAL QUALITY CONTROL

The basis of statistical quality control is the degree of *'variability'* in the size or the magnitude of a given characteristic of the product. Variation in the quality of manufactured product in the repetitive process in industry is inherent and inevitable. These variations are broadly classified as being due to two causes, *viz.,* (i) *chance causes,* and (ii) *assignable causes.*

Chance Causes. Some *"stable pattern of variation"* or *"a constant cause system"* is inherent in any particular scheme of production and inspection. This pattern results from many minor causes that behave in a random manner. The variation due to these causes is beyond the control of human hand and cannot be prevented or eliminated under any circumstances. One has got to allow for variation within this stable pattern, usually termed as *allowable variation.* The range of such variation is known as 'natural tolerance of the process'.

Assignable Causes. The second type of variation attributed to any production process is due to non-random or the so-called assignable causes and is termed as *preventable variation.* The assignable causes may creep in at any stage of the process, right from the arrival of the raw materials to the final delivery of goods. Some of the important factors of assignable causes of variation are sub-standard or defective raw materials, new techniques or operations, negligence of the operators, wrong or improper handling of machines, faulty equipment, unskilled or inexperienced technical staff, and so on. These causes can be identified and eliminated and are to be discovered in a production process before it goes wrong, *i.e.,* before the production becomes defective.

Chance causes of variation	Assignable causes of variation
(i) *Consist* of many individual causes.	*Consist* of just a few individual causes.
(ii) Any one chance cause results in only a small amount of variation.	Any one assignable cause can result in a large amount of variation.
(iii) Chance variation cannot economically be eliminated from a process.	The presence of assignable variation can be detected, and action to eliminate the causes is usually economically justified.
(iv) Some typical chance causes of variation are : — Slight vibration of a machine. — Lack of human perfection in reading instruments and setting controls. — Voltage fluctuations and variation in temperatures.	Some typical assignable causes of variation are : —Negligence of operators. —Defective raw material. —Faulty equipment. —Improper handling of machines.

S.Q.C. means planned collection and effective use of data for studying causes of variations in quality either as between processes, procedures, materials, machines, etc., or over periods of time. This cause-effect analysis is then fed back into the system with a view to continuous action on the processes of handling, manufacturing, packaging, transporting and delivery at end-use.

The main purpose of *Statistical Quality Control (S.Q.C.)* is to devise statistical techniques which would help us in separating the assignable causes from the chance causes, thus enabling us to take immediate remedial action whenever assignable causes are present. The elimination of assignable causes of erratic fluctuations is described as bringing a process under control.

> *A production process is said to be in a state of statistical control, if it is governed by chance causes alone, in the absence of assignable causes of variation.*

S.Q.C. is a productivity enhancing and regulatory technique (PERT) with three factors— Management, Methods and Mathematics. More particularly, if we want to properly design a self-regulating system for quality, we must look to the field of cybernetics for design information. There are six elements to a cybernetic or self-regulating system :

1. Management
2. Standard/Specification
3. Measurement/Comparison
4. Action on process/product/system
5. Information (Feedback System/Quality Information Service)
6. R & D with two objectives :

   (a) better quality standard or more economical quality standard.

   (b) better means for achieving the standard.

It is important to note that if any element of the system is missing or mismatched, it will not function. Further, control is two-fold—controlling the process (process control) and controlling the finished products (product control). It is expected that items from a controlled process should have a higher degree of conformity to specifications.

1·3. STATISTICAL QUALITY CONTROL (DEFINITION)

A few definitions are being reproduced below to make the term understandable :

> ❶ *"S.Q.C. may be broadly defined as that industrial management technique by means of which product of uniform acceptable quality are manufactured. It is mainly concerned with setting things right rather than discovering and rejecting those made wrong."*—**Duncan**
>
> ❷ *"S.Q.C. refers to the systematic control of those variables encountered in a manufacturing process which affect the excellence of the end product. Such variables are from the application of materials, men, machines and manufacturing conditions."*
>
> —**Bethel, Atwater and Stackman**
>
> ❸ *"Statistical Quality Control is simply a statistical method for determining the extent to which quality goals are being met without necessarily checking every item produced and for indicating whether or not the variations which occur are exceeding normal expectations. S.Q.C. also enables us to decide whether to reject or accept a particular product."* —**Grant**

1·4. BENEFITS OF STATISTICAL QUALITY CONTROL

The following are some of the benefits that result when a manufacturing process is operating in a state of statistical control :

❶ An obvious advantage of S.Q.C. is the control, maintenance and improvement in the quality standards.

❷ The act of getting a process in statistical quality control involves the identification and elimination of assignable causes of variation and possibly the inclusion of good ones, *viz.,* new material or methods. This (*a*) helps in the detection and correction of many production troubles, and (*b*) brings about a substantial improvement in the product quality and reduction of spoilage and rework.

❸ It tells us when to leave a process alone and when to take action to correct troubles, thus preventing frequent and unwarranted adjustments.

❹ If a process in control (which is doing about all we can expect of it) is not good enough, we shall have to make more or less a radical (fundamental) change in the process—just meddling (tampering) with it won't help.

❺ A process in control is predictable—we know what it is going to do and thus we can more safely guarantee the product. In the presence of good statistical control by the supplier, the previous lots supply evidence on the present lots, which is not usually the case if the process is not in control.

❻ If testing is destructive (*e.g.*, testing the breaking strength of chalk ; proofing of ammunition, explosives. crackers, etc.), a process in control gives confidence in the quality of untested product which is not the case otherwise.

❼ It provides better quality assurance at lower inspection cost.

❽ Quality control finds its applications not only in the sphere of production, but also in other areas like packaging, scrap and spoilage, recoveries, advertising, etc. Foreign trade items of developing countries like India are particularly appropriate for every type of quality control in every possible area.

❾ The very presence of a quality control scheme in a plant improves and alerts the personnel. Such a scheme is likely to breed 'quality consciousness' throughout the organisation which is of immense long-run value.

❿ S.Q.C reduce waste of time and material to the absolute minimum by giving an early warning about the occurrence of defects. Savings in terms of the factors stated above mean less cost of production and hence may ultimately lead to more profits.

Remarks 1. An SQC department is, thus, an essential part of a modern plant, and its important functions are as follows :

(*i*) Evaluation of quality standards of incoming materials, products in process and of finished goods.

(*ii*) Judging the conformity of the process to established standards and taking suitable action when deviations are noted.

(*iii*) Evaluation of optimum quality obtainable under given conditions.

(*iv*) Improvement of quality and productivity by process control and experimentation.

2. The following diagram [Fig. 1·1] gives a summary of the advantages of quality control in industry :

A substantial increase in productivity, basic savings in costs, and improvement in quality of products are attainable in any industry by means of the application of quality control techniques.

The basic pre-requisites for successful quality control applications are :

(a) an alert and progressive management,

(b) competent technical staff, searching for new methods and new economies,

(c) competitive or social pressures for technical advancement, and

(d) the training of top management staff in new statistical methods.

3. The increasing role of computers in process control and acceptance decisions not only reduces time lags, but additionally makes possible certain cost reductions. The reduced calculations that were formerly required to manually prepare control charts and make acceptance plan decisions have freed up quality control personnel to do other duties. In some cases the number of these personnel has actually been reduced. Inventory levels may be reduced because fewer lots are in quality control holding areas waiting for disposition. Reduced inventories can reduce holding costs, free up capital funds, and promote increased operating efficiency through reduced congestion.

> QUALITY CONTROL
> APPROACH AND BENEFITS
>
> *Planned collection of data, analysis and*
> *interpretation*
>
> ↓
>
> *Improvement in product quality and design*
> *Reduction in operating costs and losses*
> *Reduction in scrap*
> *Saving in excess use of materials*
> *Removing production bottlenecks*
> *Reduction in inspection*
> *Evaluation of scientific tolerances*
> *Improvement of employee morale*
> *Maintenance of operating efficiency*
> *quality consciousness*
> *Greater customer satisfaction*
>
> ↓
>
> *ENHANCED PRODUCTIVITY*
>
> **Fig. 1·1.**

The growth of computers in quality control departments has been spearheaded by a growing list of computer companies and computer software companies which provide standard quality control computer programs. These special programs are usually an integral part of larger management information systems and thus involve little out-of-pocket cost to users.

1·5. PROCESS CONTROL AND PRODUCT CONTROL

As already stated the main objective in any production process is to control and maintain a satisfactory quality level of the manufactured product so that it conforms to specified quality standards. In other words, we want to ensure that the proportion of defective items in the manufactured product is not too large. This is termed as '*process control*' and is achieved through the technique of '*Control Charts*' pioneered by W.A. Shewhart in 1924.

On the other hand, by *product control* we mean controlling the quality of the product by critical examination at strategic points and this is achieved through '*Sampling Inspection Plans*' pioneered by H.F. Dodge and H.C. Romig. Product control aims at guaranteeing a certain quality level to the consumer regardless of what quality level is being maintained by the producer. In other words, it attempts to ensure that the product marketed by sale department does not contain a large number of defective (unsatisfactory) items. Thus, product control is concerned with classification of raw materials, semi-finished goods or finished goods into acceptable or rejectable items.
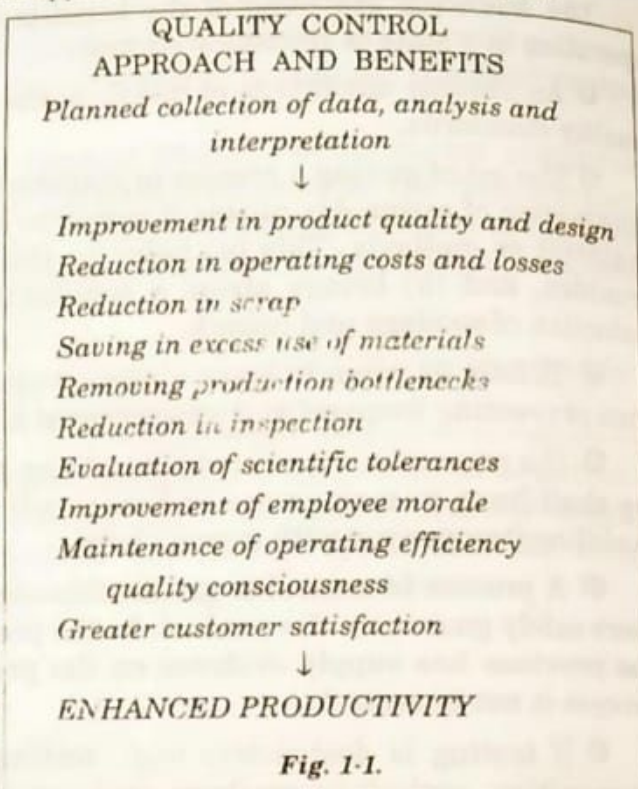
The following diagram [Fig. 1·2] summarizes the techniques of S.Q.C. :

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                        ┌─────────────────────────┐
                        │   TECHNIQUES  OF  S.Q.C. │
                        └─────────────────────────┘
                 ┌────────────────┴────────────────┐
        Process Control                      Product Control
    (by Control Chart Device)          (by Sampling Inspection Plans)
      ┌──────────┴──────────┐            ┌──────────┴──────────┐
   Variables            Attributes     Attributes          Variables
   ┌───┼───┐           ┌───┼───┐
X̄-chart R-chart σ-chart  C-chart np-chart p-chart
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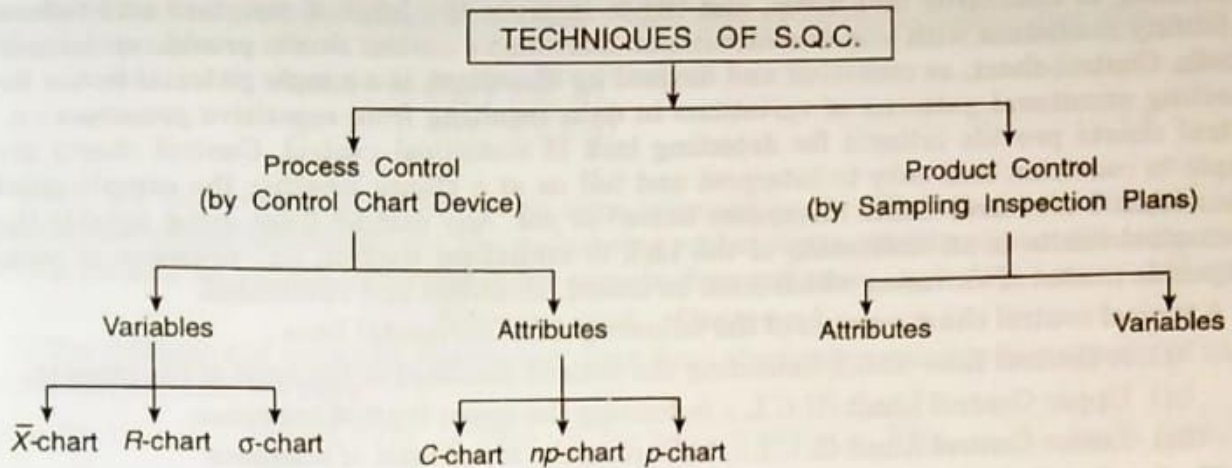
Fig. 1·2

1·5·1. Control Limits, Specification Limits and Tolerance Limits

1. Control Limits. These are limits of sampling variation of a statistical measure (*e.g.*, mean, range, or fraction-defective) such that if the production process is under control, the values of the measure calculated from different rational sub-groups will lie within these limits. Points falling outside control limits indicate that the process is not operating under a system of chance causes, *i.e.*, assignable causes of variation are present, which must be eliminated. Control limits are used in 'Control Charts'.

2. Specification Limits. When an article is proposed to be manufactured, the manufacturers have to decide upon the maximum and the minimum allowable dimensions of some quality characteristics so that the product can be gainfully utilised for which it is intended. If the dimensions are beyond these limits, the product is treated as defective and cannot be used. These maximum and minimum limits of variation of individual items, as mentioned in the product design, are known as '*specification limits*'.

3. Tolerance Limits. These are limits of variation of a quality measure of the product between which at least a specified proportion of the product is expected to lie (with a given probability), provided the process is in a state of statistical quality control. For example, we may claim with a probability of 0·99 that at least 90% of the products will have dimensions between some stated limits. These limits are also known as '*statistical tolerance limits*'.

Remark. The three wings of 'specification', 'production' and 'inspection' often display a very poor appreciation of one another's problems. Engineer, who prepare the specifications, often complain of poor quality ; the production wing grows dissatisfied with stringency of specifications and unnecessary rejections by the inspection wing and the inspection personnel complain not only about the poor quality of the manufactured products but also about the unreasonableness of the specified tolerances. Quality control techniques can be said to provide the necessary data and thereby a basis on which these three wings can discuss the common problems and reach agreements based on mutual understanding.

1·6. CONTROL CHARTS

The epoch-making discovery and development of control charts was made by a young physicist, Dr. Walter A. Shewart of Bell Telephone Laboratories, in 1924 and the following years. Based on the theory of probability and sampling, Shewhart's control charts provide a powerful tool of discovering and correcting the assignable causes of variation outside the 'stable pattern' of chance causes, thus enabling us to stabilize and control our processes at desired performances and thus bring the process under statistical control.

In industry one is faced with two kinds of problems : (*i*) to check whether the process is conforming to standards laid down, and (*ii*) to improve the level of standard and reduce variability consistent with cost considerations. Shewhart's control charts provide an answer to both. Control chart, as conceived and devised by Shewhart, is a simple pictorial device for detecting unnatural patterns of variations in data resulting from repetitive processes *i.e.*, control charts provide criteria for detecting lack of statistical control. Control charts are simple to construct and easy to interpret and tell us at a glance whether the sample point falls within 3-σ control limits (discussed below) or not. Any sample point going outside the 3-σ control limits is an indication of the lack of statistical control, *i.e.*, presence of some assignable causes of variation which must be traced, identified and eliminated.

A typical control chart consists of the following three horizontal lines :

(*i*) A Central Line (C.L.), indicating the desired standard or the level of the process.

(*ii*) Upper Control Limit (U.C.L.), indicating the upper limit of tolerance,

(*iii*) Lower Control Limit (L.C.L.), indicating the lower limit of tolerance

The control line as well as the upper and lower limits are established by computations based on the past records or current production records.

Major Parts of a Control Chart. A control chart generally includes the following four major parts :

1. *Quality Scale.* This is a vertical scale. The scale is marked according to the quality characteristics (either in variables or in attributes) of each sample.

2. *Plotted Samples.* The qualities of individual items of a sample are not shown on a control chart. Only the quality of the entire sample represented by a single value (a statistic) is plotted. The single value plotted on the chart is in the form of a dot (sometimes a small circle or a cross).

