state-space tree are generated. For backtracking, this tree is usually developed depth-first (i.e., similar to DFS). Branch-and-bound can generate nodes according to several rules: the most natural one is the so-called best-first rule explained in Section 12.2.

Section 12.3 takes a break from the idea of solving a problem exactly. The algorithms presented there solve problems approximately but fast. Specifically, we consider a few approximation algorithms for the traveling salesman and knapsack problems. For the traveling salesman problem, we discuss basic theoretical results and pertinent empirical data for several well-known approximation algorithms. For the knapsack problem, we first introduce a greedy algorithm and then a parametric family of polynomial-time algorithms that yield arbitrarily good approximations.

Section 12.4 is devoted to algorithms for solving nonlinear equations. After a brief discussion of this very important problem, we examine three classic methods for approximate root finding: the bisection method, the method of false position, and Newton's method.

Backtracking

Throughout the book (see in particular Sections 3.4 and 11.3), we have encountered problems that require finding an element with a special property in a domain that grows exponentially fast (or faster) with the size of the problem's input: a Hamiltonian circuit among all permutations of a graph's vertices, the most valuable subset of items for an instance of the knapsack problem, and the like. We addressed in Section 11.3 the reasons for believing that many such problems might not be solvable in polynomial time. Also recall that we discussed in Section 3.4 how such problems can be solved, at least in principle, by exhaustive search. The exhaustive-search technique suggests generating all candidate solutions and then identifying the one (or the ones) with a desired property.

Backtracking is a more intelligent variation of this approach. The principal idea is to construct solutions one component at a time and evaluate such partially constructed candidates as follows. If a partially constructed solution can be developed further without violating the problem's constraints, it is done by taking the first remaining legitimate option for the next component. If there is no legitimate option for the next component, no alternatives for *any* remaining component need to be considered. In this case, the algorithm backtracks to replace the last component of the partially constructed solution with its next option.

It is convenient to implement this kind of processing by constructing a tree of choices being made, called the ***state-space tree***. Its root represents an initial state before the search for a solution begins. The nodes of the first level in the tree represent the choices made for the first component of a solution, the nodes of the second level represent the choices for the second component, and so on. A node in a state-space tree is said to be ***promising*** if it corresponds to a partially constructed solution that may still lead to a complete solution; otherwise,

it is called ***nonpromising***. Leaves represent either nonpromising dead ends or complete solutions found by the algorithm. In the majority of cases, a state-space tree for a backtracking algorithm is constructed in the manner of depth-first search. If the current node is promising, its child is generated by adding the first remaining legitimate option for the next component of a solution, and the processing moves to this child. If the current node turns out to be nonpromising, the algorithm backtracks to the node's parent to consider the next possible option for its last component; if there is no such option, it backtracks one more level up the tree, and so on. Finally, if the algorithm reaches a complete solution to the problem, it either stops (if just one solution is required) or continues searching for other possible solutions.

n-Queens Problem

As our first example, we use a perennial favorite of textbook writers: the ***n-queens problem***. The problem is to place n queens on an $n \times n$ chessboard so that no two queens attack each other by being in the same row or in the same column or on the same diagonal. For $n = 1$, the problem has a trivial solution, and it is easy to see that there is no solution for $n = 2$ and $n = 3$. So let us consider the four-queens problem and solve it by the backtracking technique. Since each of the four queens has to be placed in its own row, all we need to do is to assign a column for each queen on the board presented in Figure 12.1.

We start with the empty board and then place queen 1 in the first possible position of its row, which is in column 1 of row 1. Then we place queen 2, after trying unsuccessfully columns 1 and 2, in the first acceptable position for it, which is square (2, 3), the square in row 2 and column 3. This proves to be a dead end because there is no acceptable position for queen 3. So, the algorithm backtracks and puts queen 2 in the next possible position at (2, 4). Then queen 3 is placed at (3, 2), which proves to be another dead end. The algorithm then backtracks all the way to queen 1 and moves it to (1, 2). Queen 2 then goes to (2, 4), queen 3 to (3, 1), and queen 4 to (4, 3), which is a solution to the problem. The state-space tree of this search is shown in Figure 12.2.

If other solutions need to be found (how many of them are there for the four-queens problem?), the algorithm can simply resume its operations at the leaf at which it stopped. Alternatively, we can use the board's symmetry for this purpose.

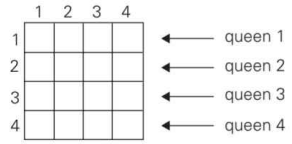
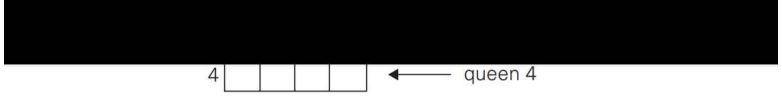
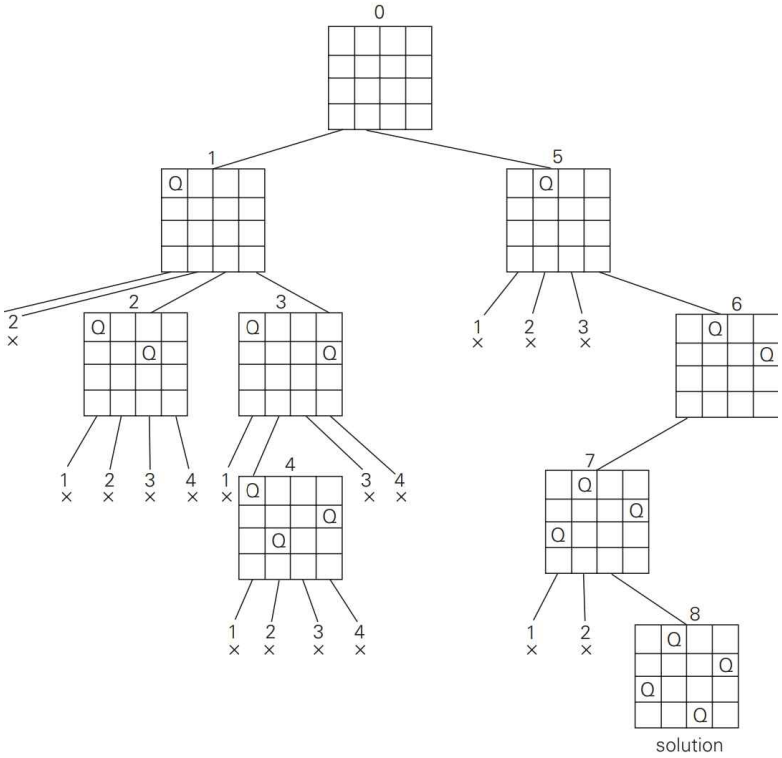


