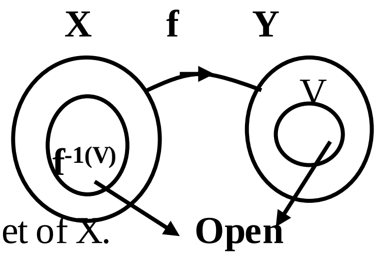
UNIT – II

CONTINUOUS FUNCTION

Definition: Continuous Function



Let X and Y be topological spaces.

A function **f : X → Y** is said to be continuous if for each open subset V of Y, the set f^{-1} (V) is an open subset of X.

Note 1:

Recall that f^{-1} (V) is the set of all points x of X for which f(x) ∈ V. It is empty if V does not intersect the image set f(X) of f.

Note 2:

Continuity of a function depends not only upon the function itself, but also on the topologies specified for its domain and range. If we wish to emphasize this fact, we can say that f is continuous relative to specified topologies on X and Y.

Example-1:

Let us consider the function like those studied in analysis **" A real valued function of a real variable".**

$f : \mathcal{R} \to \mathcal{R}$

i.e., f(x) = x

Example-2:

Let $\mathcal{R}$ denote the set of real numbers in its usual topology and let $\mathcal{R}_\ell$ denote the same set in the **lower limit topology.**

Let $f : \mathcal{R} \to \mathcal{R}_\ell$ be the identity function.

f(x) = x for every real number X. Then f is not a continuous function. The inverse image of an open set [a, b) of $\mathcal{R}_\ell$ equals itself which is not open in $\mathcal{R}$.

On the otherhand, the identity function $\mathbf{g} : \mathcal{R}_\ell \to \mathcal{R}$ is continuous, because the inverse image of (a, b) is itself, which is open in $\mathcal{R}_\ell$.

Definition : Homeomorphism

Let X and Y be a topological spaces. Let f : X → Y be a **bijection.**

If both **f** and **f^{-1}** are continuous, then **f** is called a **Homeomorphism.**

Theorem :

Statement:

Let X and Y be a topological spaces. Let f: X $\rightarrow$ Y. Then the following are equivalent.

(i) f is continuous

(ii) for every subset A of X, **f**($\bar{A}$ **)** $\subset$ **$\overline{f(A)}$**

(iii) for every closed set B of Y the set f^{-1}(B) is closed in X

(iv) for each x $\in$ X and each neighbourhood V of f(x) there is a neighbourhood U of x such that f(U) $\subset$ V

If the conclusion in (**4**) holds for the point x of X we say that **f** is **continuous** at the point x.

Proof:

Let X and Y be the topological spaces. Let f : X $\rightarrow$ Y.

(i) ⇒ (ii)

Assume that f is continuous. Let A be a subset of X.

To prove: **f**($\bar{A}$) $\subset$ $\overline{\boldsymbol{f}(\mathbf{A})}$

Let $x \in \bar{A}$. Then $f(x) \in f(\bar{A})$

if $f(x) \in f(\bar{A})$ then we have to show that

f(x) $\subset$ $\overline{\boldsymbol{f}(\mathbf{A})}$

Let V be a neighbourhood of f(x). Then f^{-1}(V) is an open set of X containing x. (∵ f is continuous)

Here $x \in \bar{A}$ and f^{-1}(V) is open.

∴ f^{-1}(V) must intersect A in some point y.

Then V intersects f(A) in the point f(y).

i.e., $f(y) \in V \cap f(A)$

$\Rightarrow V \cap f(A)$ is non empty.

$\Rightarrow$ $f(x) \in \overline{f(A)}$

∴ $f(\bar{A}) \subset \overline{f(A)}$

(ii) ⇒ (iii)

Let B closed in Y and let $A = f^{-1}(B)$

To prove: A is closed in X.

We have $A \subset \bar{A}$

if we prove $\bar{A} \subset A$ then $A = \bar{A}$

$\Rightarrow$ A is closed.

Let us prove : $\bar{A} \subset A$

Here $A = f^{-1}(B) \Rightarrow f(A) \subset B$.

Let $x \in \bar{A}$ then $f(x) \in f(\bar{A})$

$\subset \overline{f(A)}$

$\subseteq \bar{B} = B$ since B is closed

i.e., $f(x) \in B$ (or) $x \in f^{-1}(B) = A$

$\Rightarrow \bar{A} \subset A$

Hence $A = \bar{A}$

$\therefore$ $A = f^{-1}(B)$ is closed.

(iii) $\Rightarrow$ (i)

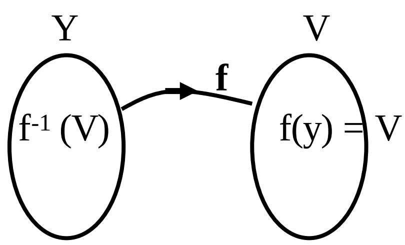
Let V be an open set in Y. Let B = Y-V, then B is closed in Y.

$\therefore$ $f^{-1}(B)$ is closed in X. (by (iii))

$f^{-1}(V) = f^{-1}(Y-B)$

$= f^{-1}(Y) - f^{-1}(B)$

$= X - f^{-1}(B)$



$\therefore$ $f^{-1}(V)$ is open.

Hence f is continuous.

(i) $\Rightarrow$(iv)

Let $x \in X$ and V be a neighbourhood of f(x). Then since f is continuous $f^{-1}(V)$ is a neighbourhood of x. Let $f^{-1}(V) = U$.

Then $f(U) = f(f^{-1}(V)) \subseteq V$

For given $x \in X$ and a neighbourhood V of f(x), there exist a neighbourhood U of x such that $f(U) \subseteq V$.

(iv)$\Rightarrow$(i)

Let V be an open set of Y.

Let $x \in f^{-1}(V)$ then $f(x) \in V$.

By hypothesis,

$\exists$ a neighbourhood U_x of x such that $f(U_x) \subseteq V$

Then, $U_x \subseteq f^{-1}(f(U_x)) \subseteq f^{-1}(V)$.

Hence $f^{-1}(V) = \cup U_x$.

since each U_x is open and union of open sets is open, $f^{-1}(V)$ is open in X.

Therefore f is continuous.

Hence the theorem.

Foot Note:

i) A is always contained in $f^{-1}(f(A))$

i.e $A \subseteq f^{-1}(f(A))$

ii) $f(f^{-1}(B)) \subseteq B$.

Result-1:

If the inverse image of every basis element is open, then f is continuous.

Proof:

Let $f : X \to Y$ and the inverse image of every basis element be open

Let $V \subseteq Y$ be a open in Y.

Then $V = \bigcup_{\alpha} B_{\alpha}$

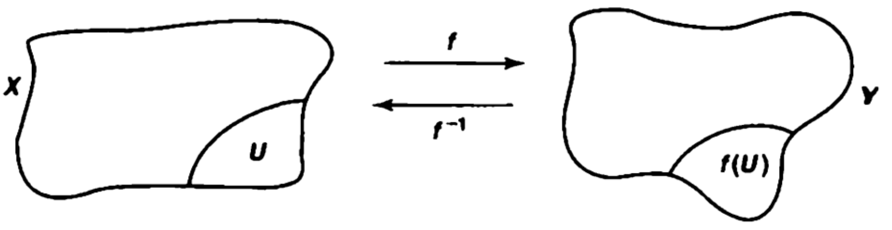
$f^{-1}(V) \subset f^{-1}\left(\bigcup_{\alpha} B_{\alpha} \right) = \bigcup_{\alpha} f^{-1}(B_{\alpha})$

$\Rightarrow$ $f^{-1}(V)$ is open in X, since each $f^{-1}(B_{\alpha})$ is open in X.

Definition: Open map

A map $f : X \to Y$ is said to be an open map if for every open set U of X, f(U) is open in Y.

$$f^{-1} : Y \to X$$



Note:

Let $f : X \to Y$, then the map $f^{-1} : Y \to X$ the inverse image of U under the map f^{-1} is same as the image of U under the map f.