3. *Sample (or Sub-group) Numbers.* The samples plotted on a control chart are numbered individually and consecutively on a horizontal line. The line is usually placed at the bottom of the chart. The samples are also referred to as sub-groups in statistical quality control. Generally 25 sub-groups are used in constructing a control chart.

4. *The Horizontal Lines.* The central line represents the average quality of the samples plotted on the chart. The line above the central line shows the upper control limit (*UCL*) which is commonly obtained by adding 3 sigma's to the average, *i.e.*, Mean + 3(S.D.). The line below the central line is the lower control limit (L.C.L.) which is obtained by subtracting 3 sigmas from the average, *i.e.*, Mean − 3(S.D.). The upper and lower control limits are usually drawn as dotted lines, and the central line is plotted as a bold (dark) line.

The adjoining diagram (*Fig. 1·3*) depicts the principle of Shewhart's control chart.

In the control chart, Upper Control Limit (U.C.L.) and Lower Control Limit
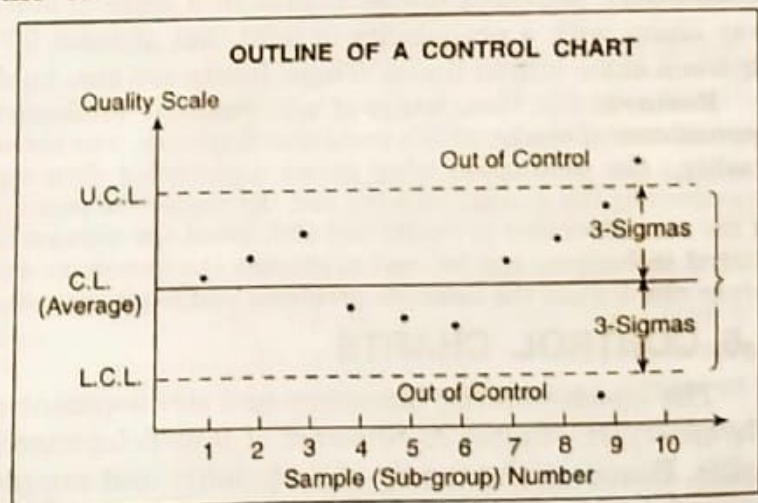

OUTLINE OF A CONTROL CHART

Fig. 1·3

(L.C.L.) are usually plotted as dotted lines and central line (C.L.) is plotted as a bold (dark) line. If t is the underlying statistic then these values depend on the sampling distribution of t and are given by :

$$\text{U.C.L.} = E(t) + \text{S.E.} (t)$$
$$\text{L.C.L.} = E(t) - 3\,\text{S.E.}(t)$$
$$\text{C.L.} = E(t)$$

1·6·1. 3σ Control Limits. 3-σ limits were proposed by **Dr. Shewhart** for his control charts from various considerations, the main being probabilistic considerations. Consider the statistic $t = t(x_1, x_2, ..., x_n)$, a function of the sample observations $x_1, x_2, ..., x_n$. Let

$$E(t) = \mu_t \quad \text{and} \quad \text{Var}(t) = \sigma_t^2$$

If the statistic t is normally distributed, then from the fundamental area property of the normal distribution, we have

$$P[\mu_t - 3\sigma_t < t < \mu_t + 3\sigma_t] = 0{\cdot}9973 \;\Rightarrow\; P[|t - \mu_t| < 3\sigma_t] = 0{\cdot}9973 \; i.e., \; P[|t - \mu_t| > 3\sigma_t] = 0{\cdot}0027$$

In other words, the probability that a random value of t goes outside the 3-σ limits, $viz.$, $\mu_t \pm 3\sigma_t$ is 0·0027, which is very small. Hence, if t is normally distributed, the limits of variation should be between $\mu_t + 3\sigma_t$ and $\mu - 3\sigma_t$ which are termed respectively the *Upper Control Limit* (U.C.L.) and *Lower Control Limit* (L.C.L.). If, for the ith sample, the observed t_i lies between the upper and lower control limits, there is nothing to worry as in such a case variation between samples is attributed to chance $i.e.$, in this case the process is in statistical control. It is only when any observed t_i falls outside the control limits, it is considered to be a danger signal indicating that some assignable cause has crept in which must be identified and eliminated.

Remarks 1. If the assumption regarding normality of the statistic t does not hold, then the above argument does not remain strictly valid. In practice, the quality characteristic can seldom be supposed to be exactly normal. For non-normal population, ($i.e.$, if the sampling distribution of statistic t is not normal) we apply *Chebychey's Inequality* in probability theory which states that for any constant $k > 0$,

$$P\left[|t - E(t)| < k\right] \geq 1 - \frac{\text{Var}(t)}{k^2} \quad\Rightarrow\quad P\left[|t - \mu_t| < 3\sigma_t\right] \geq 1 - \frac{\sigma_t^2}{9\sigma_t^2} = \frac{8}{9} = 0{\cdot}9,$$

which is also fairly high for practical purposes and the above argument holds, more or less. However, in practice, σ_t is not known and is estimated from the sample data and consequently Chebychev's inequality does not hold if σ_t is not known.

Moreover, according to the central limit theorem in probability, the statistics of observations drawn from non-normal populations will exhibit nearly normal behaviour. Of course, such behaviour will be more closely normal, the closer the underlying population is to a normal distribution, but the principle applies in any case.

Hence, even for non-normal population, 3-σ limits are almost universally used, as they have been found to be most suitable empirically in the sense that the 3-σ control charts have been found to give excellent protection against both types of wrong actions we can take, $viz.$, looking for trouble when there is none and not looking for trouble when there really is one.

2. If none of the sample points falls outside the control limits and if there is no evidence of non-random variation within the limits, it does not imply the absence of assignable causes altogether. All we can infer is that the hypothesis of random variations alone is reasonable one and from management point of view, looking for special assignable causes at this stage is unlikely to be profitable.

3. It has been emphasised strongly by Dr. Shewhart that a production process should not be adjudged in statistical control unless the random variation pattern persists for quite some time and for a sizable volume of output. More specifically he states :

"This potential state of economic control can be approached only as a statistical limit even after the assignable causes of variability have been detected and removed. Control of this kind cannot be reached

in a day. It cannot be reached in the production of a product in which only a few pieces are manufactured. It can, however, be approached scientifically in a continuing mass production."

Usually, a process should be considered in statistical control if the pattern of random variation is exhibited by a sequence of not less than twenty-five samples, each of size four.

1·7. TOOLS FOR S.Q.C.

The following four, separate but related techniques, are the most important statistical tools for data analysis in quality control of the manufactured products :

1. *Shewhart's Control Chart for Variables i.e.,* for a characteristic which can be measured quantitatively. Many quality characteristics of a product are measurable and can be expressed in specific units of measurements such as diameter of a screw, tensile strength of steel pipe, specific resistance of a wire, life of an electric bulb, etc. Such variables are of continuous type and are regarded to follow normal probability law. For quality control of such data, two types of control charts are used and technically these charts are known as :

     (a) *Charts for $\overline{X}$ (mean) and R (Range), and*

     (b) *Charts for $\overline{X}$ (Mean) and σ (standard deviation).*

2. *Shewhart's Control Chart for fraction Defective or p-Chart.* This chart is used if we are dealing with attributes in which case the quality characteristics of the product are not amenable to measurement but can be identified by their absence or presence from the product or by classifying the product as defective or non-defective.

3. *Shewhart's Control Chart for the 'Number of Defects' per unit or c-Chart.* This is usually used with advantage when the characteristic representing the quality of a product is a discrete variable, *e.g.,* (i) the number of defective rivets in an aircraft wing, and (ii) the number of surface defects observed in a roll of coated paper or a sheet of photographic film.

4. The portion of the sampling theory which deals with the quality protection given by any specified sampling acceptance procedure.

1·8. CONTROL CHARTS FOR VARIABLES

These charts may be applied to any quality characteristic that is measurable. In order to control a measurable characteristic we have to exercise control on the measure of location as well as the measure of dispersion. Usually $\overline{X}$ and R charts are employed to control the mean (location) and standard deviation (dispersion) respectively of the characteristic.

1·8·1. $\overline{X}$ and R Charts. No production process is perfect enough to produce all the items exactly alike. Some amount of variation, in the produced items, is inherent in any production scheme. This variation is the totality of numerous characteristics of the production process *viz.,* raw material, machine setting and handling, operators, etc. As pointed out earlier, this variation is the result of (i) chance causes, and (ii) assignable causes. The control limits in the $\overline{X}$ and R charts are so placed that they reveal the presence or absence of assignable causes of variation in the

     (a) average—mostly related to machine setting, and

     (b) range—mostly related to negligence on the part of the operator

Steps for $\overline{X}$ and R Charts

1. *Measurement.* Actually the work of a control chart starts first with measurements. Any method of measurement has its own inherent variability. Errors in measurement can enter

into the data by :

    (i) the use of faulty instruments,

    (ii) lack of clear-cut definitions of quality characteristics and the method of taking measurements, and

    (iii) lack of experience in the handling or use of the instrument, etc.

Since the conclusions drawn from control chart are broadly based on the variability in the measurements as well as the variability in the quality being measured, it is important that the mistakes in reading measurement instruments or errors in recording data should be minimised so as to draw valid conclusions from control charts.

2. *Selection of Samples or Sub-groups.* In order to make the control chart analysis effective, it is essential to pay due regard to the rational selection of the samples or sub-groups. The choice of the sample size n and the frequency of sampling, *i.e.*, the time between the selection of two groups, depend upon the process and no hard and fast rules can be laid down for this purpose. Usually n is taken to be 4 or 5 while the frequency of sampling depends on the state of the control exercised. Initially more frequent samples will be required (15 to 30 minutes) and once a state of control is maintained, the frequency may be relaxed. Normally 25 samples of size 4 each or 20 samples of size 5 each under control will give good estimate of the process average and dispersion.

Remark. While collecting data it may not be necessary to go exactly at the specified time, in fact this should not be practised. This is to avoid (i) the operative being careful at the time of sampling, or (ii) any periodicities of the process to coincide with sampling.

3. *Calculation of $\bar{X}$ and R for each Sub-group.* Let X_{ij}, $j = 1, 2, ..., n$ be the measurements on the ith sample ($i = 1, 2, ..., k$). The mean $\bar{X}_i$, the range R_i and the standard deviation s_i for the ith sample are given by :

$$\bar{X}_i = \frac{1}{n} \sum_j X_{ij}, \quad R_i = \max_j X_{ij} - \min_j X_{ij}, \quad s_i^2 = \frac{1}{n} \sum_j (X_{ij} - \bar{X}_i)^2 \ (i = 1, 2, ..., k) \qquad ...(1·1)$$

Next we find $\bar{\bar{X}}$, $\bar{R}$ and $\bar{s}$, the averages of sample means, sample ranges and sample standard deviations, respectively, as, follows :

$$\bar{\bar{X}} = \frac{1}{k} \sum_i \bar{X}_i, \quad \bar{R} = \frac{1}{k} \sum_i R_i, \quad \bar{s} = \frac{1}{k} \sum_i s_i \qquad ...(1·2)$$

4. *Setting of Control Limits.* It is well known that if σ is the process standard deviation (standard deviation of the universe from which samples are taken), then the standard error of sample mean is $\sigma/\sqrt{n}$, where n is the sample size, *i.e.*, S.E. $(\bar{X}_i) = \sigma/\sqrt{n}$, $(1, 2,..., k)$.

Also from the sampling distribution of range, we know that

$$E(R) = d_2.\sigma,$$

where d_2 is a constant depending on the sample size. Thus an estimate of σ can be obtained from $\bar{R}$ by the relation : $\bar{R} = d_2.\sigma \quad \Rightarrow \quad \hat{\sigma} = \bar{R}/d_2$ ...(1·3)

Also $\bar{\bar{X}}$ gives an unbiased estimate of the population mean μ, since

$$E(\bar{\bar{X}}) = \frac{1}{k} \sum_{i=1}^{k} E(\bar{X}_i) = \frac{1}{k} \sum_{i=1}^{k} \mu = \mu$$

Control Limits for $\bar{X}$-chart :

Case 1. *When standards are given, i.e.,* both μ and σ are known. The 3-σ control limits for $\bar{X}$ chart are given by :
$$E(\bar{X}) \pm 3 \text{ S.E. } (\bar{X}) = \mu \pm 3\sigma/\sqrt{n} = \mu \pm A\sigma, (A = 3/\sqrt{n}).$$

If μ' and σ' are known or specified values of μ and σ respectively, then

$$UCL_{\bar{X}} = \mu' + A\sigma' \quad \text{and} \quad LCL_{\bar{X}} = \mu' - A\sigma'$$

...(1·4)

where $A (= 3 / \sqrt{n})$ is a constant depending on n and its values are tabulated for different values of n from 2 to 25 in Table VIII in the Appendix.

Case 2. *Standards not given.* If both μ and σ are unknown, then using their estimates $\bar{\bar{X}}$ and $\hat{\sigma}$ given in Eqns. (1·2) and (1·3) respectively, we get the 3-σ control limits for the $\bar{X}$-chart as :

$$\bar{\bar{X}} \pm 3 \frac{\bar{R}}{d_2} \cdot \frac{1}{\sqrt{n}} = \bar{\bar{X}} \pm \left(\frac{3}{d_2\sqrt{n}}\right) \bar{R} = \bar{\bar{X}} \pm A_2 \bar{R}, \left(A_2 = \frac{3}{d_2 \sqrt{n}}\right)$$

$$\therefore \quad UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} \quad \text{and} \quad LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R}$$

...(1·4a)

Since d_2 is a constant depending on n, $A_2 = 3/(d_2\sqrt{n})$ also depends only on n and its values have been computed and tabulated for different values of n from 2 to 25 and are given in the Table at the end of the chapter.

If, on the other hand, the control limits are to be obtained in terms of $\bar{s}$ rather than $\bar{R}$, then an estimate of σ can be obtained from the relation. [See Remarks 1 and 2, § 1·8·4, on page 1·17] :

...(1·4b)

$$E(s) = C_2 \sigma \quad \Rightarrow \quad \bar{s} = C_2 \sigma \quad i.e., \quad \hat{\sigma} = \bar{s}/C_2$$

where
$$C_2 = \sqrt{\frac{2}{n}} \cdot \frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!} \cdot \text{ is a constant depending on } n.$$

$$\therefore \quad UCL\bar{x} = \bar{\bar{X}} + \left(\frac{3}{\sqrt{n} \, C_2}\right) \bar{s} = \bar{\bar{X}} + A_1 \bar{s} \quad \text{and} \quad LCL_{\bar{X}} = \bar{\bar{X}} - \left(\frac{3}{\sqrt{n} \, C_2}\right) \bar{s} = \bar{\bar{X}} - A_1 \bar{s}$$

...(1·4c)

The factor $A_1 = 3/\left(\sqrt{n} \, C_2\right)$ has been tabulated for different values of n from 2 to 25 in Table at the end of the chapter.

Control Limits for R-chart. R-Chart is constructed for controlling the variation in the dispersion (variability) of the product. The procedure of constructing R-chart is similar to that for the $\bar{X}$-chart and involves the following steps :

1. Compute the range $R_i = \max_j X_j - \min_j X_{ij}$, $(i = 1, 2, ..., n)$ for each sample .

2. Compute the mean of the sample ranges :

$$\bar{R} = \frac{1}{k} \sum_{i=1}^{k} R_i = \frac{1}{k}\left(R_1 + R_2 + ... + R_k\right).$$

3. **Computation of Control Limits.** The 3-σ control limits for R-chart are : $E(R) \pm 3\sigma_R$. $E(R)$ is estimated by $\bar{R}$ and σ_R is estimated from the relation :

$$\sigma_R = d_3 \, \hat{\sigma} = d_3 \cdot \frac{\overline{R}}{d_2}, \quad \text{[From (1·3)]} \qquad \qquad ...(1·5)$$

where d_2 and d_3 are constants depending on n.

$$\therefore \qquad UCL_R = E(R) + 3\sigma_R = \overline{R} + \frac{3d_3}{d_2} \overline{R} \quad \text{[From (1·5)]}$$

$$\Rightarrow \qquad UCL_R = \left(1 + \frac{3d_3}{d_2}\right) \overline{R} = D_4 \overline{R} \qquad \qquad ...(1·5a)$$

Similarly $\qquad LCL_R = \left(1 - \frac{3d_3}{d_2}\right) \overline{R} = D_3 \overline{R} \qquad \qquad ...(1·5b)$

The values of D_4 and D_3 depend only on n and have been computed and tabulated for different values of n from 2 to 25 in Table given at the end of the chapter.