FIGURE 12.1 Board for the four-queens problem.



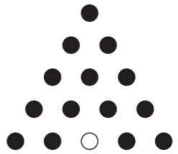
JRE 12.1 Board for the four-queens problem.



JRE 12.2 State-space tree of solving the four-queens problem by backtracking. × denotes an unsuccessful attempt to place a queen in the indicated column. The numbers above the nodes indicate the order in which the nodes are generated.

Finally, it should be pointed out that a single solution to the n-queens problem ıny $n \geq 4$ can be found in linear time. In fact, over the last 150 years mathe- cians have discovered several alternative formulas for nonattacking positions queens [Bel09]. Such positions can also be found by applying some genera rithm design strategies (Problem 4 in this section's exercises).

niltonian Circuit Problem



Design and implement a backtracking algorithm for solving the following versions of this puzzle.

a. Starting with a given location of the empty hole, find a shortest sequence of moves that eliminates 14 pegs with no limitations on the final position of the remaining peg.

b. Starting with a given location of the empty hole, find a shortest sequence of moves that eliminates 14 pegs with the remaining peg at the empty hole of the initial board.

Branch-and-Bound

Recall that the central idea of backtracking, discussed in the previous section, is to cut off a branch of the problem's state-space tree as soon as we can deduce that it cannot lead to a solution. This idea can be strengthened further if we deal with an optimization problem. An optimization problem seeks to minimize or maximize some objective function (a tour length, the value of items selected, the cost of an assignment, and the like), usually subject to some constraints. Note that in the standard terminology of optimization problems, a ***feasible solution*** is a point in the problem's search space that satisfies all the problem's constraints (e.g., a Hamiltonian circuit in the traveling salesman problem or a subset of items whose total weight does not exceed the knapsack's capacity in the knapsack problem), whereas an ***optimal solution*** is a feasible solution with the best value of the objective function (e.g., the shortest Hamiltonian circuit or the most valuable subset of items that fit the knapsack).

Compared to backtracking, branch-and-bound requires two additional items:

- a way to provide, for every node of a state-space tree, a bound on the best value of the objective function[1] on any solution that can be obtained by adding further components to the partially constructed solution represented by the node
- the value of the best solution seen so far

If this information is available, we can compare a node's bound value with the value of the best solution seen so far. If the bound value is not better than the value of the best solution seen so far—i.e., not smaller for a minimization problem

[1] This bound should be a lower bound for a minimization problem and an upper bound for a maximization problem.

and not larger for a maximization problem—the node is nonpromising and can be terminated (some people say the branch is "pruned"). Indeed, no solution obtained from it can yield a better solution than the one already available. This is the principal idea of the branch-and-bound technique.

In general, we terminate a search path at the current node in a state-space tree of a branch-and-bound algorithm for any one of the following three reasons:

- The value of the node's bound is not better than the value of the best solution seen so far.
- The node represents no feasible solutions because the constraints of the problem are already violated.
- The subset of feasible solutions represented by the node consists of a single point (and hence no further choices can be made)—in this case, we compare the value of the objective function for this feasible solution with that of the best solution seen so far and update the latter with the former if the new solution is better.

Assignment Problem

Let us illustrate the branch-and-bound approach by applying it to the problem of assigning n people to n jobs so that the total cost of the assignment is as small as possible. We introduced this problem in Section 3.4, where we solved it by exhaustive search. Recall that an instance of the assignment problem is specified by an $n \times n$ cost matrix C so that we can state the problem as follows: select one element in each row of the matrix so that no two selected elements are in the same column and their sum is the smallest possible. We will demonstrate how this problem can be solved using the branch-and-bound technique by considering the same small instance of the problem that we investigated in Section 3.4:

$$C = \begin{array}{c} \begin{array}{cccc} \text{job 1} & \text{job 2} & \text{job 3} & \text{job 4} \end{array} \\ \begin{bmatrix} 9 & 2 & 7 & 8 \\ 6 & 4 & 3 & 7 \\ 5 & 8 & 1 & 8 \\ 7 & 6 & 9 & 4 \end{bmatrix} \end{array} \begin{array}{l} \text{person } a \\ \text{person } b \\ \text{person } c \\ \text{person } d \end{array}$$

How can we find a lower bound on the cost of an optimal selection without actually solving the problem? We can do this by several methods. For example, it is clear that the cost of any solution, including an optimal one, cannot be smaller than the sum of the smallest elements in each of the matrix's rows. For the instance here, this sum is $2 + 3 + 1 + 4 = 10$. It is important to stress that this is not the cost of any legitimate selection (3 and 1 came from the same column of the matrix); it is just a lower bound on the cost of any legitimate selection. We can and will apply the same thinking to partially constructed solutions. For example, for any legitimate selection that selects 9 from the first row, the lower bound will be $9 + 3 + 1 + 4 = 17$.