The homeomorphism can be defined as a bijective correspondence

$f : X \to Y$ such that $f(U)$ is open iff U is open.

Definition : Topological property

Let $f : X \to Y$ be a homeomorphism. Any property of X that is entirely expressed in terms of the topology of X yields, through the correspondence f, the corresponding property for the space Y, such a property of X is called **topological property.**

Constructing Continuous Functions:

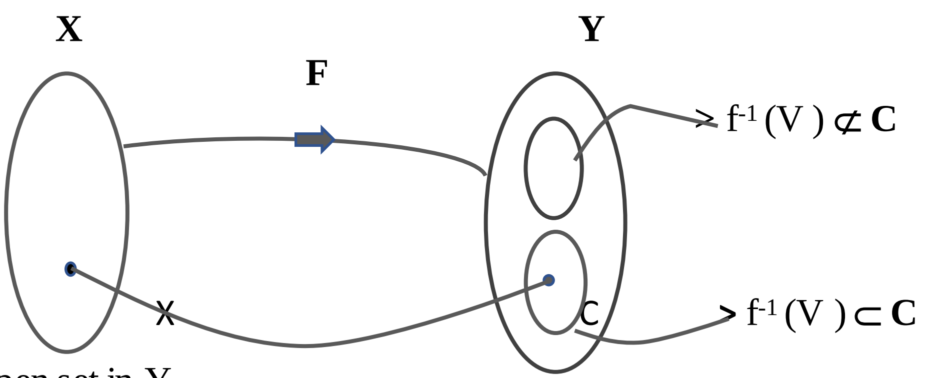
Theorem : Rules for constructing continuous function

Let X, Y and Z be topological spaces.

(a) Constant function:

If $f : X \to Y$ maps all of X into the single point y_0 of Y. Then f is continuous.

Proof:



Let V be an open set in Y.

$$\text{Then } f^{-1}(V) = \begin{cases} \phi & y_0 \notin V \\ X & y_0 \in V \end{cases}$$

$\therefore f^{-1}(V)$ is open in X. (**since** both ϕ and X are open in X)

(b) Restriction Function:

Let A be a subspace of X. Then the restriction function, restricting the domain $f/A : A \to Y$ is continuous.

Proof:

Let V be open in Y.Then $f^{-1}(V)$ is open in X, since f is continuous.
$f^{-1}(V) \cap A$ is open in A and $(f(A)^{-1}(V) = f^{-1}(V) \cap A$
Thus $(f(A))^{-1}(V)$ is open in A.
$\therefore f/A : A \to Y$ is continuous.

(c) Inclusion Function:

If A is a subspace of X the inclusion function **j :** $A \to X$ is continuous.

Proof:

Let U be open in X then $j^{-1}(U) = A \cap U$ is open in A.
(in subspace topology)
$\therefore j^{-1}(U)$ is open in A.
i.e j : $A \to X$ is continuous.

(d) Composition of continuous functions is continuous

If $f : X \to Y$ and $g: Y \to Z$ are continuous then the map $gof : X \to Z$ is continuous.

Proof:

Let V be an open set of Z then $g^{-1}(V)$ is open in Y. ($\because$ g is continuous)
Since f is continuous $f^{-1}(g^{-1}(V))$ is open in X.
And $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$
$\therefore$ for every open set V of Z, $(gof)^{-1}(V)$ is open in X.
Hence $gof : X \to Z$ is continuous.

(e) Restricting (or) Expanding the range:

(**i**) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set f(X), then the function $g: X \to Z$ obtained by restricting the range of f is continuous.

(**ii**) If Z is a space having Y as a subspace then the function $f_{\mathbf{n}}$, $h : X \to Z$ obtained by expanding the range of f is continuous.

Proof:

(i) Let $f: X \to Y$ be continuous.

If Z is a subspace of Y containing f(X)

We have $f(X) \subseteq Z \subseteq Y$.

Let $g: X \to Z$ be a function.

<u>To prove:</u> g is continuous.

Let B be open in Z.

<u>To prove:</u>

$g^{-1}(B)$ is open in X.

Since B is open in the subspace topology, $B = U \cap Z$, where U is open in Y.

$g^{-1}(B) = f^{-1}(B) = f^{-1}(U \cap Z)$

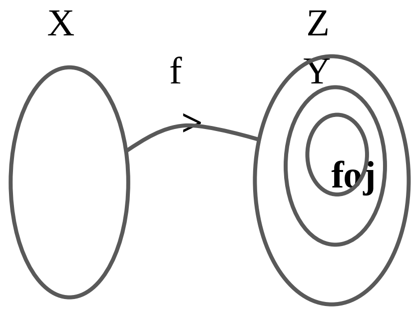
$= f^{-1}(U) \cap f^{-1}(Z)$

$= f^{-1}(U) \cap X.$

$g^{-1}(B) = f^{-1}(U)$ ....(∗)

Since U is open in Y and f is continuous,

$f^{-1}(U)$ is open in X.



i.e., $g^{-1}(B)$ is open in X (by (∗))

Hence $g: X \to Z$ is continuous.

(ii) Let Z contains Y as a subspace

given that $f: X \to Y$ is continuous.

The inclusion function $j: Y \to Z$ is also continuous.

∴ Their composition (jof) : $X \to Z$ is continuous.

i.e., The map $h: X \to Z$ is continuous.

(f) Local formulation of Continuity:

The map $f: X \to Y$ is continuous if X can be written as union of open sets U_α such that f / U_α is continuous for each α.

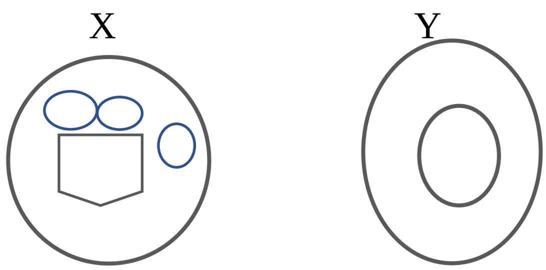
Proof:

Let $f: X \to Y$ and let $X = U_\alpha$

Given that :

$f_{|U_\alpha} : U_\alpha \to Y$ is continuous, for each α.

<u>To prove:</u>



$f|U_\alpha : X \to Y$ is continuous.

Let V be open in Y.

<u>claim:</u>

$f^{-1}(V)$ is open in X.

Since $f / U : U_\alpha \to Y$ is continuous, and V is open in Y.

$(f/U_\alpha)^{-1}(V)$ is open in U_α.

$(f/U_\alpha)^{-1}(V) = f^{-1}(V) \cap U_\alpha$ is open in U_α.

Since U_α is open in X, we have

$f^{-1}(V) \cap U_\alpha$ is open in X.

Now

$$f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha),$$

arbitrary union of open sets is open.

$\Rightarrow$ $f^{-1}(V)$ is open in X, Since $f : X \to Y$ is continuous.

Hence proved.

Pasting Lemma:

Statement:

Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ $\forall x \in A \cap B$ then **f** & **g** combine to give a continuous function.

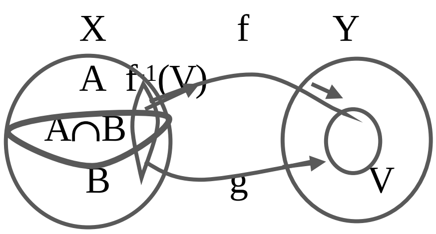
$h : X \to Y$ defined by setting

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

Proof:

Let V be a closed set in Y. Then

$h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$



Since f is continuous,

$f^{-1}(V)$ is closed in A and A closed in X

$\Rightarrow$ $f^{-1}(V)$ is closed in X.

$g^{-1}(V)$ is closed in B and B closed in X

$\Rightarrow$ $g^{-1}(V)$ is closed in X.

Since union of two closed sets is closed,

$f^{-1}(V) \cup g^{-1}(V)$ is closed in X.

i.e., $h^{-1}(V)$ is closed in X.

Therefore h is continuous.