However, if σ is *known*, then

$$UCL_R = E(R) + 3\sigma_R = d_2\,\sigma + 3d_3\,\sigma = (d_2 + 3d_3)\,\sigma = D_2\,\sigma \qquad ...(1·5c)$$

$$LCL_R = E(R) - 3\sigma_R = d_2\,\sigma - 3d_3\,\sigma = (d_2 - 3d_3)\,\sigma = D_1\sigma \qquad ...(1·5d)$$

In each case, (σ known or unknown), the central line is given by :

$$CL_R = E(R) = \overline{R} \qquad \qquad ...(1·5e)$$

Since range can never be negative, LCL_R must be greater than or equal to 0. In case it comes out to be negative, it is taken as zero.

Remark. It should be noted carefully that the control limits for $\overline{X}$ and R-charts are based on the assumption that different samples or sub-groups are of constant size n.

4. Construction of Control Charts for $\overline{X}$ and R, *i.e., plotting of Central Line and the Control Limits.* Control charts are plotted on a rectangular co-ordinate axis—vertical scale (ordinate) representing the statistical measures $\overline{X}$ and R, and horizontal scale (abscissa) representing the sample number. Hours, dates or lot numbers may also be represented on the horizontal scale. Sample points (mean or range) are indicated on the chart by points, which may or may not be joined.

For $\overline{X}$-chart, the central line is drawn as a *solid* horizontal line at $\overline{\overline{X}}$ and $UCL_{\overline{x}}$ and $LCL_{\overline{x}}$ are drawn at the computed values as *dotted* horizontal lines.

For $\overline{R}$-chart, the central line is drawn as a solid horizontal line at $\overline{R}$ and UCL_R is drawn at the computed value as a dotted horizontal line. If the sample size is seven or more ($n \geq 7$), LCL_R is drawn as dotted horizontal line at the computed value, otherwise ($n < 7$) LCL_R is taken as zero.

Remarks on $\overline{X}$ and R-Charts. We give below some of the very important remarks which should be clearly understood by the reader.

1. The values of constants $A, A_1, A_2, D_1, D_2, D_3$, and D_4, for different values of n are given in the Table at the end of the chapter.

2. $\overline{X}$-chart reveals undesirable variations between samples as far as their averages are concerned while the R-chart reveals any undesirable variation within samples.

3. For a process to be working under statistical control, points both in the $\overline{X}$ and R-charts should lie between the control limits. A process which is not in statistical quality control suggests the presence

of assignable causes of variation which throw the process out of control. These causes must be traced and eliminated so that the process may return to operation under stable statistical conditions. Reasons for the process being out of control vary from faulty tools, a sudden significant change in properties of new materials in a new consignment, breakdown of the lubrication system, faults in timing or speed mechanisms, etc. Tracing these causes is sometimes simple and straightforward, but in some cases it may be a rather lengthy and complicated business, especially when the process is subject to the combined effect of several external causes simultaneously.

If the process is found to be in statistical control, a comparison between the required specifications and the process capability may be carried out to determine whether the two are compatible. Should the specified tolerances prove to be too tight for process capability, there are three possible alternatives :

    (i) Re-evaluate the specifications : Are the tight tolerances really necessary for effective performance, or could they, perhaps, be relaxed with no detriment to the quality of the product.

    (ii) If relaxation of the specifications is not acceptable, perhaps a more accurate process should be selected for the purpose ?

    (iii) If both the previous alternatives are out of the question, a 100 per cent inspection must be undertaken to sort out the defective products.

4. *Modified Control Limits for Future Use.* If all the points in both the charts remain within trial control limits, then these limits are accepted as final, and used for maintaining control charts for subsequent production. If, however, some of the points go outside the limits in one of the charts then it is concluded that these samples were produced when the process was not in control and these samples are rejected as unusable. Then a second set of trial limits is constructed, using only the remaining samples, and using these fresh control limits, new charts are constructed and the remaining samples are plotted on the new charts. If all the sample points now remain within the new control limits, they are accepted as final otherwise the same procedure as described above is followed to get a third set of trial control limits. The control limits are accepted as final only when all the sample points on which they are based remain within these limits.

5. Should the routine results show a better degree of uniformity than that expected from the standard, there is evidence that the acceptance standard is too loose. The latest data must then be used to re-estimate current standard quality, which is then used for future control. This procedure should also be adopted at regular intervals, thereby producing a gradual improvement in quality standards.

6. The diagrammatic representation of the control procedure associated with $\overline{X}$ and R-charts is given in *Fig. 1·4 on page 1·15.*

1·8·2. Criterion for Detecting Lack of Control in $\overline{X}$ and R-Charts.

As pointed out earlier, the main object of the control chart is to indicate when a process is not in control. The criteria for detecting lack of control are, therefore, of fundamental and crucial importance. The pattern of the sample points in a control chart is the key to the proper interpretation of the working of the process. The following situations depict lack of control :

    1. *A point outside the control limits.* The probabilistic considerations provide a basis for hunting for lack of control in such a situation. A point going outside control limits is a clear indication of the presence of assignable causes of variation which must be searched and corrected. A point outside the control limits may result from an increased dispersion or change in level. Lack of uniformity may be due to the variation in the quality of raw materials, deficiency in the skill of the operators, loss of alignment among machines, change of working conditions, etc. It may be indicated by a point (or points) above the upper control limit for ranges. It may also result in points outside the control limits for means.

    2. *A run of seven or more points.* Although all the sample points are within control limits, usually the pattern of points in the chart indicates assignable causes. One such situation is a run of 7 or more points above or below the central line in the control chart. Such runs indicate shift in the process level. On R-chart a run of points above the central line is

indicative of increase in process spread and therefore represents an undesirable situation, while a run below the central line indicates an improvement in the sense that the variability has been reduced, *i.e.*, the process could hold to a closer tolerance.

CONTROL PROCEDURE FOR $\bar{X}$ AND R-CHARTS

Take adequate number of samples

↓

Calculate $\bar{X}$, R for each sample

↓

Find $\bar{\bar{X}}$, $\bar{R}$
Determine control limits
Draw $\bar{X}$ and R-charts

↓

Is process in statistical control ?

If "yes"   If "no"

↓   ↓

Is process compatible   Trace cause
with specifications ?
                                                                            ↓

                                                                      Remove causes

If "yes"   If "no"

↓   ↓   Continue sampling

Continue sampling   Study following   ↓
for process control   alternatives
                                                                    Draw $\bar{X}$ and
↓   ↓   R-charts

Detect trends in $\bar{X}$ ─(1) Review   ↓
                                       specifications
↓   ─(2) Changes in   Is process in
                                       process   control ?
Determine when to   ─(3) 100%
reset process   inspection

Fig. 1·4

3. One or more points in the vicinity of control limits or a run of points beyond some secondary limits, *e.g.*, a run of 2, 3 points beyond 2-σ limits or a run of 4, 5 points beyond 1-σ limits.

4. The sample points on $\bar{X}$ and R-charts, too close to the central line, exhibit another form of assignable-cause. This situation (*Fig. 1·5*) represents systematic differences within samples or sub-groups and results from improper selection of samples and biases in measurements.
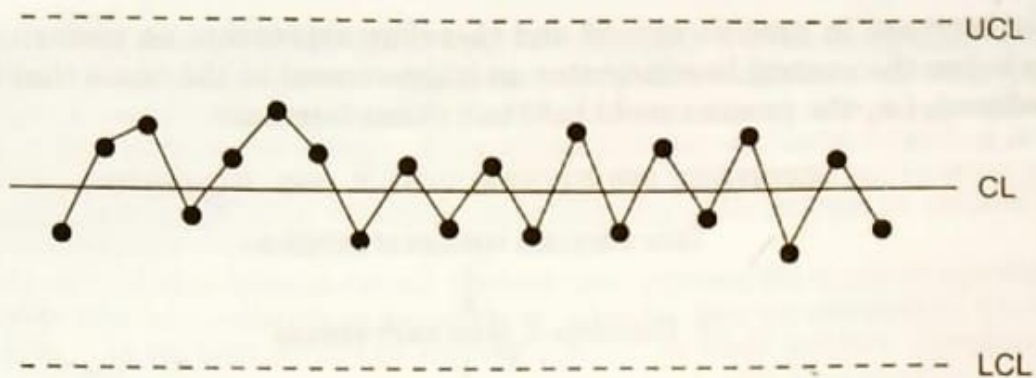
Fig. 1·5

5. *Presence of Trends*. The trends exhibited by sample points on the control chart are also an indication of assignable cause. Trend pattern [*Fig. 1·6(a)* and *Fig. 1·6(b)*], a phenomenon usually observed in engineering industry, indicates the gradual shift in the process level. Trend may be upward or downward. Tools wear and the need for resetting machines often accounts for such a shift, and it is essential to determine when machine resetting becomes desirable bearing in mind that too frequent adjustments are a serious setback to production output.

PRESENCE OF TRENDS IN CONTROL CHART
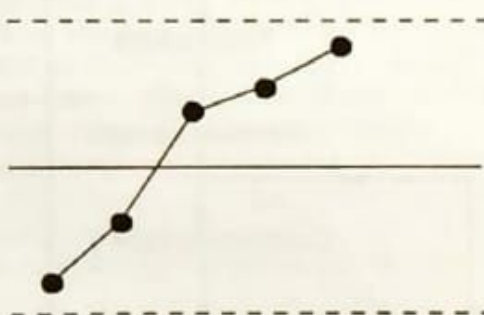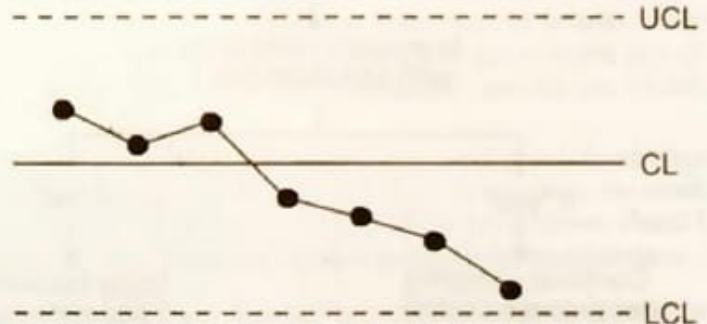
Fig. 1·6(a)   Fig. 1·6(b)

6. *Presence of Cycles*. In some cases the cyclic pattern of points in the control chart (*Fig. 1·7*) indicates the presence of assignable causes of variation. Such patterns are due to material or/and any mechanical reasons.

PRESENCE OF CYCLES IN CONTROL CHARTS

Fig. 1·7

1·8·3. Interpretation of $\overline{X}$ and R-Charts. In order to judge if a process is in control, $\overline{X}$ and R-charts should be examined together and the process should be deemed in statistical control if both the charts show a state of control. Situations exist where R-chart is in a state of control but $\overline{X}$-chart is not. We summarise below, in a tabular form, such different situations and the interpretation to be accorded to each.
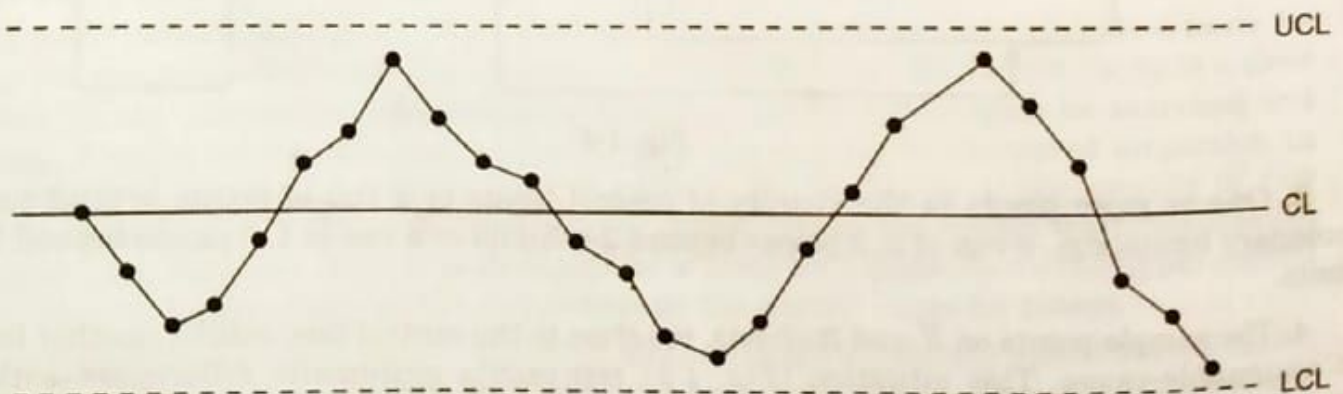
Sl. No.	Situations in		Interpretation
	R-chart	$\overline{X}$-chart	
1.	In control	Points beyond limits only on one side	Level of process has shifted
2.	In control	Points beyond limits on both the sides	Level of process is changing in erratic manner—frequent adjustments.
3.	Out of control	Points beyond limits on both sides	Variability has increased.
4.	Out of control	Out of control on one side	Both level and variability have changed.
5.	In control	Run of 7 or more points on one side of central line	Shift in process level.
6.	In control	Trend of 7 or more points. No point outside control limits	Process level is gradually changing.
7.	Runs of 7 or more points above central line	— — —	Variability has increased.
8.	Points too close to the central line	— — —	Systematic differences within sub-groups.
9.	— — —	Points too close to the central line	Systematic differences within sub-groups.

1·8·4. Control Chart for Standard Deviation (or σ-Chart). Since standard deviation is an ideal measure of dispersion, a combination of control chart for mean $(\overline{X})$ and standard deviation(s), known as $\overline{X}$ and s-charts (or $\overline{X}$ and σ-charts) is theoretically more appropriate than a combination of $\overline{X}$ and R-charts for controlling process average and process variability.

In a random sample of size n from normal population with standard deviation σ, we have

$$E(s^2) = \frac{n-1}{n}\,\sigma^2 \qquad\qquad \ldots (*)$$

and
$$E(s) = C_2\,\sigma, \text{ where } C_2 = \sqrt{\frac{2}{n}}\;\frac{[(n-2)/2]\,!}{[(n-3)/2]\,!} \qquad\qquad \ldots(**)$$

[For derivation of (*) and (**), see *Remark 1*, on next page]

The values of C_2 have been tabulated for different values of n from 2 to 25 in the Table at the end of the chapter.

Hence, in sampling from normal population, we have

$$\text{Var}\,(s) = E(s^2) - \{E(s)\}^2 = \left(\frac{n-1}{n} - C_2^2\right)\sigma^2$$

$\Rightarrow \qquad\qquad S.E.(s) = C_3\,.\,\sigma, \text{ where } C_3 = \sqrt{\frac{n-1}{n} - C_2^2}$

$$UCL_s = E(s) + 3\ S.E.(s) = (C_2 + 3C_3)\ \sigma = B_2 \cdot \sigma$$
$$LCL_s = E(s) - 3\ S.E.\ (s) = (C_2 - 3C_3)\ \sigma = B_1 \cdot \sigma \Big\} \qquad ...(1·6)$$

$$\text{Central line} = CL_s = C_2 \cdot \sigma$$

Values of B_1 and B_2 have been tabulated for different values of n.

If the value of σ is not specified or not known, then we use its estimate, based on $\bar{s}$

defined in (1·2) and given by $\hat{\sigma} = \bar{s}/C_2$. [See *Remark 2* on next page.] In this case

$$UCL_s = E(s) + 3\ S.E.(s) = \bar{s} + 3\ \frac{C_3}{C_2} \cdot \bar{s}$$

$$= \left(1 + \frac{3C_3}{C_2}\right) \bar{s} = B_4 \cdot \bar{s} \qquad ...(1·6a)$$

Similarly, we shall get

$$LCL_s = \left(1 - \frac{3C_3}{C_2}\right) \bar{s} = B_3 \cdot \bar{s} \qquad ...(1·6b)$$

and $\qquad\qquad CL_s = \bar{s} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ...(1·6c)$

where B_3 and B_4 have been tabulated for different values of n. Since s can never be negative, if LCL given by (1·6c) comes out to be negative, as will be the case for n from 2 to 5, it is taken to be zero.