One more comment is in order before we embark on constructing the problem's state-space tree. It deals with the order in which the tree nodes will be

generated. Rather than generating a single child of the last promising node as we did in backtracking, we will generate all the children of the most promising node among nonterminated leaves in the current tree. (Nonterminated, i.e., still promising, leaves are also called ***live***.) How can we tell which of the nodes is most promising? We can do this by comparing the lower bounds of the live nodes. It is sensible to consider a node with the best bound as most promising, although this does not, of course, preclude the possibility that an optimal solution will ultimately belong to a different branch of the state-space tree. This variation of the strategy is called the ***best-first branch-and-bound***.

So, returning to the instance of the assignment problem given earlier, we start with the root that corresponds to no elements selected from the cost matrix. As we already discussed, the lower-bound value for the root, denoted lb, is 10. The nodes on the first level of the tree correspond to selections of an element in the first row of the matrix, i.e., a job for person a (Figure 12.5).

So we have four live leaves—nodes 1 through 4—that may contain an optimal solution. The most promising of them is node 2 because it has the smallest lower-bound value. Following our best-first search strategy, we branch out from that node first by considering the three different ways of selecting an element from the second row and not in the second column—the three different jobs that can be assigned to person b (Figure 12.6).

Of the six live leaves—nodes 1, 3, 4, 5, 6, and 7—that may contain an optimal solution, we again choose the one with the smallest lower bound, node 5. First, we consider selecting the third column's element from c's row (i.e., assigning person c to job 3); this leaves us with no choice but to select the element from the fourth column of d's row (assigning person d to job 4). This yields leaf 8 (Figure 12.7), which corresponds to the feasible solution $\{a \rightarrow 2, b \rightarrow 1, c \rightarrow 3, d \rightarrow 4\}$ with the total cost of 13. Its sibling, node 9, corresponds to the feasible solution $\{a \rightarrow 2, b \rightarrow 1, c \rightarrow 4, d \rightarrow 3\}$ with the total cost of 25. Since its cost is larger than the cost of the solution represented by leaf 8, node 9 is simply terminated. (Of course, if

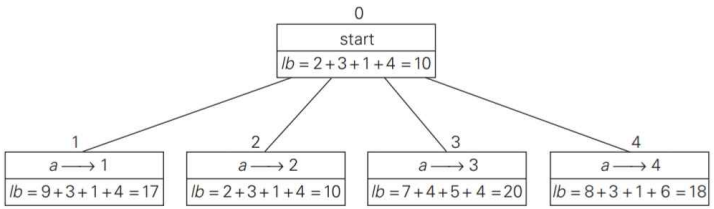


FIGURE 12.5 Levels 0 and 1 of the state-space tree for the instance of the assignment problem being solved with the best-first branch-and-bound algorithm. The number above a node shows the order in which the node was generated. A node's fields indicate the job number assigned to person a and the lower bound value, lb, for this node.

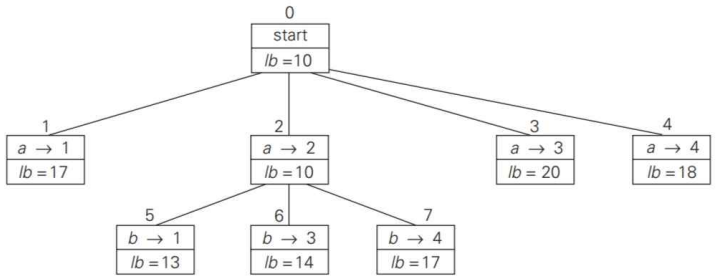


FIGURE 12.6 Levels 0, 1, and 2 of the state-space tree for the instance of the assignment problem being solved with the best-first branch-and-bound algorithm.

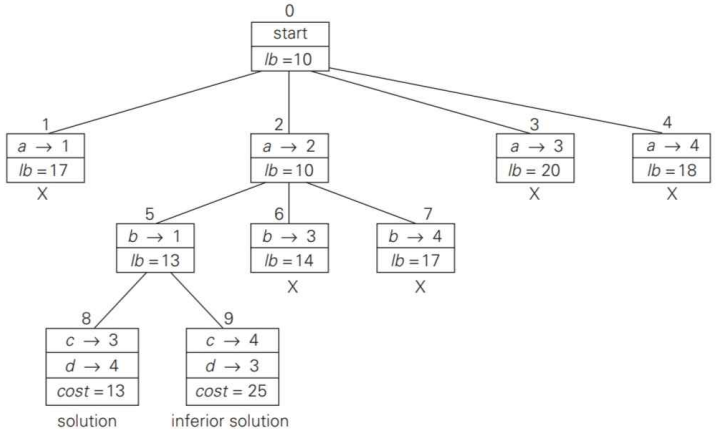


FIGURE 12.7 Complete state-space tree for the instance of the assignment proble
solved with the best-first branch-and-bound algo

its cost were smaller than 13, we would have to replace the information about the best solution seen so far with the data provided by this node.)

Now, as we inspect each of the live leaves of the last state-space tree—nodes 1, 3, 4, 6, and 7 in Figure 12.7—we discover that their lower-bound values are not smaller than 13, the value of the best selection seen so far (leaf 8). Hence, we terminate all of them and recognize the solution represented by leaf 8 as the optimal solution to the problem.

a. Prove that any algorithm for this problem must make at least $\lceil \log_3(2n+1) \rceil$ weighings in the worst case.

b. Draw a decision tree for an algorithm that solves the problem for $n = 3$ coins in two weighings.

c. Prove that there exists no algorithm that solves the problem for $n = 4$ coins in two weighings.

d. Draw a decision tree for an algorithm that solves the problem for $n = 4$ coins in two weighings by using an extra coin known to be genuine.

e. Draw a decision tree for an algorithm that solves the classic version of the problem—that for $n = 12$ coins in three weighings (with no extra coins being used).