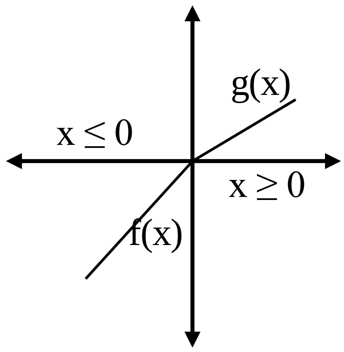
Hence the proof.

Example 1:

For Pasting Lemma

Define $h : \mathcal{R} \to \mathcal{R}$ by

$$h(x) = \begin{cases} x & \text{if } x \le 0 \\ x/2 & \text{if } x \ge 0 \end{cases}$$



$f(x) = x$, $g(x) = x/2$

$A = \{ x : x \le 0 \}$ = negative reals $\cup \{0\}$ is closed.

$B = \{ x : x \ge 0 \} = R_+ \cup \{0\}$ is closed.

and $R = A \cup B$

$A \cap B = \{0\}$

$f(0) = 0$, $g(0) = 0$.

Hence $f(0) = g(0)$.

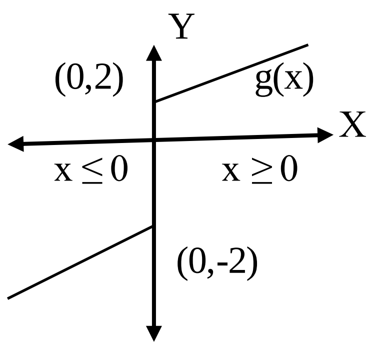
Hence by Pasting Lemma, h is continuous.

Example 2:

The pieces of the function must agree on the overlapping part of their domains in Pasting Lemma. If not the function need not be continuous.

Let h1: $\mathcal{R} \to \mathcal{R}$ defined by

$$h_1(x) = \begin{cases} x-2 & \text{if } x \le 0 \\ x+2 & \text{if } x \ge 0 \end{cases}$$



$f(x) = x-2$, $g(x) = x+2$

$A = \{x : x \le 0\}$

$= R_- \cup \{0\}$ is closed.

$B = \{x : x \ge 0\}$

$= R_+ \cup \{0\}$ is closed.

$\mathcal{R} = A \cup B$.

$A \cap B = \{0\}$

$f(0) = -2 \neq g(0) = 2$.

From the graph it is clear that h_1 is not continuous.

Example 3:

Let : $\mathcal{R} \to \mathcal{R}$

Let $\ell(x) = \begin{cases} x-2 & \text{if } x < 0, \quad x+2 \quad \text{if } x \ge 0 \end{cases}$

A = {x: x < 0}

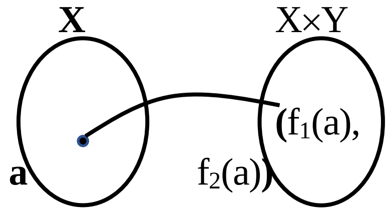
= R_- is not closed.

We define a function ℓ mapping $\mathcal{R}$ into $\mathcal{R}$ and both the pieces are continuous.

But ℓ is not continuous, the inverse image of the open set (1,3) is non-open set [0,1).

Theorem 2.4:

Maps Into Products:



Statement:

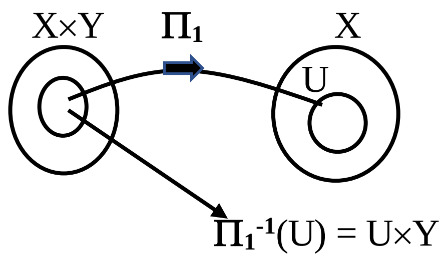
Let $f : A \to X \times Y$ be given by the equation

$f(a) = (f_1(a), f_2(a))$

Then f is continuous if and only if the function $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

The maps f_1 and f_2 are called the coordinate functions of f.

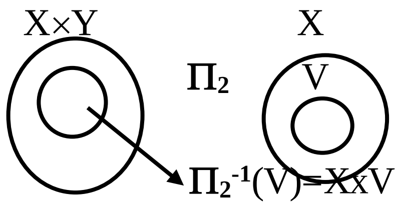
Proof:



Let $\Pi_1 : X \times Y \to X$ and

$\Pi_2 : X \times Y \to Y$ be projections onto the first and second factors, respectively.

These maps are continuous.



For $\Pi_1^{-1}(U) = U \times Y$ and

$\Pi_2^{-1}(U) = X \times V$ and these sets are open.

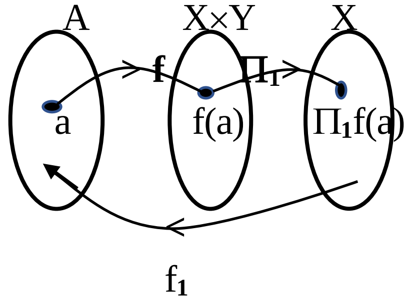
if U and V are open. Note that for each $a \in A$.

$f_1(a) = \Pi_1(f(a))$ and

$f_2(a) = \Pi_2(f(a))$.

If the function f is continuous then $\mathbf{f_1}$ and $\mathbf{f_2}$ are composites of continuous function and therefore continuous.

Conversely,



Suppose that $\mathbf{f_1}$ and $\mathbf{f_2}$ are continuous.

We show that for each basis element $U \times V$ for the topology of $X \times Y$,

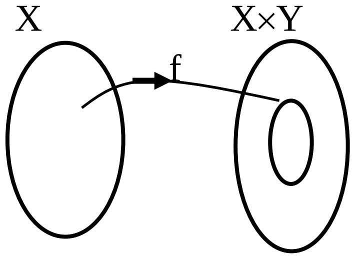
its inverse image $f^{-1}(U \times V)$ is open.

A point **a** is in $f^{-1}(U \times V)$ iff $f(a) \in U \times V$.

i.e., iff $f_1(a) \in U$ and $f_2(a) \in V$.

$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open,



so is their intersection.

Hence the proof.

Definition : Limit of the Sequence

If the sequence $\{\mathbf{x_n}\}$ of points of the Hausdorff Space X converges to a point $\mathbf{x}$ of X. We write $\mathbf{x_n} \to \mathbf{x}$ and call $\mathbf{x}$ as a limit of the sequence $\{\mathbf{x_n}\}$.

The Product Topology:

Definition: J-tuple

Let **J** be an indexed set given a set X. We define a **J-tuple** of elements of X to be the function

X: J $\to$ X if α is an element of J. We denote the value if X at α by (X(a)=) x_α rather than x(α).

Then x_α is called the α^{th} co-ordinate of X.

The function X itself is denoted by the symbol $(x_\alpha)_{\alpha \in J}$. We denote the set of all **J**-tuples of elements of X by X^J.

Definition:

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Let

$$X = \bigcup_{\alpha \in J} A_\alpha$$

The Cartesian product of this indexed family denoted by

$$\prod_{\alpha \in J} A_\alpha,$$

is defined to be the set of all **J-tuples** $(x_\alpha)_{\alpha \in J}$ elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in$ J.

i.e., it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that x(α) $\in A_\alpha$ for each α.

Note:

If all the sets A_α are equal to X, then the cartesian product $\Pi_{\alpha \in J} A_\alpha$ is just the set X^J of **J**-tuples of elements of X.

Definition: Box Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Then the basis for a topology on the product space $\Pi_{\alpha\in J}\ X_\alpha$ is the collection of all sets of the form $\Pi_{\alpha\in J}\ U_\alpha$ where U_α is open in X for each $\alpha \in J$.

The topology generated by this basis is called the **Box topology.**

Note:

The collection satisfies the first condition for a basis because $\Pi\ X_\alpha$ is itself a basis element and it satisfies the 2[nd] condition because the intersection of any two basis element is another basis element.

$$\left(\prod_{\alpha\in J} U_\alpha\right)\cap\left(\prod_{\alpha\in J} V_\alpha\right)=\prod_{\alpha\in J}(U_\alpha\cap V_\alpha).$$

Definition: Projection Mapping

Let $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$ be the function assigning to each element of the product space its β^{th} co-ordinate,

$$\pi_\beta(\ (x_\alpha)_{\alpha\ \epsilon\ J}\ ) = x_\beta$$

It is called the projection mapping associated with the index β.