Remarks 1. If X_i ; $i = 1, 2, ..., n$ is a random sample of size n from $N(\mu, \sigma^2)$ population, then the distribution of sample variance :

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2 \ ; \quad \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

is given by : $\qquad\qquad\qquad\qquad (ns^2/\sigma^2) \sim \chi_{n-1}^2 \qquad\qquad\qquad ...(1)$

$$\therefore \qquad E(ns^2/\sigma^2) = (n-1) \quad \left[\because\ \text{If}\ X \sim \chi_n^2,\ E(X) = n\right]$$

$$\Rightarrow \qquad \frac{n}{\sigma^2} E(s^2) = (n-1) \qquad \Rightarrow \qquad E(s^2) = \frac{n-1}{n} \cdot \sigma^2 \qquad ...(2)$$

Also $\qquad\qquad \text{Var}(ns^2/\sigma^2) = 2(n-1) \qquad\qquad \left[\because\ \text{If}\ X \sim \chi_n^2,\ \text{Var}(X) = 2n\right]$

$$\Rightarrow \qquad \frac{n^2}{\sigma^4} \text{Var}(s^2) = 2(n-1)\ ,\ i.e.,\quad \text{Var}(s^2) = \frac{2(n-1)}{n^2} \sigma^2 \qquad ...(3)$$

Further, if $X \sim \chi_n^2$ and $U = \sqrt{X}$, then we have

$$E(U) = E(\sqrt{X}) = \sqrt{2} \cdot \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)} \qquad ...(4)$$

We have : $\qquad\qquad X = \frac{ns^2}{\sigma^2} \sim \chi_{n-1}^2$

$$\therefore \quad E(\sqrt{X}) = E\left(\frac{ns^2}{\sigma^2}\right)^{1/2} = \frac{\sqrt{2} \cdot \Gamma[(n-1+1)/2]}{\Gamma[(n-1)/2]} \qquad \text{[From (4) with } n \text{ replaced by } (n-1)]$$

$$\Rightarrow \quad \frac{\sqrt{n}}{\sigma} E(s) = \sqrt{2}\ \frac{\Gamma(n/2)}{\Gamma[(n-1/2)]} \quad \Rightarrow \quad E(s)\ \sqrt{\frac{2}{n}}\ \frac{\Gamma(n/2)}{\Gamma[(n-1/2)]} \cdot \sigma = C_2 \sigma \qquad ...(5)$$

where $\qquad C_2 = \sqrt{\dfrac{2}{n}} \cdot \dfrac{\Gamma(n/2)}{\Gamma(n-1/2)} = \sqrt{\dfrac{2}{n}} \cdot \dfrac{[(n-2/2)]\,!}{[(n-3/2)]\,!}$ ...(6)

is a constant depending on n.

2. An estimate of $E(s)$ is provided by the average of the sample standard deviations i.e., by $\bar{s} = (s_1 + s_2 + \ldots + s_k)/k$. Hence, from (5), an estimate of the population s.d. σ is given by :

$$\bar{s} = C_2 \cdot \hat{\sigma} \qquad \Rightarrow \qquad \hat{\sigma} = \frac{\bar{s}}{C_2} \cdot \qquad \qquad \text{...(7)}$$

3. **s-Chart vs R-Chart.** In small samples the standard deviation s and range R are likely to fluctuate together, i.e., if s is small (large) R is likely to be small (large). However, for large samples a single extreme observation will have a significantly large effect on range while its effect on standard deviation will be comparatively much less. Hence for analysing or controlling variability if we use small samples, the range may be used as a substitute for standard deviation with little loss in efficiency. Since range is almost as efficient as standard deviation in small samples, it is usually preferred to standard deviation in quality control analysis because of its ease of calculations.

Example 1·1. *A machine is set to deliver the packets of a given weight. Ten samples of size five each were examined and the following results were obtained :*

Sample No. :	1	2	3	4	5	6	7	8	9	10
Mean :	43	49	37	44	45	37	51	46	43	47
Range :	5	6	5	7	7	4	8	6	4	6

Calculate the values for the central line and the control limits for the mean chart and range chart. Comment on the state of control. (Given for $n = 5$, $d_2 = 2·326$, $d_3 = 0·864$)

Solution. In the usual notations, we are given : $n = 5$, $k = 10$.

$$\bar{\bar{X}} = \frac{1}{k}\Sigma\bar{X}_i = \frac{1}{10}\left(43 + 49 + 37 + \ldots + 47\right) = 44·2$$

$$\bar{R} = \frac{1}{k}\Sigma R = \frac{1}{10}(5 + 6 + 5 + \ldots + 6) = 5·8$$

The 3-σ control limits are given by :

For $\bar{X}$-chart

$CL_{\bar{X}} = \bar{\bar{X}} = 44·2$

$UCL_{\bar{X}} = \bar{\bar{X}} + \dfrac{3\,\bar{R}}{d_2\sqrt{n}} = 44·2 + \dfrac{3 \times 5·8}{2·326\sqrt{5}}$
$\qquad = 44·2 + 3·346 = 47·546$

$LCL_{\bar{X}} = \bar{\bar{X}} - \dfrac{3\bar{R}}{d_2\sqrt{n}} = 44·2 - \dfrac{3 \times 5·8}{2·326\sqrt{5}}$
$\qquad = 44·2 - 3·346 = 40·854$

For R-Chart

$CL_R = \bar{R} = 5·8$

$UCL_R = \bar{R} + \dfrac{3d_3\,\bar{R}}{d_2} = 5·8 + \dfrac{3 \times 0·864 \times 5·8}{2·326}$
$\qquad = 5·8 + 6·463 = 12·263$

$LCL_R = \bar{R} - \dfrac{3d_3\,\bar{R}}{d_2} = 5·8 - \dfrac{3 \times 0·864 \times 5·8}{2·326}$
$\qquad = 5·8 - 6·463 = 0$ (being negative).

Comments :

$\bar{X}$-chart. Since the sample means corresponding to the sample numbers 2, 3, 6 and 7 are outside the control limits, the process average is out of control. This suggests the presence of assignable causes of variation which should be traced and corrected.

R-chart. Since none of the sample ranges lies beyond the control limits of the R-chart, the process variability is in control.

Since one of the control charts ($\bar{X}$-chart) shows lack of control, the process cannot be regarded in a state of statistical control. [See *Remark 3*, page 1·13.]

Example 1·2. *Construct a control chart for mean and the range for the following data on the basis of fuses, samples of 5 being taken every hour (each set of 5 has been arranged in ascending order of magnitude). Comment on whether the production seems to be under control, assuming that these are the first data :*

42	42	19	36	42	51	60	18	15	69	64	61
65	45	24	54	51	74	60	20	30	109	90	78
75	68	80	69	57	75	72	27	39	113	93	94
78	72	81	77	59	78	95	42	62	118	109	109
87	90	81	84	78	132	138	60	84	153	112	136

Solution.

TABLE 1·1 : CALCUALTIONS FOR SAMPLE MEANS AND RANGES

Sample No.	Sample Observations					Total	Sample Mean $(\overline{X})$	Sample Range (R)
(1)	(2)					(3)	(4) = (3) ÷ 5	(5)
1	42	65	75	78	87	347	69·4	45
2	42	45	68	72	90	317	63·4	48
3	19	24	80	81	81	285	57·0	62
4	36	54	69	77	84	320	64·0	48
5	42	51	57	59	78	287	57·4	36
6	51	74	75	78	132	410	82·0	81
7	60	60	72	95	138	425	85·0	78
8	18	20	27	42	60	167	33·4	42
9	15	30	39	62	84	230	46·0	69
10	69	109	113	118	153	562	112·4	84
11	64	90	93	109	112	468	93·6	48
12	61	78	94	109	136	478	95·6	75
						Total	859·2	716

From the above data, we get

$$\overline{\overline{X}} = \frac{1}{12} \Sigma \overline{X}_i = \frac{859 \cdot 2}{12} = 71 \cdot 6 \quad ; \quad \overline{R} = \frac{1}{12} \Sigma R_i = \frac{716}{12} = 59 \cdot 67$$

From Table given in the end of chapter, for $n = 5$, we have $A_2 = 0 \cdot 58$, $D_3 = 0$ and $D_4 = 2 \cdot 11$.

$\overline{X}$-Chart

$$UCL_{\overline{X}} = \overline{\overline{X}} + A_2 \, \overline{R} = 71 \cdot 6 + 0 \cdot 58 \times 59 \cdot 67 = 106 \cdot 21$$

$$LCL_{\overline{X}} = \overline{\overline{X}} - A_2 \, \overline{R} = 71 \cdot 60 - 0 \cdot 58 \times 59 \cdot 67 = 36 \cdot 99$$

$$CL_{\overline{X}} = \overline{\overline{X}} = 71 \cdot 60$$

R-Chart

$$UCL_R = D_4 \, \overline{R} = 2 \cdot 11 \times 59 \cdot 67 = 125 \cdot 904$$

$$LCL_R = D_3 \, \overline{R} = 0$$

$$CL_R = \overline{R} = 59 \cdot 67$$

The control charts for mean ($\overline{X}$-chart) and range (R-chart) are drawn in *Fig. 1·8(a)* and *Fig. 1·8(b)* respectively.
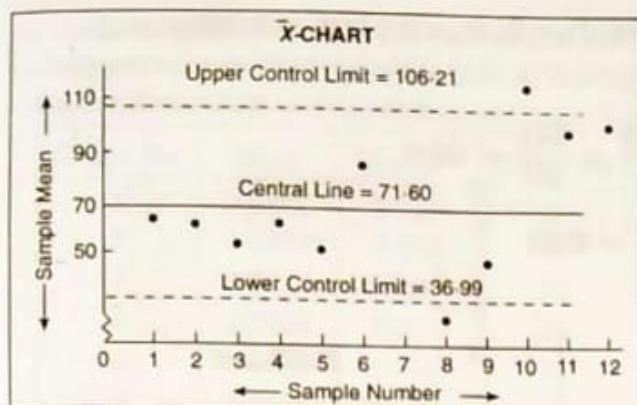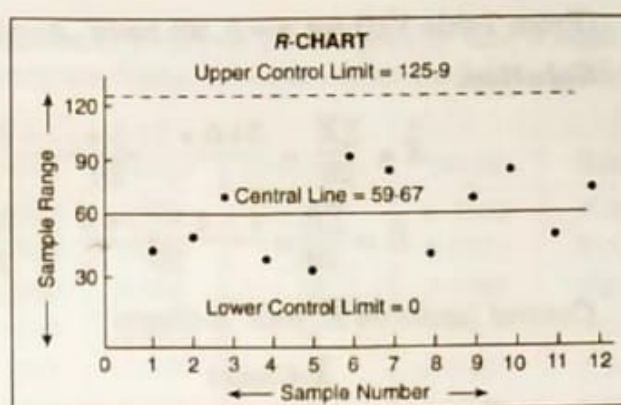
Fig. 1·8(a)

Fig. 1·8(b)

From the $\overline{X}$ chart, clearly the process average is out of control since the points (sample means) corresponding to 8th and 10th samples lie outside the control limits.

In R-chart [*Fig. 1·8(b)*], since all the sample points (ranges) fall within control limits, this chart shows that process variability is in control.

Although R-chart depicts control, the process cannot be regarded to be in statistical control since $\overline{X}$-chart shows lack of control.

The choice between $\overline{X}$ and R-charts is a managerial problem. It is better to construct R-chart first. If the R-chart indicates that the dispersion of the quality by the process is out of control, generally it is better not to construct $\overline{X}$-chart, until the quality dispersion is brought under control.

Example 1·3. *The following are the $\overline{X}$ and R values for 20 sub-groups of 5 readings. The specifications for this product characteristic are 0·4037 ± 0·0010. The values given are the last two figures of the dimension reading, i.e., 31·6 should be 0·40316 :*

Sub-group	$\overline{X}$	R	Sub-group	$\overline{X}$	R
1	34·0	4	11	35·8	4
2	31·8	4	12	38·4	4
3	30·6	2	13	34·0	14
4	33·0	3	14	35·0	4
5	35·0	5	15	33·8	7
6	32·2	2	16	31·6	5
7	33·6	5	17	33·0	5
8	32·0	13	18	28·2	3
9	33·8	19	19	31·8	9
10	37·8	6	20	35·6	6

(i) *Determine the control limits for $\overline{X}$ and R-charts for future use, eliminating all the out of control points.*

(ii) *Will be process be able to meet the specifications ?*

(iii) *Will you recommend shifting of the process centering ?*

(From Table VIII for $n = 5$, we have $A_2 = 0.58, D_3 = 0, D_4 = 2.11, d_2 = 2.326$.)

Solution.

$$\bar{\bar{X}} = \frac{\Sigma \bar{X}}{20} = \frac{34.0 + 31.8 + \ldots + 35.6}{20} = \frac{671}{20} = 33.6$$

$$\bar{R} = \frac{\Sigma R}{20} = \frac{4 + 4 + \ldots + 6}{20} = \frac{124}{20} = 6.20$$

Control limits on $\bar{X}$ and R-charts :

$\bar{X}$-Chart

$$UCL_{\bar{x}} = \bar{\bar{X}} + A_2 \bar{R} = 33.6 + (0.58 \times 6.20) = 37.20$$

$$LCL_{\bar{x}} = \bar{\bar{X}} - A_2 \bar{R} = 33.6 - (0.58 \times 6.20) = 30.00$$

$$CL_{\bar{x}} = \bar{\bar{X}} = 33.6$$

R-Chart

$$UCL_R = D_4 \bar{R} = 2.11 \times 6.20 = 13.08$$

$$LCL_R = D_3 \bar{R} = 0 \times 6.20 = 0$$

$$CL_R = \bar{R} = 6.2$$

On plotting the control charts it will be noticed that the sample points (means) corresponding to the sub-groups 10, 12 fall above $UCL_{\bar{x}}$ and sub-group 18 falls below $LCL_{\bar{x}}$ and hence the process is out of control.

Since out of control points suggest the presence of assignable causes of variations, eliminating these points, we have

$$\bar{\bar{X}}_{revised} = \frac{671.0 - (37.8 + 38.4 + 28.2)}{20 - 3} = \frac{566.6}{17} = 33.3$$

On plotting the R-chart, it will be noticed that the sample ranges for sub-groups 9 and 13 fall above UCL_R and hence the process is out of control. Eliminating these points, we have

$$\bar{R}_{revised} = \frac{124 - (19 + 14)}{20 - 2} = \frac{91}{18} = 5.06$$

(i) Revised Control Limits for Future Use :

$\bar{X}$-Chart.

$$UCL_{\bar{X}} = \bar{\bar{X}}_{revised} + A_2 \bar{R}_{revised} = 33.30 + (0.58 \times 5.06) = 33.30 + 2.93 = 36.23$$

$$LCL_{\bar{x}} = \bar{\bar{X}}_{revised} - A_2 \bar{R} = 33.30 - 2.93 = 30.37$$

$$UCL_R = D_4 \bar{R}_{revised} = 2.11 \times 5.06 = 10.67 \quad ; \quad LCL_R = D_3 \bar{R}_{revised} = 0 \times 5.06 = 0$$

(ii)

$$\hat{\sigma} = \frac{\bar{R}_{revised}}{d_2} = \frac{5.06}{2.326} = 2.175$$

Upper natural specification limit $= \bar{\bar{X}}_{revised} + 3\hat{\sigma} = 33.3 + 3(2.175) = 39.825$

Lower natural specification limit $= \bar{\bar{X}}_{revised} - 3\hat{\sigma} = 33.3 - 3(2.175) = 26.775$

In terms of actual data the natural specifications will be 0·40268 and 0·40398. *i.e.*, 0·4027 and 0·4040. Since 0·4037 ± 0·0010, *i.e.*, the specification limits are 0·4027 and 0·4047 it is evident that the process will be able to meet the specifications.

(iii) Since the process is able to meet its specifications, it is not advisable to shift the process centering provided the process remains in control at the above level ($\bar{\bar{X}}_{revised} = 33.3$), *i.e.*, 0·4033.

Example 1·4 . *The following are the mean and ranges of 20 samples of size 5 each. The data pertain to the overall length of a fragmentation bomb base manufactured during the war by the American store camp.*

Group No.	Mean	Range	Group No.	Mean	Range	Group No.	Mean	Range
1	0·8372	0·010	8	0·8344	0·003	15	0·8404	0·008
2	0·8324	0·009	9	0·8308	0·002	16	0·8372	0·011
3	0·8318	0·008	10	0·8350	0·006	17	0·8282	0·006
4	0·8344	0·004	11	0·8380	0·006	18	0·8346	0·006
5	0·8346	0·005	12	0·8322	0·002	19	0·8360	0·004
6	0·8332	0·011	13	0·8356	0·013	20	0·8374	0·006
7	0·8340	0·009	14	0·8322	0·005			

(a) *From these data, obtain the control limits for $\bar{X}$ and R-charts to control the length of bomb bases produced in the future.*

(b) *The above samples were taken every 15 minutes in order of production. The production rate was 350 units per hour and the tolerances were 0·820 and 0·840 inches.*

On the assumption that the lengths of the bomb bases are normally distributed, what percentage of the bomb base would you estimate to have length outside the tolerance limits when the process is under control at the levels indicated by the above data ?