11. *Jigsaw puzzle* A jigsaw puzzle contains n pieces. A "section" of the puzzle is a set of one or more pieces that have been connected to each other. A "move" consists of connecting two sections. What algorithm will minimize the number of moves required to complete the puzzle?

3 *P*, *NP*, and *NP*-Complete Problems

In the study of the computational complexity of problems, the first concern of both computer scientists and computing professionals is whether a given problem can be solved in polynomial time by some algorithm.

DEFINITION 1 We say that an algorithm solves a problem in polynomial time if its worst-case time efficiency belongs to $O(p(n))$ where $p(n)$ is a polynomial of the problem's input size n. (Note that since we are using big-oh notation here, problems solvable in, say, logarithmic time are solvable in polynomial time as well.) Problems that can be solved in polynomial time are called ***tractable***, and problems that cannot be solved in polynomial time are called ***intractable***.

There are several reasons for drawing the intractability line in this way. First, the entries of Table 2.1 and their discussion in Section 2.1 imply that we cannot solve arbitrary instances of intractable problems in a reasonable amount of time unless such instances are very small. Second, although there might be a huge difference between the running times in $O(p(n))$ for polynomials of drastically different degrees, there are very few useful polynomial-time algorithms with the degree of a polynomial higher than three. In addition, polynomials that bound running times of algorithms do not usually have extremely large coefficients. Third, polynomial functions possess many convenient properties; in particular, both the sum and composition of two polynomials are always polynomials too. Fourth, the choice of this class has led to a development of an extensive theory called ***computational complexity***, which seeks to classify problems according to their inherent difficulty. And according to this theory, a problem's intractability

remains the same for all principal models of computations and all reasonable input-encoding schemes for the problem under consideration.

We just touch on some basic notions and ideas of complexity theory in this section. If you are interested in a more formal treatment of this theory, you will have no trouble finding a wealth of textbooks devoted to the subject (e.g., [Sip05], [Aro09]).

P and *NP* Problems

Most problems discussed in this book can be solved in polynomial time by some algorithm. They include computing the product and the greatest common divisor of two integers, sorting a list, searching for a key in a list or for a pattern in a text string, checking connectivity and acyclicity of a graph, and finding a minimum spanning tree and shortest paths in a weighted graph. (You are invited to add more examples to this list.) Informally, we can think about problems that can be solved in polynomial time as the set that computer science theoreticians call P. A more formal definition includes in P only ***decision problems***, which are problems with yes/no answers.

DEFINITION 2 Class P is a class of decision problems that can be solved in polynomial time by (deterministic) algorithms. This class of problems is called ***polynomial***.

The restriction of P to decision problems can be justified by the following reasons. First, it is sensible to exclude problems not solvable in polynomial time because of their exponentially large output. Such problems do arise naturally—e.g., generating subsets of a given set or all the permutations of n distinct items—but it is apparent from the outset that they cannot be solved in polynomial time. Second, many important problems that are not decision problems in their most natural formulation can be reduced to a series of decision problems that are easier to study. For example, instead of asking about the minimum number of colors needed to color the vertices of a graph so that no two adjacent vertices are colored the same color, we can ask whether there exists such a coloring of the graph's vertices with no more than m colors for $m = 1, 2, \ldots$. (The latter is called the ***m-coloring problem***.) The first value of m in this series for which the decision problem of m-coloring has a solution solves the optimization version of the graph-coloring problem as well.

It is natural to wonder whether *every* decision problem can be solved in polynomial time. The answer to this question turns out to be no. In fact, some decision problems cannot be solved at all by any algorithm. Such problems are called ***undecidable***, as opposed to ***decidable*** problems that can be solved by an algorithm. A famous example of an undecidable problem was given by Alan

Turing in 1936.[1] The problem in question is called the ***halting problem***: given a computer program and an input to it, determine whether the program will halt on that input or continue working indefinitely on it.

Here is a surprisingly short proof of this remarkable fact. By way of contradiction, assume that A is an algorithm that solves the halting problem. That is, for any program P and input I,

$$A(P, I) = \begin{cases} 1, & \text{if program } P \text{ halts on input } I; \\ 0, & \text{if program } P \text{ does not halt on input } I. \end{cases}$$

We can consider program P as an input to itself and use the output of algorithm A for pair (P, P) to construct a program Q as follows:

$$Q(P) = \begin{cases} \text{halts}, & \text{if } A(P, P) = 0, \text{ i.e., if program } P \text{ does not halt on input } P; \\ \text{does not halt}, & \text{if } A(P, P) = 1, \text{ i.e., if program } P \text{ halts on input } P. \end{cases}$$

Then on substituting Q for P, we obtain

$$Q(Q) = \begin{cases} \text{halts}, & \text{if } A(Q, Q) = 0, \text{ i.e., if program } Q \text{ does not halt on input } Q; \\ \text{does not halt}, & \text{if } A(Q, Q) = 1, \text{ i.e., if program } Q \text{ halts on input } Q. \end{cases}$$

This is a contradiction because neither of the two outcomes for program Q is possible, which completes the proof.

Are there decidable but intractable problems? Yes, there are, but the number of *known* examples is surprisingly small, especially of those that arise naturally rather than being constructed for the sake of a theoretical argument.

There are many important problems, however, for which no polynomial-time algorithm has been found, nor has the impossibility of such an algorithm been proved. The classic monograph by M. Garey and D. Johnson [Gar79] contains a list of several hundred such problems from different areas of computer science, mathematics, and operations research. Here is just a small sample of some of the best-known problems that fall into this category:

Hamiltonian circuit problem Determine whether a given graph has a Hamiltonian circuit—a path that starts and ends at the same vertex and passes through all the other vertices exactly once.

Traveling salesman problem Find the shortest tour through n cities with known positive integer distances between them (find the shortest Hamiltonian circuit in a complete graph with positive integer weights).