Definition :

Let S_β denote the collection

$$S_\beta = \{\ \prod^{-1}{}_\beta\,(U_\beta\ )\ /\ \ U_\beta \text{ open in } X_\beta\ \}$$

And let S denote the union of these collections,

$$S\ = U_{\beta\ \epsilon\ J}\ S_\beta$$

The topology generated by the sub basis S is called the **product topology**.

In this topology $\prod_{\alpha\ \epsilon\ J} X_\alpha$ is called a **product space.**

Theorem

Comparison of the Box and Product topologies

Statement :

The box of topology on $\prod X_\alpha$ has an basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Proof:

Basis for product topology on $\prod X_\alpha$.

The collection **B** consist of all finite intersection of elements of S**.**

If we intersect elements belonging to the same collection of S_β then

$$\prod^{-1}_\beta(U_\beta) \cap \prod^{-1}_\beta(V_\beta) = \prod^{-1}_\beta (U_\beta \cap V_\beta)$$

Thus the intersection of two elements of S_β or finitely many such elements is again an element of S_β .

So let us intersect elements from different sets S_β . Let β_1 , β_2 , …… β_n be a finite set of distinct indices from the index set J .

Let U_β be an open set in $X_{\beta i}$, i = 1, 2, …. n . Then

$$\prod^{-1}_{\beta 1}(U_{\beta 1}) \cap \prod^{-1}_{\beta 2}(U_{\beta 2}) \cap \ldots\ldots \cap \prod^{-1}_{\beta n}(U_{\beta n})$$

is the finite intersection of subbasis elements so it belongs to **B .**

Let $\beta = \prod^{-1}_{\beta 1}(U_{\beta 1}) \cap \prod^{-1}_{\beta 2}(U_{\beta 2}) \cap \ldots\ldots \cap \prod^{-1}_{\beta n}(U_{\beta n})$

Let $x = (x_\alpha)_{\alpha \in J} \in \beta$

$\Leftrightarrow (x_\alpha)_{\alpha \in J} \in \prod^{-1}_{\beta 1}(U_{\beta 1}) \cap \prod^{-1}_{\beta 2}(U_{\beta 2}) \cap \ldots\ldots \cap \prod^{-1}_{\beta n}(U_{\beta n})$

$\Leftrightarrow (x_\alpha)_{\alpha \in J} \in \prod^{-1}_{\beta i}(U_{\beta i})$, i = 1, 2, …… n

$\Leftrightarrow \prod_{\beta i} ((x_\alpha))_{\alpha \in J} \in U_{\beta i}$

$\Leftrightarrow X_{\beta i} \in U_{\beta i}$

There is no intersection on α^{th} co-ordinates of x if α is not one of the indices

$\beta_1 , \beta_2 , \ldots\ldots \beta_n$

$$x \in B \Leftrightarrow \prod U_\alpha$$

Where U_α is open in X_α for all α and $V_\alpha = X_\alpha$ if $\alpha \neq \beta_1 , \beta_2 , \ldots\ldots \beta_n$.

Thus $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α .

Hence the theorem

Example :

i) For finite product $\prod_{\alpha=1}^{n} X\alpha$ the two topologies are precisely the same .

ii) The box topology is in general finer than the product topology for any basis element of the form $\prod U_\alpha$ where U_α is open in X_α is contained in $\prod U_\alpha$,where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α .

Theorem :

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form $\prod_{\alpha=J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α will serve as a basis for box topology on $\prod_{\alpha=J} X_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely indices α and $B_\alpha = X_\alpha$ for all the remaining indices , will serve as a basis for the product topology $\prod_{\alpha=J} X_\alpha$

Proof:

Box Topology

Let $((X_\alpha))_{\alpha=J} \in W$ and W be an open set in $\prod_{\alpha=J} X_\alpha$. For box topology on $\prod X_\alpha$ there exist a basis element $\prod_{\alpha=J} U_\alpha$ where each U_α open un X_α such that

$$(X_\alpha) \in \prod U_\alpha \subset W$$

Since $\mathcal{B}_\alpha$ generates X_α , for each $X_\alpha \in U_\alpha$ there exist $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subset U_\alpha$.

Hence $$(x_\alpha)_{\alpha=J} \in \prod B_\alpha \subset \prod U_\alpha \subset W$$

Hence by theorem " Let (X, τ) be a topological space . Suppose that **C** is a collection of open sets of X such that for each open set U of x such that x $\in$ **C** $\subset$ U . Then **C** is a basis for the topological basis for the box topology on $\prod_{\alpha=J} X_\alpha$ " .

Product Topology :

Take $U = \prod U_\alpha$

Let $(X_\alpha)_{\alpha=J} \in W$ and W be an open set in $\prod_{\alpha=J} X_\alpha$. For the product topology on $\prod X_\alpha$ there exist a basis element $\prod_{\alpha=J} U_\alpha$ where each U_α is open in X_α and $U_{\alpha} = X_\alpha$ except for finitely many $_\alpha$,s.

$(x_\alpha) \in U \subset W$

Since $\mathcal{B}_\alpha$ generates X_α for each $x_\alpha \in U_\alpha$ there exist $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha$ U_α (except for finitely many α's note that $U_\alpha = X_\alpha$)

Hence $(x_\alpha) \in \prod B_\alpha \subset \prod U_\alpha = U \subset W$

Hence by above stated theorem $\prod_{\alpha = J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α is a basis for the product topology on $\prod X_\alpha$.

Hence the proof.

Example :

Consider Euclidean n – space $\mathbb{R}^n$. A basis for $\mathbb{R}$ consists of all open intervals in $\mathbb{R}$, hence a basis for the topology of $\mathbb{R}^n$ consists of all products of the form

$$(a_1 , b_1) \times (a_2 , b_2) \times \ldots\ldots\ldots \times (a_n , b_n)$$

Since $\mathbb{R}^n$ is a finite product, the box and product topologies agree whenever we consider $\mathbb{R}^n$, we will assume that it is given this topology unless we specifically state otherwise.

Theorem

Let A_α be a subspace of X_α for each $\alpha \in J$, then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are give the box topology or if both products are given the product topology.

Proof:

Box Topology:

Consider $\prod X_\alpha$ and $\prod A_\alpha$ with box topologies . Let $\prod U_\alpha$, U_α is open in X_α for all α be a general basis element of $\prod X_\alpha$. That implies $U_\alpha \subseteq X_\alpha$ for all with U_α is open in X_α,

$\rightarrow U_\alpha \cap A_\alpha \subseteq A_\alpha$ is open in A_α Since each A_α is a subspace

$\rightarrow \prod_{\alpha = J} (U_\alpha \cap A_\alpha) \subseteq \prod A_\alpha$ is a basis element for $\prod A_\alpha$

But $\prod (U_\alpha \cap A_\alpha) = (\prod U_\alpha) \cap (\prod A_\alpha)$

Therefore $(\prod U_\alpha) \cap (\prod A_\alpha)$ is a basis of $\prod A_\alpha$ with $\prod U_\alpha$ is basis element for $\prod X_\alpha$. So $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ in box topology.

Product Topology :

Suppose both products are given product topologies.

Claim :

$\prod A_\alpha$ is a subspace of $\prod X_\alpha$. Let $\prod U_\alpha$ be a general basis element of $\prod X_\alpha$ where U_α is open in X_α .

For finitely many α's say $\beta_1 , \beta_2 , \ldots\ldots \beta_n$ and $U_\alpha = X_\alpha$ for the remaining α's. Since each A_α is a subspace of X_α , $U_{\beta_i} \cap A_{\beta_i}$ is open in A_{β_i} , i = 1, 2, …. n and $X_\alpha \cap A_\alpha$ is open in A_α for the remaining α's.

Let $V_\alpha = X_\alpha \cap A_\alpha$ if $\alpha = \beta_i$, i = 1, 2, ……. n

A_α if $\alpha \neq \beta_i$

Then $\prod V_\alpha$ is a general basis element of $\prod A_\alpha$. Therefore $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ in product topology .