[*For* $n = 5$, *use* $A_2 = 0·58$, $D_3 = 0$, $D_4 = 2·12$, $d_2 = 2·326$]

Solution. (a) $\bar{\bar{X}} = \dfrac{1}{20} \sum\limits_{i=1}^{20} \bar{X}_i = \dfrac{16·6796}{20} = 0·83398$; $\bar{R} = \dfrac{1}{20} \sum\limits_{i=1}^{20} R_i = \dfrac{0·133}{20} = 0·00665$

We are given that for $n = 5$, $A_2 = 0·58$, $D_3 = 0$ and $D_4 = 2·12$.

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} = 0·83398 + 0·58 \times 0·00665 = 0·837837$$

$$CL_{\bar{X}} = \bar{\bar{X}} \qquad\qquad = 0·83398$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R} = 0·83398 - 0·58 \times 0·00665 = 0·830123$$

We see that values of $\bar{X}$ corresponding to the sub-groups 11 and 17 are outside the control limits. So excluding these values, we get

$$\bar{\bar{X}}' = \frac{15·0134}{18} = 0·834077 \quad ; \quad \bar{R}' = \frac{0·121}{18} = 0·00672$$

So the new control limits for $\bar{X}$ chart are :

$$\bar{\bar{X}}' \pm A_2 \bar{R}' = 0·834077 \pm 0·58 \times 0·00672 = 0·837975 \text{ and } 0·831179,$$ which may be regarded as the final limits of $\bar{X}$-chart since no value except the rejected ones is outside these limits.

$$UCL_R = D_4 \bar{R} = 2·12 \times 0·00665 = 0·014088$$

$$CL_R = \bar{R} = 0·00665$$

$$LCL_R = D_3 \bar{R} = 0$$

From the given values of R, we see that the values corresponding to all the sub-groups fall inside the control limits, which may be taken as the final control limits.

(b) We are given that :

Upper Tolerance Limit (U.T.L.) = 0·840″ ; Lower Tolerance Limit (L.T.L.) = 0·820″

Assuming that the process is in control, the random variable X (the length of the bomb bases) is normally distributed with mean and s.d. given by

$$\hat{\mu} = \bar{\bar{X}}' = 0\cdot834077 \quad \text{and} \quad \hat{\sigma} = \frac{\bar{R}'}{d_2} = \frac{0\cdot00672}{2\cdot326} = 0\cdot002889$$

Hence, the proportion 'p' of defectives when the process is in control is given by :

$$p = P\,[X \text{ lies outside tolerance limits}] = 1 - P\,[0\cdot82 < X < 0\cdot84] \qquad \qquad \dots(*)$$

$$p_1 = P(0\cdot82 < X < 0\cdot84) = P\left(\frac{0\cdot82 - 0\cdot834077}{0\cdot002889} < Z < \frac{0\cdot84 - 0\cdot834077}{0\cdot002889}\right), \text{ where } Z \sim N(0,\,1).$$

$$= P(-4\cdot8726 < Z < 2\cdot0502) \doteq P(-4\cdot87 < Z < 2\cdot05)$$
$$= P(-4\cdot87 < Z < 0) + P(0 < Z < 2\cdot05)$$
$$= P(0 < Z < 4\cdot87) + P(0 < Z < 2\cdot05) \qquad \text{[By symmetry]}$$
$$= 0\cdot5000 + 0\cdot4798 = 0\cdot9798 \qquad \text{[From Normal Table]}$$

Substituting in (*), we obtain

$$p = 1 - 0\cdot9798 = 0\cdot0202$$

Hence, the per-cent fraction defective when process is in control is : $100 \times 0\cdot0202 = 2\cdot02$.

Example 1·5. *Twenty-five samples of size 5 were taken, and the average and range of the breaking strength of the units in each sample were ascertained. The sum of the sample averages was found to be 255 kg and that of the sample ranges to be 50 kg.*

Three-sigma trial control limits are calculated for the X and R-charts, and it is found that two sample averages and three sample ranges exceed these limits. The sample averages are 12·8 kg and 8·2 kg and the sample ranges are 6, 5 and 7 kg.

After the control limits have been revised, it is found that no sample range but one sample average exceeds these limits. The value of the average is 9·2 kg. Consequently, a third set of trial control limits is computed. Then values are such that all the remaining points fall within these limits.

(a) What are the values of the trial and final control limits ?

(b) If the specifications call for a minimum breaking strength 9·0 kg and it is assumed that the universe is normally distributed, what percentage of defective output can be expected if the indicated universe continues to be produced ?

(c) What are the natural specifications of this process ?

Solution. (a) In the usual notations, we are given :

$$n = 5, \quad k = 25, \quad A_2 = 0\cdot58, \quad D_3 = 0, \quad D_4 = 2\cdot11, \quad d_2 = 2\cdot326.$$

The values of final control limits will be such that no point goes beyond the control limits on either $\bar{X}$-chart or R-chart.

$$\bar{\bar{X}} = \frac{\Sigma\bar{X}}{k} = \frac{255}{25} = 10\cdot2 \text{ kg} \quad ; \quad \bar{R} = \frac{\Sigma R}{k} = \frac{50}{25} = 2 \text{ kg}.$$

First set of trial control limits.

$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} = 10·2 + (0·58 \times 2) = 11·36$ | $UCL_R = D_4 \bar{R} = 2·11 \times 2 = 4·22$

$UCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R} = 10·2 - (0·58 \times 2) = 9·04$ | $LCL_R = D_3 \bar{R} = 0 \times 2 = 0$

Discarding the out of control points on $\bar{X}$ chart,

$$\bar{\bar{X}}_{revised} = \bar{\bar{X}}' = \frac{255 - (12·8 + 8·2)}{25 - 2} = \frac{234}{23} = 10·174 \; ; \bar{R}_{revised} = \bar{R}' = \frac{50 - (6 + 5 + 7)}{25 - 3} = \frac{32}{22} = 1·455$$

Second set of trial control limits.

$UCL_{\bar{X}} = \bar{\bar{X}}' + A_2 \bar{R}' = 10·174 + (0·58 \times 1·455) = 11·018$ | $UCL_R = D_4 \bar{R}' = 2·11 \times 1·455 = 3·070$

$LCL_{\bar{X}} = \bar{\bar{X}}' - A_2 \bar{R}' = 10·174 - (0·58 \times 1·455) = 9·330$ | $LCL_R = D_3 \bar{R}' = 0 \times 1·455 \quad = 0$

Since no point on R chart now falls outside the control limits, these are adopted as final control limits for R-chart.

Also on $\bar{X}$-chart, one point (9·2 kg.) falls below $UCL_{\bar{X}}$. Discarding this point,

$$\bar{\bar{X}}'_{revised} = \bar{\bar{X}}'' = \frac{234 - 9·2}{23 - 1} = 10·218$$

$$UCL_{\bar{X}} = \bar{\bar{X}}'' + A_2 \bar{R}' = 10·218 + (0·58 \times 1·455) = 11·062$$

$$LCL_{\bar{X}} = \bar{\bar{X}}'' - A_2 \bar{R}' = 10·218 - (0·58 \times 1·455) = 9·374$$

Since no point $(\bar{X})$ now falls outside the control limits, these are adopted as final control limits for $\bar{X}$-chart.

(b)
$$\sigma' = \hat{\sigma} = \frac{\bar{R}'}{d_2} = \frac{1·455}{2·326} = 0·626$$

Lower natural specification limit of the process is :

$$\bar{\bar{X}}'' - 3\sigma' = 10·218 - (3 \times 0·626) = 8·34$$

Since the lower specification limit for the breaking strength of unit is given to be : $L = 9·0$ kg, some products are likely to fall below L.

We want : $\qquad p = P(X < L) = P(X < 9)$

The variable X (the breaking strength of the unit) is normally distributed with mean ($\hat{\mu}$) and s.d. ($\hat{\sigma}$), given by :

$$\hat{\mu} = \bar{\bar{X}}'' = 10·218 \qquad \text{and} \qquad \hat{\sigma} = \sigma' = 0·626$$

$\therefore \qquad p = P(X < 9) = P\left(Z < \frac{9 - 10·218}{0·626}\right),$ where $Z \sim N(0, 1)$

$$= P(Z < -1·945) = P(Z > 1·945) \qquad \text{[By symmetry]}$$

$$= 0·5 - P(0 < Z < 1·945)$$

$$= 0·5 - \frac{1}{2}\left(0·4738 + 0·4744\right) = 0·5 - 0·4741 = 0·0259$$

Hence, about 2·59% of the product will be defective.

(c) Upper natural specification limit $= \hat{\mu} + 3\hat{\sigma} = 10 \cdot 218 + (3 \times 0 \cdot 626) = 12 \cdot 096$

Lower natural specification limit $= \hat{\mu} - 3\hat{\sigma} = 10 \cdot 218 - (3 \times 0 \cdot 626) = 8 \cdot 34$.

Example 1·6. (i) *Control charts for $\overline{X}$ and R are maintained on dissolved iron content of a certain solution in parts per million (ppm). After 2·5 hourly samples have been drawn and analysed, the data is organised into 25 sub-groups of 5 measurements each, maintaining the time order of sampling. From these data, $\Sigma \overline{X} = 390 \cdot 8$ and $\Sigma R = 84$. Find the values of 3-sigma control limits for $\overline{X}$ and R-charts, and estimate the value of σ' for this process under the assumption that the process is in control.*

(ii) *The specification on this process calls for a no more than 18 ppm dissolved iron solution. Assuming a normal distribution underlies the process and that the process continues to be in statistical control with no change in average or dispersion, what proportion of the sample measurements may be expected to exceed this specification ?*

Solution.

(i) $\overline{\overline{X}} = \dfrac{1}{k}\Sigma\overline{X} = \dfrac{390 \cdot 8}{25} = 15 \cdot 632$ and $\overline{R} = \dfrac{1}{k}\Sigma R = \dfrac{84}{25} = 3 \cdot 36$

For sub-groups of size $n = 5$, the factors are : $d_2 = 2 \cdot 326$, $A_2 = 0 \cdot 58$, $D_3 = 0$, $D_4 = 2 \cdot 11$.

Control Limits :

$\overline{X}$-chart	R-chart
$UCL_{\overline{X}} = \overline{\overline{X}} + A_2\overline{R} = 15 \cdot 632 + 0 \cdot 58(3 \cdot 36) = 17 \cdot 581$	$UCL_{\overline{R}} = D_4\overline{R} = 2 \cdot 11 \times 3 \cdot 36 = 7 \cdot 09$
$LCL_{\overline{X}} = \overline{\overline{X}} - A_2\overline{R} = 15 \cdot 632 - 0 \cdot 58(3 \cdot 36) = 13 \cdot 683$	$LCL_R = D_3R = 0$

The estimate of the process s.d. σ is given by :

$$\hat{\sigma} = \sigma' = \frac{\overline{R}}{d_2} = \frac{3 \cdot 36}{2 \cdot 326} = 1 \cdot 445.$$

(ii) We want $P(X \geq 18)$, where X is normally distributed with mean $(\hat{\mu})$ and s.d. $(\hat{\sigma})$ given by :

$$\hat{\mu} = \overline{\overline{X}} = 15 \cdot 632 \quad \text{and} \quad \hat{\sigma} = \sigma' = 1 \cdot 445$$

$\therefore$ $P(X \geq 18) = P\left(Z \geq \dfrac{18 - 15 \cdot 632}{1 \cdot 445}\right)$, where $Z \sim N(0, 1)$

$= P(Z \geq 1 \cdot 6387) \approx P(Z \geq 1 \cdot 64)$

$= 0 \cdot 5 - P(Z < 1 \cdot 64) = 0 \cdot 5 - 0 \cdot 4495 = 0 \cdot 0505$ [From Table IV]

Hence, about 5·05% of the samples may be expected to exceed the specification limit of 18 ppm.

Example 1·7. (a) *Show that p_n, the probability of the mean of a random sample of size n exceeding $UCL = \mu' + 3(\sigma'/\sqrt{n})$, when the population mean has shifted to $\mu' + k\sigma'$ is $G(3 - k\sqrt{n})$,*

where

$$G(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}u^2}\, du$$

(b) *If the rth sample mean is the first to exceed UCL, show that $E(r) = 1/p_n$.*

Solution *(a)*. We are given :

$$G(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \int_x^{\infty} \phi(u)\, du,$$

where $\phi(\bullet)$ is the *p.d.f.* of standard normal distribution.

∴ $G(x) = P(Z > x)$, where $Z \sim N(0, 1)$, is a standard normal variate.   ...(1)

We want

$$p_n = P\left(\overline{X} > UCL_{\overline{X}}\right) = P\left\{\overline{X} > \mu' + 3\left(\sigma'/\sqrt{n}\right)\right\} \qquad \text{...(2)}$$

If $\overline{X}$ is the mean of a sample of size n from a population with mean μ' and standard deviation σ' then, $\overline{X}$ is normally distributed if the population is normal and $\overline{X}$ is asymptotically normally distributed (by Central Limit Theorem) if the population is not normal, with mean μ' and standard deviation $\sigma'/\sqrt{n}$.

After the shift of the mean to $\mu' + k\sigma'$ has taken place,

$$\overline{X} \sim N\left(\mu' + k\sigma',\ \frac{\sigma'^2}{n}\right) \qquad \Rightarrow \qquad Z = \frac{\overline{X} - (\mu' + k\sigma')}{\sigma'/\sqrt{n}} \sim N(0, 1)$$

When $\overline{X} = \mu' + \dfrac{3\sigma'}{\sqrt{n}}$, $Z = \dfrac{\mu' + \dfrac{3\sigma'}{\sqrt{n}} - \mu' - k\sigma'}{\sigma'/\sqrt{n}} = 3 - k\sqrt{n}$

∴ $p_n = P\left(Z > 3 - k\sqrt{n}\right)$ [From (2)]

        $= G\left(3 - k\sqrt{n}\right)$ [From (1)]

(b) If $\overline{X}_i$ is the mean of the *i*th sample of size n, then

$$p_n = P(\overline{X}_i > UCL) \qquad \Rightarrow \qquad q_n = P(\overline{X}_i \leq LCL) = 1 - p_n$$

If *r*th sample mean is the first to exceed the *UCL*, the preceding $(r - 1)$ sample means must be $\leq UCL$. Thus, if Y is the random variable such that $Y = r\ (r = 1, 2, ...)$ implies that the *r*th sample mean is the first to exceed *UCL*, then Y follows the geometric distribution with probability function :

$$f(r) = P(Y = r) = q_n^{r-1} \cdot p_n \quad (r = 1, 2, ...)$$

∴ $E(Y) = \sum_{r=0}^{\infty} r\, f(r) = \sum r q_n^{r-1}\, p_n = p_n\left[1 + 2q_n + 3q_n^2 + ...\right]$ ... *(iii)*

Let $S = 1 + 2q_n + 3q_n^2 + 4q_n^3 + ...$

∴ $q_n S = \quad q_n + 2q_n^2 + 3q_n^3 + ...$

$$(1 - q_n) S = 1 + q_n + q_n^2 + q_n^3 + ... = \frac{1}{1 - q_n} \quad \Rightarrow \quad S = \frac{1}{(1 - q_n)^2} = \frac{1}{p_n^2}$$

Substituting in (3), we get $E(Y) = \dfrac{p_n}{p_n^2} = \dfrac{1}{p_n}.$

Aliter. [From (3) onwards] :

$$E(Y) = p_n \left[1 + 2 q_n + 3 q_n^2 + 4 q_n^3 + \ldots\right]$$

$$= p_n (1 - q_n)^{-2} = \frac{p_n}{(1 - q_n)^2} = \frac{1}{p_n}.$$

[Using binomial expansion with a negative index]

Example 1·8. *Show that the probability that at least one of the two points $\overline{X}$ and R goes outside the control limits is :*

$$1 - \left[G\left(\sqrt{n} T - 3\rho\right) - G\left(\sqrt{n} T + 3\rho\right)\right] \cdot \left[P\left(\frac{R}{\sigma} \le D_2\, \rho\right) - P\left(\frac{R}{\sigma} \le D_1\, \rho\right)\right]$$

where $\quad \rho = \sigma'/\sigma, \; T = (\mu' - \mu)/\sigma \quad and \quad G(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_x^{\infty} e^{-\frac{1}{2} t^2}\, dt,$

assuming that the control charts are based on μ' as population mean and σ' as population standard deviation, where the actual values of these parameters are μ and σ respectively.