[1] This was just one of many breakthrough contributions to theoretical computer science made by the English mathematician and computer science pioneer Alan Turing (1912–1954). In recognition of this, the ACM—the principal society of computing professionals and researchers—has named after him an award given for outstanding contributions to theoretical computer science. A lecture given on such an occasion by Richard Karp [Kar86] provides an interesting historical account of the development of complexity theory.

Knapsack problem Find the most valuable subset of n items of given positive integer weights and values that fit into a knapsack of a given positive integer capacity.

Partition problem Given n positive integers, determine whether it is possible to partition them into two disjoint subsets with the same sum.

Bin-packing problem Given n items whose sizes are positive rational numbers not larger than 1, put them into the smallest number of bins of size 1.

Graph-coloring problem For a given graph, find its chromatic number, which is the smallest number of colors that need to be assigned to the graph's vertices so that no two adjacent vertices are assigned the same color.

Integer linear programming problem Find the maximum (or minimum) value of a linear function of several integer-valued variables subject to a finite set of constraints in the form of linear equalities and inequalities.

Some of these problems are decision problems. Those that are not have decision-version counterparts (e.g., the m-coloring problem for the graph-coloring problem). What all these problems have in common is an exponential (or worse) growth of choices, as a function of input size, from which a solution needs to be found. Note, however, that some problems that also fall under this umbrella can be solved in polynomial time. For example, the Eulerian circuit problem—the problem of the existence of a cycle that traverses all the edges of a given graph exactly once—can be solved in $O(n^2)$ time by checking, in addition to the graph's connectivity, whether all the graph's vertices have even degrees. This example is particularly striking: it is quite counterintuitive to expect that the problem about cycles traversing all the edges exactly once (Eulerian circuits) can be so much easier than the seemingly similar problem about cycles visiting all the vertices exactly once (Hamiltonian circuits).

Another common feature of a vast majority of decision problems is the fact that although solving such problems can be computationally difficult, checking whether a proposed solution actually solves the problem is computationally easy, i.e., it can be done in polynomial time. (We can think of such a proposed solution as being randomly generated by somebody leaving us with the task of verifying its validity.) For example, it is easy to check whether a pro[illegible] Hamiltonian circuit for a given graph with n vertices. All [illegible] the list contains $n + 1$ vertices of the graph in question, that the first n vertices are distinct whereas the last one is the same as the first, and that every consecutive pair of the list's vertices is connected by an edge. This general observation about decision problems has led computer scientists to the notion of a nondeterministic algorithm.

DEFINITION 3 A ***nondeterministic algorithm*** is a two-stage procedure that takes as its input an instance I of a decision problem and does the following.

Nondeterministic ("guessing") stage: An arbitrary string S is generated that can be thought of as a candidate solution to the given instance I (but may be complete gibberish as well).

Deterministic ("verification") stage: A deterministic algorithm takes both I and S as its input and outputs yes if S represents a solution to instance I. (If S is not a solution to instance I, the algorithm either returns no or is allowed not to halt at all.)

We say that a nondeterministic algorithm solves a decision problem if and only if for every yes instance of the problem it returns yes on some execution. (In other words, we require a nondeterministic algorithm to be capable of "guessing" a solution at least once and to be able to verify its validity. And, of course, we do not want it to ever output a yes answer on an instance for which the answer should be no.) Finally, a nondeterministic algorithm is said to be ***nondeterministic polynomial*** if the time efficiency of its verification stage is polynomial.

Now we can define the class of *NP* problems.

DEFINITION 4 Class *NP* is the class of decision problems that can be solved by nondeterministic polynomial algorithms. This class of problems is called ***nondeterministic polynomial***.

Most decision problems are in *NP*. First of all, this class includes all the problems in *P*:

$$P \subseteq NP.$$

This is true because, if a problem is in P, we can use the deterministic polynomial-time algorithm that solves it in the verification-stage of a nondeterministic algorithm that simply ignores string S generated in its nondeterministic ("guessing") stage. But *NP* also contains the Hamiltonian circuit problem, the partition problem, decision versions of the traveling salesman, the knapsack, graph coloring, and many hundreds of other difficult combinatorial optimization problems cataloged in [Gar79]. The halting problem, on the other hand, is among the rare examples of decision problems that are known not to be in *NP*.

This leads to the most important open question of theoretical computer science: Is P a proper subset of *NP*, or are these two classes, in fact, the same? We can put this symbolically as

$$P \stackrel{?}{=} NP.$$

Note that $P = NP$ would imply that each of many hundreds of difficult combinatorial decision problems can be solved by a polynomial-time algorithm, although computer scientists have failed to find such algorithms despite their persistent efforts over many years. Moreover, many well-known decision problems are known to be "*NP*-complete" (see below), which seems to cast more doubts on the possibility that $P = NP$.

NP-Complete Problems

Informally, an *NP*-complete problem is a problem in *NP* that is as difficult as any other problem in this class because, by definition, any other problem in *NP* can be reduced to it in polynomial time (shown symbolically in Figure 11.6).

Here are more formal definitions of these concepts.

DEFINITION 5 A decision problem D_1 is said to be ***polynomially reducible*** to a decision problem D_2, if there exists a function t that transforms instances of D_1 to instances of D_2 such that:

1. t maps all yes instances of D_1 to yes instances of D_2 and all no instances of D_1 to no instances of D_2
2. t is computable by a polynomial time algorithm

This definition immediately implies that if a problem D_1 is polynomially reducible to some problem D_2 that can be solved in polynomial time, then problem D_1 can also be solved in polynomial time (why?).