Hence the proof.

Theorem :

If each space X_α is Hausdorff space then $\prod X_\alpha$ is Hausdorff space in both the box and product topologies.

Proof :

Claim : $\prod_{\alpha = J} X_\alpha$ is Hausdorff

Let $(x_\alpha)_{\alpha = J} \neq (y_\alpha)_{\alpha = J}$ in $\prod_{\alpha = J} X_\alpha$. Then there exist atleast one $\beta \in J$ such that $x_\beta \neq y_\beta$. Now X_β is a Hausdorff and $x_\beta \neq y_\beta$ in X_β .

There exist two open sets U_β and V_β in X_β such that $x_\beta \in U_\beta$, $y_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \Phi$

Now consider the projection

$$\prod_\beta : \prod X_\alpha \rightarrow X_\beta$$

$\prod^{-1}_\beta (U_\beta)$ is open in $\prod X_\alpha$ and $(x_\alpha) \in \prod^{-1}_\beta (U_\beta)$

Similarly,

$\prod^{-1}_\beta (V_\beta)$ is open in $\prod X_\alpha$ and $(y_\alpha) \in \prod^{-1}_\beta (V_\beta)$ and

$$\prod^{-1}_\beta (U_\beta) \cap \prod^{-1}_\beta (V_\beta) = \prod^{-1}_\beta (U_\beta \cap V_\beta) = \Phi$$

In either topology this result holds good. If $\prod X_\alpha$ is given box topology then

$\prod^{-1}_{\beta} (U_\beta) = \prod U_\alpha$ where U_α is open in X_α foll all α .

If $\prod X_\alpha$ is given product topology then $\prod^{-1}_{\beta} (U_\beta) = \prod U_\alpha$ where U_α is open in X_α for finitely many α's and $U_\alpha = X_\alpha$ for the remaining α's.

Hence the proof.

Theorem :

Let $\{ X_\alpha \}$ be an indexed family of spaces. Let $A_\alpha \subset X_\alpha$ for each α . Then $\prod X_\alpha$ is given either the product topology or the box topology , then $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$

Proof :

$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$

$x = (x_\alpha) \in \prod \bar{A}_\alpha$

Claim :

$x \in \overline{\prod A_\alpha}$

Let $U = \prod U_\alpha$ be a basis element for either topology that contain x.

$x_\alpha \in U_\alpha$ for all α and so U_α intersects A_α as $x_\alpha \in \bar{A}_\alpha$.

Let $y_\alpha \in U_\alpha \cap A_\alpha$ for all α . Then $y = (y_\alpha) \in \prod U_\alpha$ and $\prod A_\alpha$.

Since U is arbitrary, every basis element about x intersects $\prod A_\alpha$ .Therefore

$x \in \overline{\prod A_\alpha} \rightarrow \prod \bar{A}_\alpha$ □ $\overline{\prod A_\alpha}$

Conversely ,

Let $x = (x_\alpha) \in \prod \bar{A}_\alpha$ in either topology.

To Prove :

$x_\alpha \in \overline{A_\alpha}$, for all α .

So that $(x_\alpha) \in \prod \bar{A}_\alpha$ choose a particular index β .

To Prove :

$x_\beta \in \overline{A_\beta}$

Let V_β be an open set of X_β containing x_β .

$\prod^{-1}_{\beta} (V_\beta)$ is open in $\prod X_\alpha$ in either topology and $X_\alpha \in \prod^{-1}_{\beta} (V_\beta)$.

Therefore $\prod^{-1}_{\beta} (V_\beta) \cap \prod A_\alpha \neq \Phi$

Let $y = (y_\alpha) \in \prod^{-1}_{\beta} (U_\beta) \cap \prod A_\alpha$ for the index β , $y_\beta \in A_\alpha$

$(y_\alpha) \in \prod^{-1}_\beta (V) \rightarrow (y_\alpha) \in V_\beta$

ie , $y_\alpha \in V_\beta$

ie , $y_\alpha \in V_\beta \cap A_\beta$

Therefore $(X_\alpha) \in \prod \bar{A}_\alpha$

$\rightarrow \overline{\prod A_\alpha} = \prod \bar{A}_\alpha$

Hence $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$

Hence Proved

Theorem :

Let $f : A \rightarrow \prod X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \rightarrow X_\alpha$ for each α . Let $\prod X_\alpha$ have the product topology . Then the function f is continuous if and only if each function f_α is continuous .

Proof :

Let $f : A \rightarrow \prod X_\alpha$ is given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \rightarrow X_\alpha$ for each α .

Claim :

f_α is continuous .

Let $\prod_\beta$ be the projection of the product onto it's the β^{th} factor. The function $\prod_\beta$ is continuous.Now, suppose that the function $f: A \rightarrow \prod X_\alpha$ is continuous. The function f_β equals the composite $\prod_\beta$ o f being the composite of two continuous functions is continuous.

Conversely, suppose that each f_α is continuous. To prove that f is continuous , it is enough to prove that inverse image of every sub-basis element is open in A. A typical sub basis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi^{-1}_\beta(U_\beta)$, where β is some index and U_β is open in X_β.

Now , $f^{-1}(\pi^{-1}_\beta(U_\beta)) = (\pi_\beta O\ f)^{-1}(U_\beta)$

$= f^{-1}_\beta(U_\beta)$

Because $f_\beta = \pi_\beta O\ f$. Since f_β is continuous, this set is open in A.

$\rightarrow$ f is continuous

Hence the proof.

Note :

The above theorem fails if $\prod X_\alpha$ is given box topology.

Example :

Consider $\mathbb{R}^w$ be the countably infinite product of $\mathbb{R}$ with itself recall that,

$$\mathbb{R}^w = \prod_{n \in \mathbb{Z}_+} X_n$$

Where $X_n \in \mathbb{R}_n$ for each n. Let us define a function $f : \mathbb{R} \rightarrow \mathbb{R}^w$ by the equation

$$f(t) = (t, t, t, \ldots.)$$

The n^{th} co – ordinate function f is the function $f_n(t) = t$ .Each of the coordinate functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ is continuous .

Therefore the function f is continuous if $\mathbb{R}^w$ is given by the product topology . But f is not continuous if $\mathbb{R}^w$ is given by the box topology. Consider the example , the basis element

$$B = (-1, 1) \times (-½ , ½) \times (-⅓ , ⅓) \ldots\ldots\ldots$$

for the box topology , we assert that $f^{-1}(B)$ is not open in $\mathbb{R}$.

If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point U . This means that $f(-\delta, \delta) \square B$, so that applying π_n to both sides of the induction ,

$$f_n(-\delta, \delta) = (-\delta, \delta) \square (-1/n, 1/n)$$

for all n, a contradiction.

Metric Topology :Definition :

A metric on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. (Triangle inequality) $d(x, y) + d(y,z) \geq d(x, z)$ for all $x, y, z \in X$

Example: 1

Given a set , define $d(x, y) = 1$ if $x \neq y$

$$d(x, y) = 0 \text{ if } x = y$$

Its trivial to check that d is a metric .

The topology induced is the discrete topology . The basis element B (x , 1), for example, consist of the point x alone .

Example : 2

The standard metric on the real number ℝ is defined by

$$d(x, y) = |x - y|$$

$$d(x, y) \geq 0 \text{ iff } x = y$$

$$|x - y| = |y - x| \text{ and}$$

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x,y) + d(y,z)$$

Definition :

Given a metric d on X the number d (x, y) is often called the distance between x and y in the metric d.

Given $\epsilon > 0$, consider the set , $B_d(x, \epsilon) = \{ y / d(x, y) < \epsilon \}$ of all points y whose distance from x is less than ϵ . It is called the ϵ - ball centered at x .

Note :

In ℝ the topology induced by the metric $d(x, y) = |x - y|$ is the same as the order topology. Each basis element (a, b) for the order topology is a basis element for the metric topology. Indeed $(a, b) = B(x, \epsilon)$ where x = a+b/2 and ϵ = b-a/2 and conversely , each ϵ- ball $B(x, \epsilon)$ equals an open interval $(x-\epsilon, x+\epsilon)$.