Solution. As in *Example 1·7*,

$$G(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} t^2}\, dt = P(Z \ge x), \qquad \ldots(*)$$

where Z is a standard normal variate, *i.e.*, $Z \sim N(0, 1)$.

Probability (p_1) that all the points on the R-chart lie within control limits is given by :

$$p_1 = P(D_1\sigma' \le R \le D_2\sigma') = P(R \le D_2\sigma') - P(R \le D_1\sigma')$$

$$\left[\text{For } a < b, \; P(a \le X \le b) = P(X \le b) - P(X \le a)\right]$$

$$= P\left(\frac{R}{\sigma} \le D_2 \frac{\sigma'}{\sigma}\right) - P\left(\frac{R}{\sigma} \le D_1 \frac{\sigma'}{\sigma}\right), \; \sigma > 0$$

$$= P\left(\frac{R}{\sigma} \le D_2\, \rho\right) - P\left(\frac{R}{\sigma} \le D_1\, \rho\right) \qquad \ldots(**)$$

Since the control limits are based on μ' as the population mean and σ' as the population standard deviation,

$$UCL_{\overline{X}} = \mu' + 3\sigma'/\sqrt{n} \quad\text{and}\quad LCL_{\overline{X}} = \mu' - 3\sigma'/\sqrt{n}$$

The probability (p_2) that all the sample points on $\overline{X}$-chart lie within the control limits is :

$$p_2 = P\left(LCL_{\overline{X}} \le \overline{X} \le UCL_{\overline{X}}\right) = P\left[\mu' - 3\frac{\sigma'}{\sqrt{n}} \le \overline{X} \le \mu' + 3\frac{\sigma'}{\sqrt{n}}\right]$$

We are further given that the actual value of population mean is μ and standard deviation is σ. Hence, as discussed in *Example 1·7*, the distribution of $\overline{X}$ is exactly normal with mean μ and s.d. σ if the population is normal and is asymptotically normal otherwise

i.e., $$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad\Rightarrow\quad Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

$$\therefore \quad p_2 = P\left[\frac{\mu' - \dfrac{3\sigma'}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \le Z \le \frac{\mu' + \dfrac{3\sigma'}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}}\right], \text{ where } Z \sim N(0, 1).$$

$\Rightarrow$ $$p_2 = P\left[\frac{\sqrt{n}\,(\mu' - \mu)}{\sigma} - 3\frac{\sigma'}{\sigma} \le Z \le \frac{\sqrt{n}\,(\mu' - \mu)}{\sigma} + 3\frac{\sigma'}{\sigma}\right]$$

$$= P\left[\sqrt{n}T - 3\rho \le Z \le \sqrt{n}T + 3\rho\right], \text{ where } T = \frac{\mu' - \mu}{\sigma}, \quad \rho = \frac{\sigma'}{\sigma}$$

$$= P\left(Z \ge \sqrt{n}\,T - 3\rho\right) - P\left(Z \ge \sqrt{n}\,T + 3\rho\right)$$

$$[\because \quad P(a \le Z \le b) = P(Z \ge a) - P(Z \ge b)]$$

$$= G\left(\sqrt{n}\,T - 3\rho\right) - G\left(\sqrt{n}T + 3\rho\right) \qquad\qquad [\text{From } (^*)]$$

Assuming normality of the distribution so that $\overline{X}$ and R are independently distributed, the probability (p) that none of the two points $\overline{X}$ and R goes outside the control limits, is given by the compound probability theorem as :

$$p = p_1.p_2 = \left[P\left(\frac{R}{\sigma} \le D_2\rho\right) - P\left(\frac{R}{\sigma} \le D_1\rho\right)\right].\left[G\left(\sqrt{n}\,T - 3\rho\right) - G\left(\sqrt{n}\,T + 3\rho\right)\right]$$

$\therefore$ $P($at least one of the two points $\overline{X}$ and R goes outside the control limits$) = 1 - p$, as desired.

Example 1·9. p_n *is the probability of the mean of a sample of size n falling outside the control limits. Show that*

(a) *The probability that at most x samples are to be taken for r points to go out of control is*

$$1 - \sum_{s=0}^{r-1}\left[{}^xC_s\,p_n^s\,(1 - p_n)^{x-s}\right]$$

(b) *The probability that exactly x samples are to be taken for r points to go out of control is*

$$\left(p_n / 1 - p_n\right)^r . \, {}^{x-1}C_{r-1}(1 - p_n)^x, \, x \ge r$$

Solution. Let Y be the random variable which represents the number of points (sample means) falling outside the control limits in x samples. Then Y is a binomial variate with parameters (x, p_n) and probability function :

$$g(s) = P(Y = s) = {}^xC_s\,p_n^s\,(1 - p_n)^{x-s}, \quad (s = 0, 1, 2, ..., x)$$

Hence, the probability 'p' that in x samples, the number of points going out of the control limits is greater than or equal to r is given by :

$$p = P(Y \ge r) = 1 - P(Y < r)$$

$$= 1 - \sum_{s=0}^{r-1}{}^xC_s\,p_n^s\,(1 - p_n)^{x-s} \qquad\qquad\qquad ...(^*)$$

p implies that the rth point goes outside the control limits either at or before the xth sample. In other words, for the rth point to go outside the control limit, the sample size required is $\le x$. Hence, the R.H.S. of (*) gives the required probability that *at most x samples* are to be taken for at least r points to go out of control.

(b) the event 'E' that *exactly x samples* are required for r points to go out of control limits, happens if

(*i*) the *r* the point goes out of the control limits at the *x*th sample and the probability of this is p_n, and

(*ii*) In the remaining $(x - 1)$ samples, exactly $(r - 1)$ points go out of the control limits, and its probability is :

$$^{x-1}C_{r-1}\, p_n^{\,r-1}\,(1 - p_n)^{x-r}$$

Hence, by the compound probability theorem, the required probability is given by

$$P(E) = P(i).\, P(ii) = p_n.\,^{x-1}C_{r-1}\, p_n^{\,r-1}\,(1 - p_n)^{x-r}$$

$$= \left(p_n/1 - p_n\right)^r\, ^{x-1}C_{r-1}\,(1 - p_n)^x\; ; x \geq r$$

1·9. CONTROL CHART FOR ATTRIBUTES

In spite of wide applications of-$\overline{X}$ and R-(or σ) charts as a powerful tool of diagnosis of sources of trouble in a production process, their use is restricted because of the following imitations :

1. They are charts for variables only, *i.e.*, for quality characteristics which can be measured and expressed in numbers.

2. In certain situations they are impracticable and un-economical, *e.g.*, if the number of measurable characteristics, each of which could be a possible candidate for $\overline{X}$ and R charts, is too large, say 30,000 or so then obviously there can't be 30,000 control charts.

As an alternative to $\overline{X}$ and R-charts, we have the control chart for attributes which can be used for quality characteristics :

(*i*) which can be observed only as attributes by classifying an item as defective or non-defective *i.e.*, conforming to specifications or not, and

(*ii*) which are actually observed as attributes even though they could be measured as variables, *e.g.*, go and no-go gauge test results.

There are two control charts for attributes :

(*a*) Control chart for fraction defective (*p*-chart) or the number of defectives (*np* or *d* chart).

(*b*) Control chart for the number of defects per unit (*c*-chart).

1·9·1. Control Chart for Fraction Defective (*p*-chart). While dealing with attributes, a process will be adjudged in statistical control if all the samples or sub-groups are ascertained to have the same population proportion *P*.

If '*d*' is the number of defectives in a sample of size *n*, then the sample proportion defective is $p = d/n$. Hence, *d* is a binomial variate with parameters *n* and *P*.

$$\therefore \qquad E(d) = nP \quad \text{and} \quad \text{Var}(d) = nPQ, \quad Q = 1 - P$$

Thus $\quad E(p) = E(d/n) = \dfrac{1}{n} E(d) = P \quad$ and $\quad \text{Var}(p) = \text{Var}(d/n) = \dfrac{1}{n^2}\text{Var}(d) = \dfrac{PQ}{n} \qquad \ldots(1.7)$

Thus, the 3-σ control limits for *p*-chart are given by :

$$E(p) \pm 3\,S.E.\,(p) = P \pm 3\sqrt{PQ/n} = P \pm A\sqrt{PQ} \qquad \ldots(1.8)$$

where $A = 3/\sqrt{n}$ has been tabulated for different values of *n*.

Case (i) Standards specified. If P' is the given or known value of P, then

$$UCL_p = P' + A\sqrt{P'(1-P')} \quad ; \quad LCL_p = P' - A\sqrt{P'(1-P')} \quad ; \quad CL_p = P' \quad ...(1\cdot8a)$$

Case (ii) Standards not specified. Let d_i be the number of defectives and p_i the fraction defective for the ith sample ($i = 1, 2, ..., k$) of size n_i. Then the population proportion P is estimated by the statistic $\bar{p}$ given by :

$$\bar{p} = \frac{\Sigma d_i}{\Sigma n_i} = \frac{\Sigma n_i p_i}{\Sigma n_i} \qquad ...(1\cdot8b)$$

It may be remarked here that $\bar{p}$ is an unbiased estimate of P, since

$$E(\bar{p}) = \Sigma E(d_i)/\Sigma n_i = \left[\Sigma(n_i P)/\Sigma n_i \right] = P$$

In this case

$$UCL_p = \bar{p} + A\sqrt{(p,)(1-\bar{p})} \quad ; \quad LCL_p = \bar{p} - A\sqrt{\bar{p}(1-\bar{p})} \quad ; \quad CL_p = \bar{p} \quad ...(1\cdot8c)$$

1·9·2. Control Chart for Number of Defectives (d-chart). If instead of p, the sample proportion defective, we use d, the number of defectives in the sample, then the 3-σ control limits for d-chart are given by :

$$E(d) \pm 3\ S.E.\ (d) = nP \pm 3\sqrt{nP(1-P)} \qquad ...(1\cdot9)$$

Case (i) Standards specified. If P' is the given value of P then

$$UCL_d = nP' + 3\sqrt{nP'(1-P')} \quad ; \quad LCL_d = nP' - 3\sqrt{nP'(1-P')} \quad ; \quad CL_d = nP' \quad ...(1\cdot9a)$$

Case (ii) Standards not specified. Using $\bar{p}$ as an estimate of P as in ($1\cdot8b$), we get

$$UCL_d = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} \quad ; \quad LCL_d = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} \quad ; \quad CL_d = n\bar{p} \quad ...(1\cdot9b)$$

Since p cannot be negative, if LCL as given by above formulae comes out to be negative, then it is taken to be zero.

Remarks 1. p and d-charts for Fixed Sample Size. If the sample size remains constant for each sample *i.e.,* if $n_1 = n_2 = ... = n_k = n$, (say), then using ($1\cdot8b$) an estimate of the population proportion p is given by :

$$\hat{p} = \bar{p} = \frac{\sum\limits_{i=1}^{k} d_i}{\sum\limits_{i} n_i} = \frac{\Sigma d_i}{nk} = \frac{n\sum\limits_{i=1}^{k} p_i}{nk} = \frac{1}{k}\sum\limits_{i=1}^{k} p_i \qquad ...(1\cdot9c)$$

In this case, the same set of control limits can be used for all the samples inspected and it is immaterial if one uses p-chart or d-chart.

2. p and d-charts for Variable Sample Size. Method I. If the number of items inspected (n) in each sample varies, for p-chart separate control limits have to be computed for each sample while the central line is invariant whereas for d-chart control limits as well as the central line has to be computed for each sample. This type of limits are known as *variable control limits*. In such a situation p-chart is relatively simple and is preferred to d-chart which becomes very confusing.

Method 2. As pointed out in *Remark 2*, if n varies, separate control limits are calculated for each sample. Since S.E. $(p) = \sqrt{PQ/n}$, it should be noted that smaller the sample size wider the control band and *vice versa*. If the sample size does not vary appreciably then a

single set of control limits based on the average sample size $\left(\sum_{i=1}^{k} n_i / k\right)$ can be used. For practical purposes, this holds good for situations in which the largest sample size does not exceed the smallest sample size by more than 20% of the smallest sample size.

Alternatively, for all sample sizes two sets of limits, one based on the largest sample size and the other based on the smallest sample size can be used. The largest sample size gives the smallest control band which is called *inner band* and the smallest sample size gives the largest control band which is called *outer band*. Points falling within the inner band indicate the process in control while points lying outside the outer band are indicative of the presence of assignable causes of variation which must be searched and rectified. For other points, action should be based on the exact control limits.

Method 3. Another procedure is to standardise the variate, *i.e.*, instead of plotting p or d on the control chart, we plot the corresponding standardised values, *viz.*,

$$Z = \frac{p - p'}{\sqrt{P'Q'/n}} \quad \text{or} \quad \frac{p - \bar{p}}{\sqrt{\bar{p}(1-\bar{p})/n}} \qquad \qquad ...(1\cdot10)$$

according as P is given or not, the symbols having their usual meanings. This stabilises our variable and the resulting chart is called *stabilised p*-chart or *d*-chart. In this case the control limits as well as the central line for p and d-charts are invariant with n (*i.e.*, they are constants independent of n) being given by :

$$UCL = 3, \qquad CL = 0, \qquad LCL = -3 \qquad\qquad ...(1\cdot10a)$$

Hence, the problem of variable control limits can be solved with a little more computational work discussed above.

Interpretations of p-chart. 1. From the *p*-chart a process is judged to be in statistical control in the same way as is done for $\bar{X}$ and R charts. If all the sample points fall within the control limits without exhibiting any specific pattern, the process is said to be in control. In such a case, the observed variations in the fraction defective are attributed to the stable pattern of chance causes and the average fraction defective $\bar{p}$ is taken as the standard fraction defective P.

2. Points outside the *UCL* are termed as *high spots*. These suggest deterioration in the quality and should be regularly reported to the production engineers. The reasons for such deterioration could possibly be known and removed if the details of conditions under which data were collected, were known. Of particular interest and importance is, if there was any change of inspection or inspection standards.

3. Points below *LCL* are called *low spots*. Such points represent a situation showing improvement in the product quality. However, before taking this improvement for guaranteed, it should be investigated if there was any slackness in inspection or not.

4. When a number of points fall outside the control limits, a revised estimate of P should be obtained by eliminating all the points that fall above *UCL* (it is assumed that points that fall below *LCL* are not due to faulty inspection). The standard fraction defective P should be revised periodically in this way.

Remark. The interpretation for the control chart for number of defects (*d*-chart) is same as that for *p*-chart.

Example 1·10. *The following are the figures of defectives in 22 lots each containing 2,000 rubber belts :*

425,   430,   216,   341,   225,   322,   280,   306,   337,   305,   356,
402,   216,   264,   126,   409,   193,   326,   280,   389,   451,   420

Draw control chart for fraction defective and comment on the state of control of the process.

Solution. Here we have a fixed sample size $n = 2{,}000$ for each lot. If d_i and p_i are respectively the number of defectives and the sample fraction defective for the ith lot, then

$$p_i = \frac{d_i}{2{,}000} \cdot (i = 1, 2, ..., 22)$$

which are given in Table 1·2.

TABLE 1·2 : COMPUTATIONS FOR C.C. FOR FRACTION DEFECTIVE

S. No.	d	p = (d/2000)	S. No.	d	p = (d/2000)
1	425	0·2125	12	402	0·2010
2	430	0·2150	13	216	0·1080
3	216	0·1080	14	264	0·1320
4	341	0·1705	15	126	0·0630
5	225	0·1125	16	409	0·2045
6	322	0·1610	17	193	0·0965
7	280	0·1400	18	326	0·1630
8	306	0·1530	19	280	0·1400
9	337	0·1685	20	389	0·1945
10	305	0·1525	21	451	0·2255
11	356	0·1780	22	420	0·2100
Total	3,543	1·7715		3,476	1·7380

In the usual notations, we have

$$\bar{p} = \frac{\Sigma p_i}{k} = \frac{1·7715 + 1·7380}{22} = \frac{3·5095}{22} = 0·1595 \quad \Rightarrow \quad \bar{q} = 1 - \bar{p} = 0·8405$$

$$\left[\text{Or} \quad \bar{p} = \frac{\Sigma d_i}{nk} = \frac{3543 + 3476}{2000 \times 22} = \frac{7019}{44000} = 0·1595 \right]$$

3-σ control limits for p-chart are given by :

$$\bar{p} \pm 3 \sqrt{\bar{p}\,\bar{q}/n} = 0·1595 \pm 3\sqrt{0·1595 \times 0·8405/2000}$$

$$= 0·1595 \pm 3\sqrt{0·000067} = 0·1595 \pm 0·0246$$

$\therefore$ $UCL_p = 0·1595 + 0·0246 = 0·1841$; $LCL_p = 0·1595 - 0·0246 = 0·1349$; $CL_p = \bar{p} = 0·1595$

The control chart for fraction defective (p-chart) is drawn in Fig. 1·9.