DEFINITION 6 A decision problem D is said to be ***NP-complete*** if:

1. it belongs to class *NP*
2. every problem in *NP* is polynomially reducible to D

The fact that closely related decision problems are polynomially reducible to each other is not very surprising. For example, let us prove that the Hamiltonian circuit problem is polynomially reducible to the decision version of the traveling

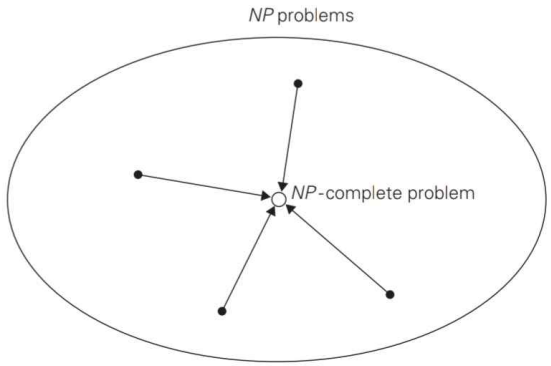


FIGURE 11.6 Notion of an *NP*-complete problem. Polynomial-time reductions of *NP* problems to an *NP*-complete problem are shown by arrows.

salesman problem. The latter can be stated as the existence problem of a Hamiltonian circuit not longer than a given positive integer m in a given complete graph with positive integer weights. We can map a graph G of a given instance of the Hamiltonian circuit problem to a complete weighted graph G' representing an instance of the traveling salesman problem by assigning 1 as the weight to each edge in G and adding an edge of weight 2 between any pair of nonadjacent vertices in G. As the upper bound m on the Hamiltonian circuit length, we take $m = n$, where n is the number of vertices in G (and G'). Obviously, this transformation can be done in polynomial time.

Let G be a yes instance of the Hamiltonian circuit problem. Then G has a Hamiltonian circuit, and its image in G' will have length n, making the image a yes instance of the decision traveling salesman problem. Conversely, if we have a Hamiltonian circuit of the length not larger than n in G', then its length must be exactly n (why?) and hence the circuit must be made up of edges present in G, making the inverse image of the yes instance of the decision traveling salesman problem be a yes instance of the Hamiltonian circuit problem. This completes the proof.

The notion of *NP*-completeness requires, however, polynomial reducibility of *all* problems in *NP*, both known and unknown, to the problem in question. Given the bewildering variety of decision problems, it is nothing short of amazing that specific examples of *NP*-complete problems have been actually found. Nevertheless, this mathematical feat was accomplished independently by Stephen Cook in the United States and Leonid Levin in the former Soviet Union.[2] In his 1971 paper, Cook [Coo71] showed that the so-called ***CNF-satisfiability problem*** is *NP*-complete. The CNF-satisfiability problem deals with boolean expressions. Each boolean expression can be represented in conjunctive normal form, such as the following expression involving three boolean variables x_1, x_2, and x_3 and their negations denoted $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$, respectively:

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3)\&(\bar{x}_1 \vee x_2)\&(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3).$$

The CNF-satisfiability problem asks whether or not one can assign values *true* and *false* to variables of a given boolean expression in its CNF form to make the entire expression *true*. (It is easy to see that this can be done for the above formula: if $x_1 = \mathit{true}$, $x_2 = \mathit{true}$, and $x_3 = \mathit{false}$, the entire expression is *true*.)

Since the Cook-Levin discovery of the first known *NP*-complete problems, computer scientists have found many hundreds, if not thousands, of other examples. In particular, the well-known problems (or their decision versions) mentioned above—Hamiltonian circuit, traveling salesman, partition, bin packing, and graph coloring—are all *NP*-complete. It is known, however, that if $P \neq NP$ there must exist *NP* problems that neither are in P nor are *NP*-complete.

As it often happens in the history of science, breakthrough discoveries are made independently and almost simultaneously by several scientists. In fact, Levin introduced a more general notion than *NP*-completeness, which was not limited to decision problems, but his paper [Lev73] was published two years after Cook's.

For a while, the leading candidate to be such an example was the problem of determining whether a given integer is prime or composite. But in an important theoretical breakthrough, Professor Manindra Agrawal and his students Neeraj Kayal and Nitin Saxena of the Indian Institute of Technology in Kanpur announced in 2002 a discovery of a deterministic polynomial-time algorithm for primality testing [Agr04]. Their algorithm does not solve, however, the related problem of factoring large composite integers, which lies at the heart of the widely used encryption method called the ***RSA algorithm*** [Riv78].

Showing that a decision problem is *NP*-complete can be done in two steps. First, one needs to show that the problem in question is in *NP*; i.e., a randomly generated string can be checked in polynomial time to determine whether or not it represents a solution to the problem. Typically, this step is easy. The second step is to show that every problem in *NP* is reducible to the problem in question in polynomial time. Because of the transitivity of polynomial reduction, this step can be done by showing that a known *NP*-complete problem can be transformed to the problem in question in polynomial time (see Figure 11.7). Although such a transformation may need to be quite ingenious, it is incomparably simpler than proving the existence of a transformation for every problem in *NP*. For example, if we already know that the Hamiltonian circuit problem is *NP*-complete, its polynomial reducibility to the decision traveling salesman problem implies that the latter is also *NP*-complete (after an easy check that the decision traveling salesman problem is in class *NP*).

The definition of *NP*-completeness immediately implies that if there exists a deterministic polynomial-time algorithm for just one *NP*-complete problem, then every problem in *NP* can be solved in polynomial time by a deterministic algorithm, and hence $P = NP$. In other words, finding a polynomial-time algorithm

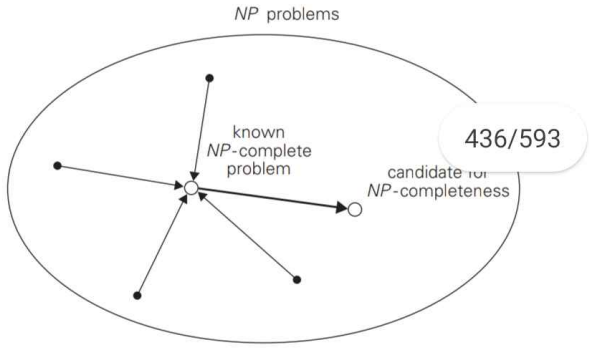
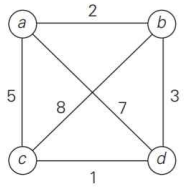


FIGURE 11.7 Proving *NP*-completeness by reduction.