Definition :

Metric Topology :

If d is a metric on the set X, then the collection of all ϵ - ball $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X , called the metric topology induced by d .

Result 1 :

If y is a point of the basis element $B(x, \epsilon)$, then there is a basis element $B(y, \delta)$ centered at y that is contained in $B(x, \epsilon)$.

Proof :

Define δ to be the positive number ϵ - d(x, y). Then B (y, δ) < B (x, ϵ) for if z ϵ B (y, δ) then d (x, z) < ϵ - d(x, y) from which we conclude that

$$d (x, z) \leq d (x, y) + d (y, z) < \epsilon$$

Hence the result .

Result 2 :

B = { B_d (x, ϵ) / x ϵ X and ϵ > 0 } is a basis.

Proof :

First condition for a basis :

x ϵ B (x, ϵ) for any ϵ > 0.

Second condition for a basis :

Let B_1 and B_2 be two basis elements . Let y ϵ $B_1 \cap B_2$. We can choose the number $\delta_1 > \delta_2$ so that B(y, δ_1) $\leq B_1$ and B(y, δ_2) $\leq B_2$.

Letting δ be the smaller of δ_1 and δ_2 . We can conclude that

B(y, δ) $\leq B_1 \cap B_2$.

Result 3 :

A set U is open in the metric topology induced by d iff for each y ϵ U , there is a δ > 0 such that B_d (y, δ) $\subseteq$ U .

Proof :

Clearly this condition implies that U is open .

Conversely , If U is open it contains a basis element B = B_d (x, ϵ) containing y, and B in turn contains a basis element B_d (y, δ) centered at y.

Hence the result.

Definition :

If X is a topological spaces . X is said to be metrizable if there exist a metric d on the set X that induces the topology of X . A metric space is a **metrizable space** X together with a specific metric d that gives the topology of X .

Definition :

Let X be a **metric space** with metric d . A subset A of X is said to be bounded if there is some number M such that $d(a_1, a_2) \leq M$ for every pair a_1 , a_2 of points of A .

Definition :

If A is bounded and non – empty the diameter of A is defined to be the number .

$$\text{diam } A = \sup \{d(a_1, a_2) \,/\, a_1, a_2 \in A\}$$

Theorem :

Let X be a metric space with metric d . Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation

$$\bar{d}(x, y) = \min \{ d(x, y), 1 \}$$

Then $\bar{d}$ is a metric that induces the same topology as d . The metric $\bar{d}$ is called the standard bounded metric corresponding to d .

Proof :

d is a metric .

- $\bar{d}(x, y) = \min \{ d(x, y), 1 \} \geq 0$
  $\bar{d}(x, y) = 0 \leftrightarrow \min \{ d(x, y), 1 \} = 0$

  $\leftrightarrow d(x, y) = 0$

  $\leftrightarrow x = y$ (d is a metric)

- $\bar{d}(x, y) = \min \{ d(x, y), 1 \}$

  $= \min \{ d(x, y), 1 \}$

  $= \bar{d}(y, x)$

Claim :

$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

Suppose $\bar{d}(x, y) = d(x, y)$, $\bar{d}(y, z) = d(y, z)$

And $d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z)$

Also $\bar{d}(x, z) \leq d(x, z)$ (by defn)

$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

Suppose $d(x, y) \geq 1$ or $d(y, z) \geq 1$ then

R.H.S of our claim is atleast 1 and L.H.S of our claim is atmost 1

The equality holds.

Hence đ is metric space.

Hence the theorem

Definition :

Given $x = (x_1, x_2, \ldots\ldots x_n)$ in $\mathbb{R}^n$, we define the norm of x by the equation

$$\|x\| = (x_1^2 + x_2^2 + \ldots\ldots + x_n^2)^{1/2}$$

The euclidean metric d on $\mathbb{R}^n$ is given by the equation

$$d(x, y) = \|x - y\| = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2]^{1/2}$$

$$= [\textstyle\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2}$$

Where $x = (x_1, x_2, \ldots\ldots x_n)$, $y = (y_1, y_2, \ldots\ldots y_n)$.

Definition :

The square metric ρ on $\mathbb{R}^n$ is given by the equation

$$\rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots\ldots, |x_n - y_n|\}$$

Relation between euclidean metric d and sequence metric ρ in $\mathbb{R}^n$ is

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}(\rho(x, y))$$

Theorem :

Let d and d′ be two metrices on the set X . Let τ and τ′ be the topologies they induce respectively. Then τ′ is finer than τ iff for each x in X and each ε > 0 there exist a δ > 0 such that

$$B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$$

Proof :

Suppose $\tau \subset \tau'$

Then by lemma

" Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies τ and τ′ respectively on X . Then the following are equivalent

(i) $\tau \subset \tau'$

(ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B'$ □ $\mathcal{B}''$."

Given the basis element

$B_d(x, \varepsilon)$ for τ, by the lemma, there exist a basis element B' for the topology τ' such that

$$x \in B' \subset B_d(x, \varepsilon)$$

within B' we can find a ball $B'(x, \delta)$ such that

$$x \in B_{d'}(x, \delta) \subset B' \subset B_d(x, \varepsilon)$$

Conversely ,

Suppose that ϵ -δ condition holds. given a basis element B for τ containing x, we can find within B a ball $B_d(x, \varepsilon)$ contained at x.

ie , $x \in B_d(x, \varepsilon) \subset B$

By hypothesis , $x \in B_d(x, \delta) \subset B_d(x, \varepsilon)$

By lemma , τ' is finer than τ.

Hence the theorem

Theorem :

The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric δ are the same as the product topology on $\mathbb{R}^n$.

Proof :

Let $X = (x_1, x_2, \ldots.. x_n)$ and $y = (y_1, y_2, \ldots.. y_n)$ be two points of $\mathbb{R}^n$.

Let τ_d be the topology induced by the metric d and τ_ρ be the topology , induced by the metric ρ.

Claim : $\tau_\rho = \tau_d$

To prove :

$$\tau_\rho \subseteq \tau_d \quad \text{and } \tau_d \subseteq \tau_\rho$$

ie , To prove : i) $B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$

ii) $B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x, \varepsilon)$

i) Let $y \in B_d(x, \varepsilon)$
$\rightarrow d(x, y) < \epsilon$

$\rightarrow \rho(x, y) < \epsilon$ $\quad (\rho(x, y) \leq d(x, y) < \epsilon)$

$\rightarrow y \in B_\rho (x, \varepsilon)$

$\rightarrow B_d (x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$

Therefore $\tau_\rho \subseteq \tau_d$ (by theorem)

ii) Let $y \in B_\rho (x, \varepsilon /\sqrt{n})$

$\rightarrow \rho(x, y) < \epsilon /\sqrt{n}$

$\rightarrow \sqrt{n}\, \rho(x, y) < \epsilon$

$\rightarrow d(x, y) < \epsilon$

$\rightarrow y \in B_d (x, \varepsilon)$

$\rightarrow B_\rho (x, \varepsilon /\sqrt{n}) \subseteq B_d (x, \varepsilon)$

$\rightarrow \tau_d \subseteq \tau_\rho$

We get , $\tau_d = \tau_\rho$

Claim :

$\tau = \tau_\rho$

To prove that the product topology is same as the topology induced by the square metric ρ .

First let prove $\tau \subseteq \tau_\rho$, where τ is the product topology on $\mathbb{R}^n$. Let $B = (a_1, b_1) \times (a_2, b_2) \times \ldots\ldots \times (a_n, b_n)$ be a basis element of τ with $x \in X$ where $X = (x_1, x_2, \ldots.. x_n)$.