From the p-chart, we find that the sample points (fraction defectives) corresponding to the sample numbers 1, 2, 3, 5, 12, 13, 14, 15, 16, 17, 20, 21 and 22, fall outside the control limits. Hence, the process cannot be regarded in statistical control.
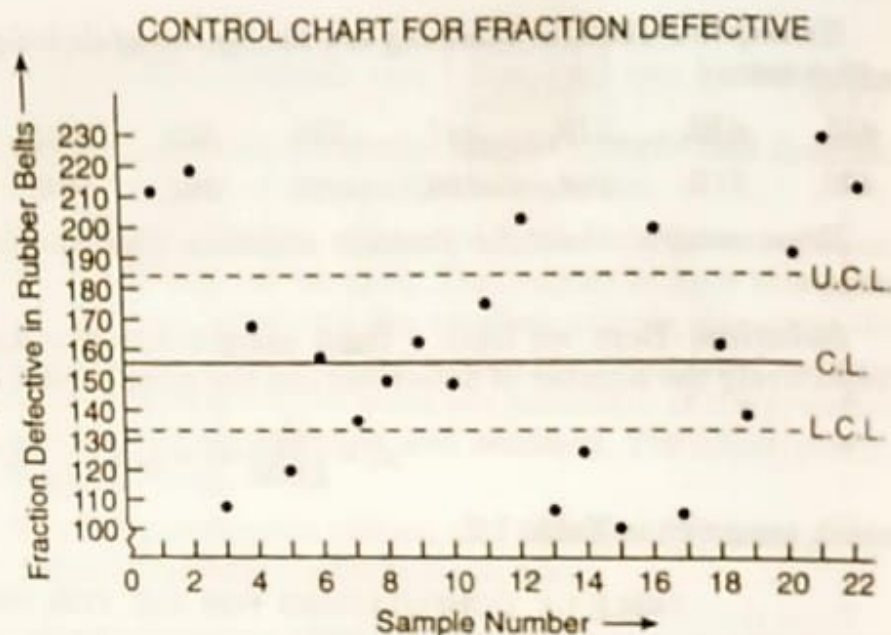

CONTROL CHART FOR FRACTION DEFECTIVE

Fig. 1·9

Example 1·11. *From the following inspection results, construct 3-σ control limits for p chart :*

Date Sept.	No. of Defectives	Date Sept.	No. of Defectives	Date Sept.	No. of Defectives
1	22	11	70	21	66
2	40	12	80	22	50
3	36	13	44	23	46
4	32	14	22	24	32
5	42	15	32	25	42
6	40	16	42	26	46
7	30	17	20	27	30
8	44	18	46	28	38
9	42	19	28	29	40
10	38	20	36	30	24

The sub-groups, from which the defectives were taken out, were of the same size, i.e., 1,000 items each.

Without constructing the control chart, comment on the state of control of the process. If the process is out of control, then suggest the revised control limits for future use.

Solution. Here we have a fixed sample size for each lot. If d_i and p_i are respectively the number of defectives and the sample fraction defective for the ith lot then

$$p_i = \frac{d_i}{1,000} \quad \cdot \quad (i = 1, 2, ..., 30)$$

which are given in *Table 1·3* :

TABLE 1·3 : CALCULATIONS FOR CONTROL LIMITS FOR p-CHART

Date Sept.	No. of defectives	Fraction defectives	Date Sept.	No. of defectives	Fraction defective	Date Sept.	No. of defectives	Fraction defectives
1	22	0·022	11	70	0·070	21	66	0·066
2	40	0·040	12	80	0·080	22	50	0·050
3	36	0·036	13	44	0·044	23	46	0·046
4	32	0·032	14	22	0·022	24	32	0·032
5	42	0·042	15	32	0·032	25	42	0·042
6	40	0·040	16	42	0·042	26	46	0·046
7	30	0·030	17	20	0·020	27	30	0·030
8	44	0·044	18	46	0·046	28	38	0·038
9	42	0·042	19	28	0·028	29	40	0·040
10	38	0·038	20	36	0·036	30	24	0·024
Total	366	0·366	Total	420	0·420	Total	414	0·414

From the above table, we have

$$\Sigma d_i = 366 + 420 + 414 = 1,200 \quad ; \quad \Sigma p_i = 0\cdot366 + 0\cdot420 + 0\cdot414 = 1\cdot200 \; ; \; n = 1000, \; k = 30$$

$$\bar{p} = \frac{\Sigma d}{nk} = \frac{1200}{1000 \times 30} = 0\cdot040 \qquad \text{or} \qquad \bar{p} = \frac{\Sigma p_i}{k} = \frac{1\cdot2}{30} = 0\cdot040$$

3-σ Control Limits for p-Chart : $CL_p = \bar{p} = 0\cdot040$

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0\cdot040 + 3\sqrt{\frac{0\cdot04(1-0\cdot04)}{1000}}$$

$$= 0\cdot040 + 3\sqrt{0\cdot0000384} = 0\cdot040 + 3 \times 0\cdot0062 = 0\cdot0586$$

$$LCL_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0\cdot040 - 0\cdot0186 = 0\cdot0214.$$

We observe that the sample points (fraction defectives) on 11th, 12th, 17th and 21st September were 0·070, 0·080, 0·020 and 0·066 respectively and these fall outside the control limits. Hence the process is not in a state of statistical control.

Revised Control Limits : The revised control limits are obtained on eliminating these four samples and considering the remaining 30 − 4 = 26 samples.

Based on the remaining 26 samples, we get

$$CL_p = \bar{p}' = \frac{\Sigma d - 70 - 80 - 20 - 66}{1000 \times 26} = \frac{1200 - 236}{1000 \times 26} = \frac{964}{26000} = 0\cdot0371$$

or

$$CL_p = \bar{p}' = \frac{\Sigma p - 0\cdot07 - 0\cdot08 - 0\cdot02 - 0\cdot066}{26}$$

$$= \frac{1\cdot200 - 0\cdot236}{26} = \frac{0\cdot964}{26} = 0\cdot0371$$

$$UCL_p = \bar{p}' + 3\sqrt{\frac{\bar{p}'\bar{q}'}{n}} = 0\cdot0371 + 3\sqrt{\frac{0\cdot0371 \times 0\cdot9629}{1000}}$$

$$= 0\cdot0371 + 3 \times \sqrt{0\cdot0000357} = 0\cdot0371 + 3 \times 0\cdot0060$$

$$= 0\cdot0371 + 0\cdot0180 = 0\cdot0551$$

$$LCL_p = \bar{p}' - 3\sqrt{\frac{\overline{p'}\,\overline{q'}}{n}} = 0.0371 - 0.0180 = 0.0191$$

Since none of the remaining 26 sample points (fraction defectives) lies outside the revised control limits, $[LCL_p = 0.0191$ and $UCL_p = 0.0551]$, these may be regarded as the control limits for p-chart for the future production from this process.

Example 1·12. *20 samples each of size 10 were inspected. The number of defectives detected in each of them is given below :*

Samples No.	:	1	2	3	4	5	6	7	8	9	10
No. of defectives	:	0	1	0	3	9	2	0	7	0	1
Sample No.	:	11	12	13	14	15	16	17	18	19	20
No. of defectives	:	1	0	0	3	1	0	0	2	1	0

Construct the 'number of defectives' chart and establish quality standard for the future.

Solution. Here we have samples of fixed size $n = 10$. The total number of defectives in all the 20 samples is :

$$\Sigma d = 0 + 1 + 0 + 3 + 9 + \ldots + 2 + 1 + 0 = 31$$

An estimate of the process fraction defective is given by :

$$\bar{p} = \frac{\Sigma d}{nk} = \frac{31}{10 \times 20} = 0.155 \quad \Rightarrow \quad \bar{q} = 1 - \bar{p} = 0.845$$

The 3-σ control limits for 'number defectives' chart (np-chart) are given by :

$$CL_{np} = n\bar{p} = 1.55$$

$$UCL_{np} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} = 1.55 + 3\sqrt{1.55 \times 0.845} = 1.55 + 3.43 = 4.98$$

$$LCL_{np} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} = 1.55 - 3\sqrt{1.55 \times 0.845} = 0 \text{ (Negative)}$$

The control chart for the 'number of defectives' is obtained on plotting the number of defectives against the corresponding sample number and is given in *Fig. 1·10*.
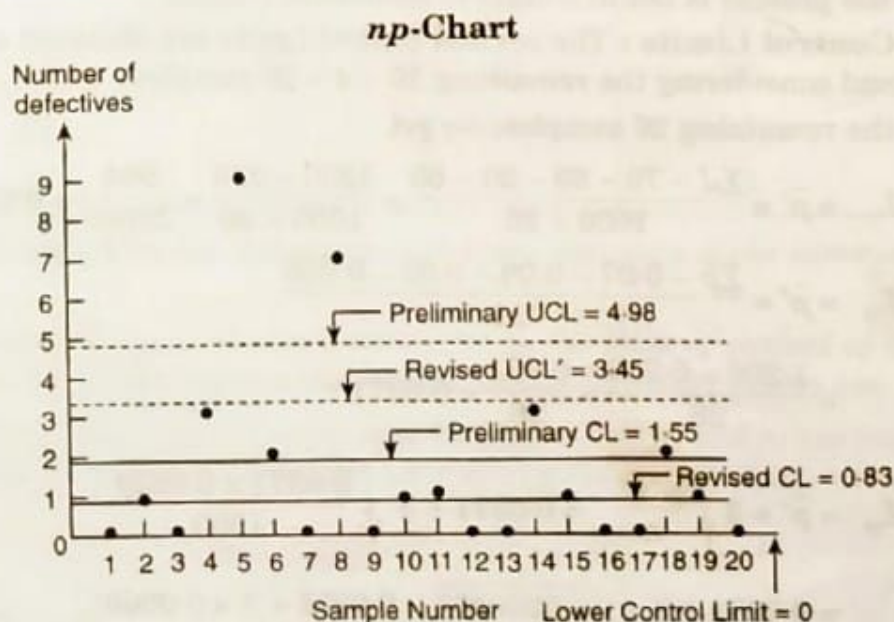
np-Chart

Fig. 1·10. np-chart.

Since two points corresponding to 5th and 8th samples lie outside the control limits, we conclude that the process is not in a state of statistical control. To establish quality standards for the future, we eliminate these points.

Deleting sample numbers 5 and 8, we compute CL, UCL and LCL based on the remaining $20 - 2 = 18$ samples as follows :

Revised Control Limits for Future :

$$\bar{p}' = \frac{\Sigma d - 9 - 7}{10 \times 18} = \frac{31 - 16}{180} = \frac{15}{180} = 0.083 \qquad \Rightarrow \qquad \bar{q}' = 1 - \bar{p}' = 0.917$$

$$CL' = n\bar{p}' = 10 \times 0.083 = 0.83$$

$$UCL' = n\bar{p}' + 3\sqrt{n\bar{p}'(1-\bar{p}')} = 0.83 + 3\sqrt{0.83 \times 0.917} = 0.83 + 2.62 = 3.45$$

$$LCL' = np' - 3\sqrt{n\bar{p}'(1-\bar{p}')} = 0.38 - 3\sqrt{0.83 \times 0.917} = -0.178 \doteq 0.$$

These revised values are shown in the (on page 1.36) as the revised UCL' and LCL' respectively. No sample values other than those which have been deleted, fall outside the new limits. We take these new limits, alongwith the new central line, as standards for controlling product in the future.

Example 1·13. *The following data give the number of defectives in 10 independent samples of varying sizes from a production process :*

Sample No. :	1	2	3	4	5	6	7	8	9	10
Sample size :	2,000	1,500	1,400	1,350	1,250	1,760	1,875	1,955	3,125	1,575
No. of defectives :	425	430	216	341	225	322	280	306	337	305

Draw the control chart for fraction defective and comment on it.

Solution. Since we have variable sample size, we can draw the control chart for fraction defective in the following three ways.

Method 1. *Variable Control Limits.* In this case we calculate 3-σ limits for each sample separately by using the formula :

$$UCL = \bar{p} + 3\sqrt{\bar{p}\bar{q}/n_i} \quad \text{and} \quad LCL = \bar{p} - 3\sqrt{\bar{p}\bar{q}/n_i}, \quad \text{where} \quad \bar{p} = \frac{\Sigma d_i}{\Sigma n_i} = \frac{3,187}{17,790} = 0.1791...(*)$$

and d_i = No. of defectives in the ith sample, n_i = sample size of the ith sample

$\therefore \quad \bar{q} = 1 - \bar{p} = 0.8209.$ Thus, $\bar{p}\bar{q} = 0.1791 \times 0.8209 = 0.1470231$

For computation of the variable control limits, see the calculation in *Table 1·4.*

TABLE 1·4 : COMPUTATIONS FOR p-CHART (VARIABLE CONTROL LIMITS)

n	d	$p = d/n$	$\bar{p}\bar{q}/n$	$\sqrt{\bar{p}\bar{q}/n}$	$3 \times \sqrt{\bar{p}\bar{q}/n}$	UCL	LCL
2,000	425	·2125	·0000735	·008573	·025719	0.205	0.153
1,500	430	·2867	·000098	·009899	·029698	0.209	0.149
1,400	216	·1543	·000105	·010247	·030741	0.210	0.148
1,350	341	·2526	·000109	·010440	·031321	0.210	0.148
1,250	225	·1800	·000118	·010863	·032588	0.212	0.147

1,760	322	·1829	·000084	·009138	·027413	.207	.152
1,875	280	·1495	·000078	·008854	·026562	.206	.153
1,955	306	·1565	·000075	·008672	·026015	.205	.153
3,125	337	·1078	·000047	·006856	·020567	.200	.159
1,575	305	·1937	·000093	·009659	·028977	.208	.150
17,790	3,187						

From the chart [Fig. 1·11], it is obvious that a number of sample points corresponding to sample numbers 1, 2, 4, 7 and 9 are outside the respective control limits. Hence the process is not in a state of statistical control. This suggests the presence of some assignable causes of variations, which should be detected and eliminated.
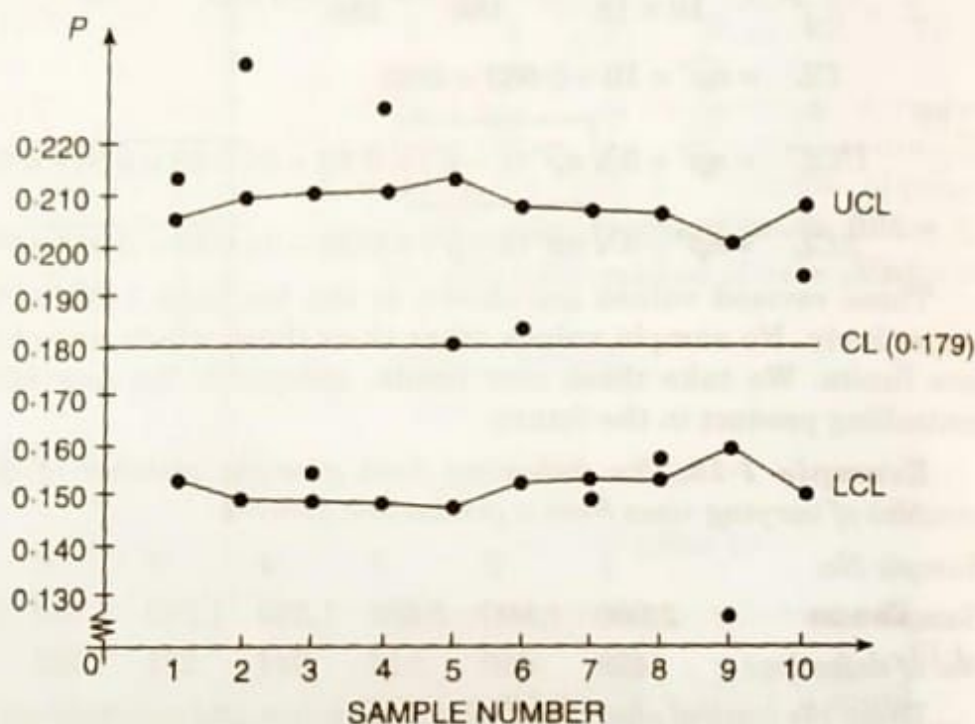

Fig. 1·11

Method 2. Here we set up two sets of control limits, one based on the maximum sample size $n = 3,125$ (corresponding to 9th sample) and the other based on the minimum sample size $n = 1,250$ (corresponding to 5th sample). From the table, we note that the corresponding sets of control limits are :

For $n = 3,125$; $UCL = 0·1997$ and $LCL = 0·1585$; For $n = 1,250$; $UCL = 0·2117$ and $LCL = 0·1465$.