6. **a.** Suggest a more sophisticated bounding function for solving the knapsack problem than the one used in the section.
   b. Use your bounding function in the branch-and-bound algorithm applied to the instance of Problem 5.
7. Write a program to solve the knapsack problem with the branch-and-bound algorithm.
8. **a.** Prove the validity of the lower bound given by formula (12.2) for instances of the traveling salesman problem with symmetric matrices of integer intercity distances.
   b. How would you modify lower bound (12.2) for nonsymmetric distance matrices?
9. Apply the branch-and-bound algorithm to solve the traveling salesman problem for the following graph:



   (We solved this problem by exhaustive search in Section 3.4.)
10. As a research project, write a report on how state-space trees are used for programming such games as chess, checkers, and tic-tac-toe. The two principal algorithms you should read about are the minimax algorithm and alpha-beta pruning.

12.3 Approximation Algorithms for *NP*-Hard Problems

In this section, we discuss a different approach to handling difficult problems of combinatorial optimization, such as the traveling salesman problem and the knapsack problem. As we pointed out in Section 11.3, the decision versions of these problems are *NP*-complete. Their optimization versions fall in the class of ***NP-hard problems***—problems that are at least as hard as *NP*-complete problems.[2] Hence, there are no known polynomial-time algorithms for these problems, and there are serious theoretical reasons to believe that such algorithms do not exist. What then are our options for handling such problems, many of which are of significant practical importance?

2. The notion of an *NP*-hard problem can be defined more formally by extending the notion of polynomial reducibility to problems that are not necessarily in class *NP*, including optimization problems of the type discussed in this section (see [Gar79, Chapter 5]).

If an instance of the problem in question is very small, we might be able to solve it by an exhaustive-search algorithm (Section 3.4). Some such problems can

If an instance of the problem in question is very small, we might be able to solve it by an exhaustive-search algorithm (Section 3.4). Some such problems can be solved by the dynamic programming technique we demonstrated in Section 8.2. But even when this approach works in principle, its practicality is limited by dependence on the instance parameters being relatively small. The discovery of the branch-and-bound technique has proved to be an important breakthrough, because this technique makes it possible to solve many large instances of difficult optimization problems in an acceptable amount of time. However, such good performance cannot usually be guaranteed.

There is a radically different way of dealing with difficult optimization problems: solve them approximately by a fast algorithm. This approach is particularly appealing for applications where a good but not necessarily optimal solution will suffice. Besides, in real-life applications, we often have to operate with inaccurate data to begin with. Under such circumstances, going for an approximate solution can be a particularly sensible choice.

Although approximation algorithms run a gamut in level of sophistication, most of them are based on some problem-specific heuristic. A ***heuristic*** is a common-sense rule drawn from experience rather than from a mathematically proved assertion. For example, going to the nearest unvisited city in the traveling salesman problem is a good illustration of this notion. We discuss an algorithm based on this heuristic later in this section.

Of course, if we use an algorithm whose output is just an approximation of the actual optimal solution, we would like to know how accurate this approximation is. We can quantify the accuracy of an approximate solution s_a to a problem of minimizing some function f by the size of the relative error of this approximation,

$$re(s_a) = \frac{f(s_a) - f(s^*)}{f(s^*)},$$

where s^* is an exact solution to the problem. Alternatively, since $re(s_a) = f(s_a)/f(s^*) - 1$, we can simply use the ***accuracy ratio***

$$r(s_a) = \frac{f(s_a)}{f(s^*)}$$

as a measure of accuracy of s_a. Note that for the sake of scale uniformity, the accuracy ratio of approximate solutions to maximization problems is usually computed as

$$r(s_a) = \frac{f(s^*)}{f(s_a)}$$

to make this ratio greater than or equal to 1, as it is for minimization problems.

Obviously, the closer $r(s_a)$ is to 1, the better the approximate solution is. For most instances, however, we cannot compute the accuracy ratio, because we typically do not know $f(s^*)$, the true optimal value of the objective function. Therefore, our hope should lie in obtaining a good upper bound on the values of $r(s_a)$. This leads to the following definitions.

DEFINITION A polynomial-time approximation algorithm is said to be a ***c*-approximation algorithm**, where $c \geq 1$, if the accuracy ratio of the approximation it produces does not exceed c for any instance of the problem in question:

$$r(s_a) \leq c. \tag{12.3}$$

The best (i.e., the smallest) value of c for which inequality (12.3) holds for all instances of the problem is called the ***performance ratio*** of the algorithm and denoted R_A.

The performance ratio serves as the principal metric indicating the quality of the approximation algorithm. We would like to have approximation algorithms with R_A as close to 1 as possible. Unfortunately, as we shall see, some approximation algorithms have infinitely large performance ratios ($R_A = \infty$). This does not necessarily rule out using such algorithms, but it does call for a cautious treatment of their outputs.

There are two important facts about difficult combinatorial optimization problems worth keeping in mind. First, although the difficulty level of solving most such problems exactly is the same to within a polynomial-time transformation of one problem to another, this equivalence does not translate into the realm of approximation algorithms. Finding good approximate solutions is much easier for some of these problems than for others. Second, some of the problems have special classes of instances that are both particularly important for real-life applications and easier to solve than their general counterparts. The traveling salesman problem is a prime example of this situation.