Now for each i there is an ε_i such that

$(x_i - \varepsilon_i , x_i + \varepsilon_i) \subseteq (a_i , b_i)$

Thus $B_\rho (x, \varepsilon) \subseteq B$ for $y \in B_\rho (x, \varepsilon)$

$\rightarrow \rho(x, \epsilon) < \epsilon$

$\rightarrow \max \{ |x_1 - y_1| , |x_1 - y_1|, \ldots\ldots |x_n - y_n| \} < \epsilon$

$\rightarrow |x_i - y_i| < \epsilon < \epsilon_i$, for all $i = 1, 2, \ldots.. n$

$\rightarrow y_i \in \epsilon (x_i - \varepsilon_i , x_i + \varepsilon_i)$ for all i

$\rightarrow y_i \in \epsilon (a_i , b_i)$ for all i

$\rightarrow y \in B$

$\rightarrow \tau \subseteq \tau_\rho$ ………(1)

Conversely ,

To Prove :

$\tau_\rho \subseteq \tau$

Let B_ρ (x, ε) be a basis element for ρ topology given the element y ϵ B_ρ (x, ε)

We need to find a basis element B for product topology such that

$$y \in B \subseteq B_\rho (x, \varepsilon)$$

Now $B_\rho (x, \varepsilon) = ((x_1 - \varepsilon, x_1 + \varepsilon) (x_2 - \varepsilon , x_2 + \varepsilon) \ldots\ldots. (x_n - \varepsilon , x_n + \varepsilon))$

Which is itself a basis element of the product topology .

$$y \in B = B_\rho (x, \varepsilon)$$

Hence $\tau_\rho \subseteq \tau$ ……..(2)

From (1) and (2)

$$\tau_\rho = \tau$$

Definition :

Given a indexed set J and given points $x = (x_\alpha)_{\alpha \varepsilon J}$ and $y = (y_\alpha)_{\alpha \varepsilon J}$ of R^J , a metric $\bar{\rho}$ on R^J defined by the equation

$$\bar{\rho} (x, y) = \sup \{ \bar{d} (x_\alpha , y_\alpha) \}_{\alpha \varepsilon J}$$

where $\bar{d} (x, y) = \min \{ |x - y| , 1 \}$ the standard bounded metric on ℝ.

$\bar{\rho}$ is a metric on R^J called the uniform metric on R^J. The topology it induces is called the **uniform topology**.

Theorem :

The uniform topology on R^J is finer than the product topology and coarser than the box topology.

Proof :

(i) Let τ_ρ be the product topology on R^J . τ_B be box topology on R^J and τ_ρ be the uniform topology on R^J. The theorem states that

$$\tau_\rho \subseteq \tau_{\bar{\rho}} \subseteq \tau_B$$

First let us prove :

$$\tau_\rho \subseteq \tau_{\bar{\rho}}$$

Let $x = (x_\alpha)_{\alpha \varepsilon J}$ and $U = \Pi U_\alpha$ be a basis element of τ_ρ with $(x_\alpha)_{\alpha \varepsilon J} \in U$

Let $\alpha_1 , \alpha_2 , \alpha_3 , \ldots.. \alpha_n$ be the induces for which $U_\alpha \neq R$ then for each i , we can choosen an $\varepsilon_i > 0$ such that

$B_{\bar{d}}(x_i , \varepsilon_i) \subseteq U_i$ for all $i = \alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$

Where U_i is open in R .

Let $\varepsilon = \min \{ \varepsilon_1 , \varepsilon_2 , \ldots\ldots \varepsilon_n \}$

Then $B_{\bar{\rho}}(x ,\varepsilon) \subseteq U$ for $Z \in B_{\bar{\rho}}(x ,\varepsilon)$

$\rightarrow \bar{\rho}(x , z) < \varepsilon$

$\rightarrow \sup \{ đ (x_\alpha , z_\alpha) \}_{\alpha \varepsilon J} < \varepsilon$

$\rightarrow \bar{d}(x_\alpha , z_\alpha) < \varepsilon < \varepsilon_i$ for all α

$\rightarrow z_\alpha \in B_{\bar{d}}(x_\alpha , \varepsilon)$

$\rightarrow Z \in U$

$\rightarrow Z \in \Pi U_\alpha$

Thus $\tau_\rho \subseteq \tau_{\bar{\rho}}$

(ii) To Prove :

$\tau_{\bar{\rho}} \subseteq \tau_\rho$

Let $B_p (x, \varepsilon)$ be a basis element of $\tau_{\bar{\rho}}$. Then the neighbourhood

$U = \Pi (x_\alpha - \varepsilon / 2 , x_\alpha + \varepsilon / 2)$ is $x \in U \subseteq B_{\bar{p}}(x, \varepsilon)$ for $y \in U$

$\rightarrow U_\alpha \in (x_\alpha - \varepsilon / 2 , x_\alpha + \varepsilon / 2)$ for all α

$\rightarrow \bar{d}(x_\alpha , y_\alpha) < \varepsilon / 2$ for all α

$\rightarrow \sup \{ \bar{d}(x_\alpha , y_\alpha) \}_{\alpha \varepsilon J} < \varepsilon / 2$

$\rightarrow \bar{p}(x ,y) < \varepsilon / 2 < \varepsilon$

$\rightarrow y \in B_{\bar{p}}(x, \varepsilon)$

The topologies τ_ρ, $\tau_{\bar{\rho}}$ and τ_B in R^J are different if J is infinite.

Theorem :

Let $\bar{d}(a,b) = \min \{ |a - b| , 1 \}$ be the standard bounded metric on R . If x and y are two points of R^w, Define

$D (x,y) = \sup \{ \bar{d}(x_i , y_i) / i \}$

Then D is a metric that induces the product topology on R^w

Proof :

$D (x,y) = \sup \{ \bar{d}(x_i , y_i) / i \}$ is a metric.

Since each $\bar{d}(x_i, y_i) \geq 0, D(x,y) \geq 0$

$D(x,y) = 0 \Leftrightarrow \bar{d}\,(x_i, y_i) = 0 \quad \forall\, i$

$$\Leftrightarrow x_i = y_i \,\forall\, i$$

$$\Leftrightarrow x = y$$

$$D(x,y) = sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} = \sup\left\{\frac{\bar{d}(y_i, x_i)}{i}\right\}$$

$$= D(x,y)$$

Since d is metric

$$\bar{d}(x_i, z_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i)$$

$$\leq \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\bar{d}(y_i, z_i)}{i}$$

$$\leq\ sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} + sup\left\{\frac{\bar{d}(y_i, z_i)}{i}\right\}$$

$$\bar{d}(x_i, z_i) \leq D(x,y) + D(y,z)$$

$$sup\left\{\frac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(x,y) + D(y,z)$$

$\therefore D$ is a metric on $\mathbb{R}$.

Claim: $\tau_D = \tau_p$ (we have topology induced by D be τ_D

To prove: $\tau_D \subseteq \tau_p$

Let U be open in metric topology τ_D and $x \in U$

To prove that $\exists$ an open set V in $\tau_p \ni x \in V \subseteq U$.

Since $x \in U, \exists\, a\ \varepsilon > 0, B_D\,(x, \varepsilon) \subset U$.

Choose N large enough ${}^1\!/_N < \varepsilon$.

Let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \ldots\ldots\ldots \times (x_N - \varepsilon, x_N + \varepsilon)$

We assert that $V \subset B_D\,(x, \varepsilon)$ for if $y \in R^w$ then $\frac{\bar{d}(x_i, y_i)}{i} < {}^1\!/_N\, \forall\, i > N$

$$\therefore D(x,y) = max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \frac{\bar{d}(x_2, y_2)}{2}, \ldots\ldots\ldots\ldots\ldots \frac{\bar{d}(x_N, y_N)}{N}\right\}$$

If $y \in V$ then $D(x,y) < \varepsilon \Rightarrow y \in B_D\,(x, \varepsilon)$

$\therefore x \in V \subset B_D\ (x, \varepsilon) \subset V$

$\tau_D \subseteq \tau_p$

Consider the basis element $U = \prod_{j \in Z_+} U_i$ for the product topology, where U_i is open in R for $i = (\alpha_1, \alpha_2, \dots \dots . . \alpha_{n)}$ and $U_i = R$ for all other indices i.