From the control chart [Fig. 1·12], we find that the sample points corresponding to sample number 1, 2, 4 and 9 lie outside
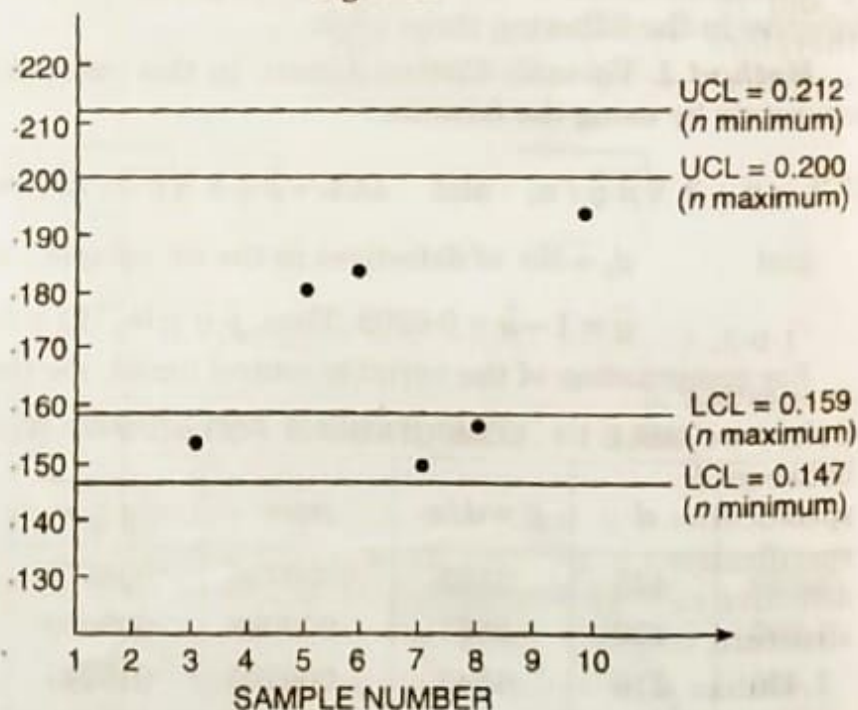

Fig. 1·12

the outer band (based on minimum sample size), and hence the process is out of statistical control.

Method 3. We standardise the statistic p by the formula :

$$Z_i = \frac{p_i - E(p_i)}{S.E.\ (p_i)} = \frac{p_i - \bar{p}}{\sqrt{\bar{p}\bar{q}/n}} \qquad \dots(*)$$

where $\bar{p}$ is computed in (*) and plot the Z-values against the corresponding sample number. Since n is large, $Z_i \sim N(0, 1)$ and hence

$$UCL_z = 3 \ ; \quad LCL_z = -3 \ ; \quad CL_z = 0$$

TABLE 1·5 : COMPUTATIONS OF Z-VALUES

p	$p - \bar{p}$	$\sqrt{\bar{p}\bar{q}/n}$	$Z = \dfrac{p - \bar{p}}{\sqrt{\bar{p}\bar{q}/n}}$
·2126	·0334	·0086	3·8841
·2867	·1076	·0099	10·8686
·1543	−·0248	·0102	−2·4313
·2526	·0735	·0104	5·25
·1800	·0009	·0109	0·0826
·1829	·0038	·0091	0·4176
·1495	−·0296	·0089	−3·2558
·1565	−·0226	·0087	−2·5977
·1078	−·0713	·0069	−10·3333
·1937	·0146	·0097	1·5052

The control chart is drawn in *Fig. 1·13.*

Since a number of sample values (Z) except for sample numbers 3, 5, 6, 8 and 10 lie-outside the limits ± 3, therefore the process is out of statistical control.

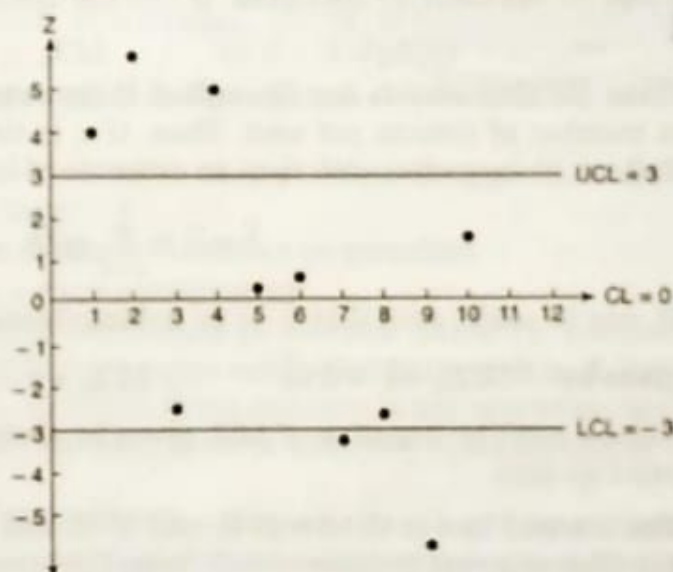**Fig. 1·13**

1·9·3. Control Chart for Number of Defects per Unit (c - Chart). The field of application of c-chart is much more restricted as compared to $\bar{X}$ and R charts or p-chart. Before we embark upon to discuss the theory behind c-chart, it is imperative to distinguish between defect and detective. An article which does not conform to one or more of the specifications, is termed as *defective* while any instance of article's lack of conformity to specifications is a *defect*. Thus, every defective contains one or more of the defects, *e.g.,* a defective casting may further be examined for blow holes, cold shuts, rough surface, weak structure, etc.

Unlike d or np-chart which applies to the number of defectives in a sample, c-chart applies to the number of defects per unit. Sample size for c-chart may be a single unit like a radio, or group of units or it may be a unit of fixed time, length, area, etc. For example, in

case of surface defects, area of the surface is the sample size ; in case of casting defects, a single part (such as base plate, side cover) is the sample size. However, defined sample size should be constant in the sense that different samples have essentially equal opportunity for the occurrence of defects.

Control Limits for c-chart. In many manufacturing or inspection situations, the sample size n i.e., the area of opportunity is very large (since the opportunities for defects to occur are numerous) and the probability p of the occurrence of a defect in any one spot is very small such that np is finite. In such situations from statistical theory we know that the pattern of variations in data can be represented by Poisson distribution, and consequently 3-σ control limits based on Poisson distribution are used. Since for a Poisson distribution, mean and variance are equal, if we assume that c is Poisson variate with parameter, λ, we get

$$E(c) = \lambda \quad \text{and} \quad \text{Var}(c) = \lambda$$

Thus 3-σ control limits for c-chart are given by :

$$\left.\begin{aligned}
UCL_c &= E(c) + 3\sqrt{\text{Var}(c)} = \lambda + 3\sqrt{\lambda} \\
LCL_c &= E(c) - 3\sqrt{\text{Var}(c)} = \lambda - 3\sqrt{\lambda} \\
CL_c &= \lambda
\end{aligned}\right\} \qquad \text{...(1·11)}$$

Case (i) Standards specified. If λ' is the specified value of λ, then

$$UCL_c = \lambda' + 3\sqrt{\lambda'} \quad ; \quad LCL_c = \lambda' - 3\sqrt{\lambda'} \quad ; \quad CL_c = \lambda' \qquad \text{...(1·12)}$$

Case (ii) Standards not Specified. If the value of λ is not known, it is estimated by the mean number of defects per unit. Thus, if c_i is the number of defects observed on the ith ($i = 1, 2, ..., k$) inspected unit, then an estimate of λ is given by :

$$\hat{\lambda} = \bar{c} = \sum_{i=1}^{k} c_i / k \qquad \text{...(1·12a)}$$

It can be easily seen that $\bar{c}$ is an unbiased estimate of λ. The control limits, in this case, are given by : $UCL_c = \bar{c} + 3\sqrt{\bar{c}} \quad ; \quad LCL_c = \bar{c} - 3\sqrt{\bar{c}} \quad ; \quad CL_c = \bar{c} \qquad \text{...(1·12b)}$

Since c can't be negative, if LCL given by above formulae comes out to be negative, it is regarded as zero.

The central line is drawn at $\bar{c}$, and UCL and LCL are drawn at the values given by (1·12a). The observed number of defects on the inspected units are then plotted on the control chart. The interpretations for c-chart are similar to those of p-chart.

Remark. Usually k, the number of samples (inspected units), is taken from 20 to 25. Normal approximation to Poisson distribution may be used provided $\bar{c} < 5$.

1·9·4. c- Chart for Variable Sample Size or u-Chart. In this case instead of plotting c, the statistic $u = c/n$ is plotted, n being the sample size which is varying. If n_i is the sample size and c_i the total number of defects observed in the ith sample, then

$$u_i = c_i / n_i, (i = 1, 2, ..., k), \qquad \text{...(1·13)}$$

gives the average number of defects per unit for the ith sample.

In this case an estimate of λ, the mean number of defects per unit in the lot, based on all the k-samples is given by :

$$\hat{\lambda} = \bar{u} = \frac{1}{k} \sum_{i=1}^{k} u_i \qquad \text{...(1·13a)}$$

We know that if $\overline{X}$ is the mean of a random sample of size n then S.E. $(\overline{X}) = \sigma/\sqrt{n}$. Hence, the standard error of the average number of defects per unit is given by :

$$S.E.\,(u) = \sqrt{\lambda/n} = \sqrt{\overline{u}/n} \quad ; \quad \text{[On using (1·13a)]} \qquad \qquad ...(1·13b)$$

Hence, 3-σ control limits for u-chart (or c - Chart for variable sample size) are given by :

$$UCL_u = \overline{u} + 3\sqrt{\overline{u}/n} \quad ; \quad LCL_u = \overline{u} - 3\sqrt{\overline{u}/n} \quad ; \quad CL_u = \overline{u} \qquad ...(1·13c)$$

As is obvious, control limits will vary for each sample. The central line, however, will be same. The interpretation of these charts is similar to the p-chart or d-chart.

Applications of c-chart

The universal nature of Poisson distribution as the law of small numbers makes the c-chart technique quite useful. In spite of the limited field of application of c-chart (as compared to $\overline{X}, R, p$-charts), there do exist situations in industry where c-chart is definitely needed. Some of the representative types of defects to which c-chart can be applied with advantage are :

1. c is number of imperfections observed in a bale of cloth.

2. c is the number of surface defects observed in (i) roll of coated paper or a sheet of photographic film, and (ii) a galvanised sheet or a painted, plated or enamelled surface of given area.

3. c is the number of defects of all types observed in aircraft sub-assemblies or final assembly.

4. c is the number of breakdowns at weak spots in insulation in a given length of insulated wire subject to a specified test voltage.

5. c is the number of defects observed in stains or blemishes on a surface.

6. c is the number of soiled packages in a given consignment.

7. c-chart has been applied to sampling acceptance procedures based on number of defects per unit, e.g., in case of inspection of fairly complex assembled units such as T.V. sets, aircraft engines, tanks, machine-guns, etc., in which there are very many opportunities for the occurrence of defects of various types and the total number of defects of all types found by inspection is recorded for each unit.

8. c-chart technique can be used with advantage in various fields other than industrial quality control, e.g., it has been applied (i) to accident statistics (both of industrial accidents and highway accidents), (ii) in chemical laboratories, and (iii) in epidemiology.

Example 1·14. *In welding of seams, defects included pinholes, cracks, cold taps, etc. A record was made of the number of defects found in one seam each hour and is given below.*

1·12·2005	8 A.M.	2	2·12·2005	8 A.M.	5	3·12·2005	8 A.M.	6
	9 A.M.	4		9 A.M.	3		9 A.M.	4
	10 A.M.	7		10 A.M.	7		10 A.M.	3
	11 A.M.	3		11 A.M.	11		11 A.M.	9
	12 A.M.	1		12 A.M.	6		12 A.M.	7
	1 P.M.	4		1 P.M.	4		1 P.M.	4
	2 P.M.	8		2 P.M.	9		2 P.M.	7
	3 P.M.	9		3 P.M.	9		3 P.M.	12

Draw the control chart for number of defects and give your comments.

Solution. Average number of defects per sample is : $\bar{c} = \frac{1}{k} \Sigma c = \frac{1}{24} \times 144 = 6$

The control limits and the central line, therefore are as follows :

$UCL_c = \bar{c} + 3\sqrt{\bar{c}} = 6 + 3\sqrt{6} = 13 \cdot 35$

$LCL_c = \bar{c} - 3\sqrt{\bar{c}} = 6 - 3\sqrt{6} = -1 \cdot 35$

$CL_c = \bar{c} = 6$

Because the number of defects cannot be negative, so we consider the lower limit to be zero, *i.e.*, $\bar{c}$ is allowed to vary between 0 and 13·35.

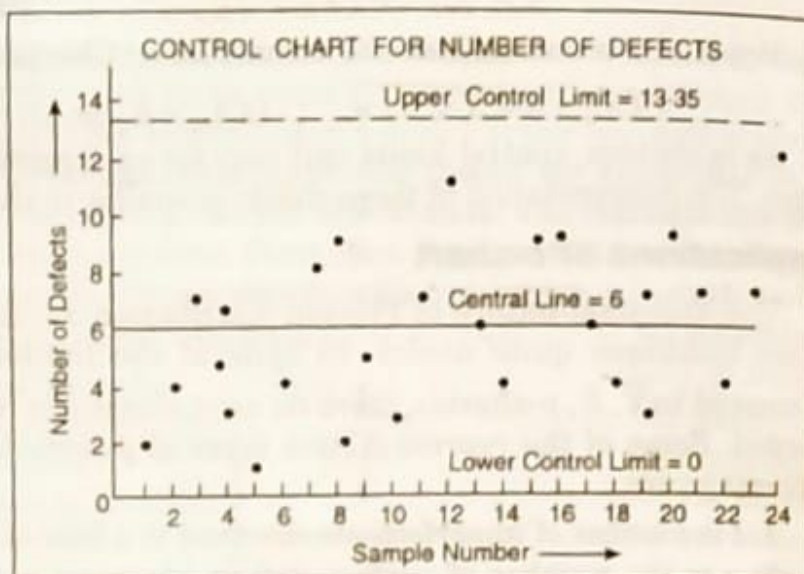The control chart is drawn in Fig. 1·14.

Since none of the 24 points falls outside the control limits, process average may be regarded in state of statistical control.

Fig. 1·14

Example 1·15. *The number of defects on 20 items are given below :*

Item No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of defects :	2	0	4	1	0	8	0	1	2	0	6	0	2	1	0	3	2	1	0	2

Devise a suitable control scheme for the future.

Solution. The control chart to be used is the c-chart.

$$\bar{c} = \text{Average number of defects/item} = \frac{1}{k} \Sigma c = \frac{1}{20} \times 35 = 1 \cdot 75$$

$CL_c = \bar{c} = 1 \cdot 75$; $UCL_c = \bar{c} + 3\sqrt{\bar{c}} = 1 \cdot 75 + 3\sqrt{1 \cdot 75} = 5 \cdot 71$; $LCL_c = \bar{c} - 3\sqrt{\bar{c}} = 1 \cdot 75 - 3 \cdot 96 \simeq 0$.

Since two sample points corresponding to 6th and 11th samples lie outside the control limits, we conclude that the process is not in a state of statistical control. To establish quality standards for the future, we eliminate these 'out of control' sample points. Deleting sample numbers 6 and 11, we compute the new control limits CL', UCL' and LCL' for the remaining 18 samples as follows :

$$\bar{c}' = \frac{\Sigma c - 8 - 6}{20 - 2} = \frac{35 - 14}{18} = \frac{21}{18} = 1 \cdot 17$$

$CL' = \bar{c}' = 1 \cdot 17$, $\quad UCL' = \bar{c}' + 3\sqrt{\bar{c}'} = 1 \cdot 17 + 3\sqrt{1 \cdot 17} = 4 \cdot 15$;

$LCL' = \bar{c}' - 3\sqrt{\bar{c}'} = 1 \cdot 17 - 3\sqrt{1 \cdot 17} = 0$

It may be noted that now no sample points (c-values) other than those which have been deleted, fall outside the new control limits. We take these new limits, alongwith the new central line, as standards for controlling production in the future.

1·10. NATURAL TOLERANCE LIMITS AND SPECIFICATION LIMITS

A process in statistical control implies that the control charts for both the mean and range show complete homogeneity and in such a case, a measure of the variation of the individual products is given by the standard deviation (σ), estimate by $\bar{R}/d_2$ or $\bar{s}/C_2$ from