Approximation Algorithms for the Traveling Salesman Problem

We solved the traveling salesman problem by exhaustive search in Section 3.4, mentioned its decision version as one of the most well-known *NP*-complete problems in Section 11.3, and saw how its instances can be solved by a branch-and-bound algorithm in Section 12.2. Here, we consider several approximation algorithms, a small sample of dozens of such algorithms suggested over the years for this famous problem. (For a much more detailed discussion of the topic, see [Law85], [Hoc97], [App07], and [Gut07].)

But first let us answer the question of whether we should hope to find a polynomial-time approximation algorithm with a finite performance ratio on all instances of the traveling salesman problem. As the following theorem [Sah76] shows, the answer turns out to be no, unless $P = NP$.

THEOREM 1 If $P \neq NP$, there exists no c-approximation algorithm for the traveling salesman problem, i.e., there exists no polynomial-time approximation algorithm for this problem so that for all instances

$$f(s_a) \leq cf(s^*)$$

for some constant c.

PROOF By way of contradiction, suppose that such an approximation algorithm A and a constant c exist. (Without loss of generality, we can assume that c is a positive integer.) We will show that this algorithm could then be used for solving the Hamiltonian circuit problem in polynomial time. We will take advantage of a variation of the transformation used in Section 11.3 to reduce the Hamiltonian

PROOF By way of contradiction, suppose that such an approximation algorithm A and a constant c exist. (Without loss of generality, we can assume that c is a positive integer.) We will show that this algorithm could then be used for solving the Hamiltonian circuit problem in polynomial time. We will take advantage of a variation of the transformation used in Section 11.3 to reduce the Hamiltonian circuit problem to the traveling salesman problem. Let G be an arbitrary graph with n vertices. We map G to a complete weighted graph G' by assigning weight 1 to each edge in G and adding an edge of weight $cn + 1$ between each pair of vertices not adjacent in G. If G has a Hamiltonian circuit, its length in G' is n; hence, it is the exact solution s^* to the traveling salesman problem for G'. Note that if s_a is an approximate solution obtained for G' by algorithm A, then $f(s_a) \leq cn$ by the assumption. If G does not have a Hamiltonian circuit in G, the shortest tour in G' will contain at least one edge of weight $cn + 1$, and hence $f(s_a) \geq f(s^*) > cn$. Taking into account the two derived inequalities, we could solve the Hamiltonian circuit problem for graph G in polynomial time by mapping G to G', applying algorithm A to get tour s_a in G', and comparing its length with cn. Since the Hamiltonian circuit problem is *NP*-complete, we have a contradiction unless $P = NP$. ■

Greedy Algorithms for the TSP The simplest approximation algorithms for the traveling salesman problem are based on the greedy technique. We will discuss here two such algorithms.

Nearest-neighbor algorithm

The following well-known greedy algorithm is based on the ***nearest-neighbor*** heuristic: always go next to the nearest unvisited city.

Step 1 Choose an arbitrary city as the start.

Step 2 Repeat the following operation until all the cities have been visited: go to the unvisited city nearest the one visited last (ties can be broken arbitrarily).

Step 3 Return to the starting city.

EXAMPLE 1 For the instance represented by the graph in Figure 12.10, with a as the starting vertex, the nearest-neighbor algorithm yields the tour (Hamiltonian circuit) s_a: $a - b - c - d - a$ of length 10.

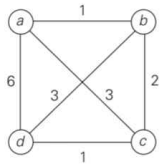


FIGURE 12.10 Instance of the traveling salesman problem.

The optimal solution, as can be easily checked by exhaustive search, is the tour s^*: $a - b - d - c - a$ of length 8. Thus, the accuracy ratio of this approximation is

$$r(s_a) = \frac{f(s_a)}{f(s^*)} = \frac{10}{8} = 1.25$$

(i.e., tour s_a is 25% longer than the optimal tour s^*). ■

Unfortunately, except for its simplicity, not many good things can be said about the nearest-neighbor algorithm. In particular, nothing can be said in general about the accuracy of solutions obtained by this algorithm because it can force us to traverse a very long edge on the last leg of the tour. Indeed, if we change the weight of edge (a, d) from 6 to an arbitrary large number $w \geq 6$ in Example 1, the algorithm will still yield the tour $a - b - c - d - a$ of length $4 + w$, and the optimal solution will still be $a - b - d - c - a$ of length 8. Hence,

$$r(s_a) = \frac{f(s_a)}{f(s^*)} = \frac{4 + w}{8},$$

which can be made as large as we wish by choosing an appropriately large value of w. Hence, $R_A = \infty$ for this algorithm (as it should be according to Theorem 1).

Multifragment-heuristic algorithm

Another natural greedy algorithm for the traveling salesman problem considers it as the problem of finding a minimum-weight collection of edges in a given complete weighted graph so that all the vertices have degree 2. (With this emphasis on edges rather than vertices, what other greedy algorithm does it remind you of?) An application of the greedy technique to this problem leads to the following algorithm [Ben90].

Step 1 Sort the edges in increasing order of their weights. (Ties can be broken arbitrarily.) Initialize the set of tour edges to be constructed to the empty set.

Step 2 Repeat this step n times, where n is the number of cities in the instance being solved: add the next edge on the sorted edge list to the set of tour edges, provided this addition does not create a vertex of degree 3 or a cycle of length less than n; otherwise, skip the edge.

Step 3 Return the set of tour edges.

As an example, applying the algorithm to the graph 473/593 iel $\{(a, b), (c, d), (b, c), (a, d)\}$. This set of edges forms the sam e pr duced by the nearest-neighbor algorithm. In general, the multifragment-heuristic algorithm tends to produce significantly better tours than the nearest-neighbor algorithm, as we are going to see from the experimental data quoted at the end of this section. But the performance ratio of the multifragment-heuristic algorithm is also unbounded, of course.