Let $x \in U$. Let us find an set V of the metric topology such that

x $\in V \subset U$. Choose an interval $(x_i - \varepsilon_i, x_i + \varepsilon_i)$ in R centered about xi and lying in $U_{i,}$ for $i = \alpha_1, . . . , \alpha_n$; choose each $\varepsilon_i \leq 1$.

Then define

$\varepsilon = \min\{\varepsilon_i /i \mid i = \alpha_1, . . . , \alpha_n\}$.

$x \in B_D\ (x, \varepsilon) \subset U$.

For y $\in B_D\ (x, \varepsilon)$,

$\Rightarrow D(x, y) < \varepsilon \Rightarrow \frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon$ for all i ,

$\Rightarrow \bar{d}(x_i, y_i) \leq D(x, y) < \varepsilon(i) < \varepsilon_i$ for $i = \alpha_1, . . . , \alpha_n$

$\Rightarrow \min\{|x_i - y_i|, 1\} < \varepsilon_1 < 1$

$\Rightarrow y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i \ \forall i = \alpha_1, . . . , \alpha_n$

obviously, $y_i \in R$ for other indices i,

$\therefore y \in U$

Hence, $x \in B_D\ (x, \varepsilon) \subset U$ (basis element of τ_P)

$\therefore \tau_P \subseteq \tau_D$

Hence $\tau_P = \tau_D$

Hence the theorem.

Theorem:

Let $f : X \to Y$. Let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$d_X(x, y) < \delta \Rightarrow d_Y\ (f(x), f(y)) < \varepsilon$.

Proof.

Suppose that f is continuous.

Given x and ε, consider the set

$f^{-1}(B(f(x),\varepsilon))$,

which is open in X and contains the point x.

$\therefore \exists\ \delta > 0 \ni B(x,\delta) \subset f^{-1}(B(f(x),\varepsilon))$

Then $y \in B(x,\delta) \Rightarrow y \in f^{-1}B(f(x),\varepsilon))$

i.e $y \in B(x,\delta) \Rightarrow f(y) \in B(f(x),\varepsilon)$

i.e $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Thus $\varepsilon - \delta$ conditions holds

Conversely,

Suppose $\varepsilon - \delta$ conditions holds

Claim:

$f : X \to Y$ is continuous

Let V be open in Y; we

show that $f-1(V)$ is open in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$,

As V is open, there exists $\varepsilon > 0$ such that $B(f(x),\varepsilon) \subseteq V$

By the ε -δ condition, there is a δ-ball $B(x, \delta)$ centered at x such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.

$\Rightarrow$ Thus $x \in B(x, \delta)$ contained in $f^{-1}(V)$, so that $f^{-1}(V)$ is open in X.

Hence continuity and ε -δ conditions are equivalent.

Hence the theorem.

The Sequence lemma :

Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Proof:

Suppose that $x_n \to x$, where $x_n \in A$ then for every neighborhood of x contains a point of $A \Rightarrow x \in \bar{A}$

by Theorem "Let A be a subset of the topological space X.

Then $x \in \bar{A}$ iff every open set V containing x intersects A.

Supposing the topology of X is given by a basis then $x \in \bar{A}$ iff every basis element B containing x intersects A"

Conversely, suppose that X is metrizable and $x \in \bar{A}$. Let d be a metric for the topology of X. For each positive integer n, take the neighborhood $B_d(x, 1/n)$ of radius $1/n$ of x, and choose $x_n \ni x_n \in Bd\left(x, \frac{1}{n}\right)$

Then (x_n) is a sequence of A.

Claim: x_n converges to x

Any open set U containing x is such that $x \in B(x, \varepsilon) \subset U$

if we choose N so that $1/N < \varepsilon$, then $\frac{1}{N+3} < \frac{1}{N+2} < \frac{1}{N+1} < \frac{1}{N} < \varepsilon$ and so that U contains x_i for all $i \geq N$.

Hence the proof.

Theorem:

Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Proof:

Assume that f is continuous. Let $x_n \to x$.

Claim: $f(x_n) \to f(x)$.

Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, and so there is an N such that $x_n \in f^{-1}(V)$ for $n \geq N$.

Then $f(x_n) \in V$ for $n \geq N$.

$f(x_n) \to f(x)$

conversely,

Suppose that the convergent sequence condition is satisfied.

Let X is metrizable and $A \subset X$;

To prove : f is continuous

We prove, $f(\bar{A}) \subseteq \overline{f(A)}$. Let $x \in \bar{A}$, then there is a sequence x_n in x such that $x_n \to x$ (by the sequence lemma).

By hypothesis, $f(x_n)$ converges to $f(x)$. As $f(x_n))$ is in (A) ; $f(x_n) \in \overline{f(A)}$

Hence $f(\bar{A}) \subseteq \overline{f(A)}$

i.e, f is continuous

Hence the theorem.

Theorem:

The addition, subtraction and multiplication operations are continuous function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.

Proof:

Let the addition operation be defined by

$$f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \, , \, f_1(x, y) = x + y$$

The multiplication operation be defined by

$$f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \, , \, f_2(x, y) = xy$$

Let us use the metric:

$$d(a, b) = |a - b| \text{ on } \mathbb{R} \text{ and let the metric on } \mathbb{R}^2 \text{ given by,}$$

$$\varrho\big((x, y), (x_0, y_0)\big) = \max\{|x - x_0|, |y - y_0|\}$$

Let $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and $\varepsilon > 0$ be given if $\varrho\big((x, y), (x_0, y_0)\big) < \delta$. Then,

$$\max\{|x - x_0|, |y - y_0| < \delta$$

i.e., $|x - x_0| < \delta$ and $|y - y_0| < \delta$.

To prove: f_1 is continuous

Choose $\delta < \frac{\varepsilon}{2}$

$$\begin{aligned} d\big(f_1(x, y), f_1(x_0, y_0)\big) &= |f_1(x, y) - f_1(x_0, y_0)| \\ &= |(x + y) - (x_0 + y_0)| \\ &= |x - x_0 + y - y_0| \\ &= |x - x_0| + |y - y_0| \end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Thus $\varrho\big((x,y),(x_0,y_0)\big) < \delta$.

$\Rightarrow d\big(f_1(x,y), f_1(x_0,y_0)\big) < \varepsilon \quad \forall x,y \in \mathbb{R}\times\mathbb{R}$

Thus f_1 is continuous.

To prove: f_2 is continuous

Let $(x_0,y_0) \in \mathbb{R}\times\mathbb{R}$ and $0 < \varepsilon < 1$ be given.

Choose $\delta = \frac{\varepsilon}{1+|x_0|+|y_0|}$

$$\begin{aligned}
d\big(f_1(x,y), f_1(x_0,y_0)\big) &= |f_2(x,y) - f_2(x_0,y_0)| \\
&= |(xy) - (x_0y_0)| \\
&= |xy - x_0y + x_0y - xy_0 + xy_0 + x_0y_0 - x_0y_0 - x_0y_0| \\
&= |x(y-y_0) - x_0(y-y_0) + y_0(x-x_0) - x_0(y-y_0)| \\
&\le |(x-x_0)(y-y_0)| + |x_0||(y-y_0)| + |y_0||(x-x_0)| \\
&\le |(x-x_0)||(y-y_0)| + |x_0||(y-y_0)| + |y_0||(x-x_0)| \\
&< \delta^2 + |x_0|\delta + |y_0|\delta \quad \text{if } \varrho\big((x,y),(x_0,y_0)\big) < \delta \\
&< \delta + |x_0|\delta + |y_0|\delta \\
&< \delta\,(1 + |x_0| + |y_0|) \\
&= \frac{\varepsilon}{1+|x_0|+|y_0|}\,(1 + |x_0| + |y_0|) \\
&< \varepsilon
\end{aligned}$$

Thus $\varrho\big((x,y),(x_0,y_0)\big) < \delta$

$\Rightarrow d\big(f_2(x,y), f_2(x_0,y_0)\big) < \varepsilon$

Hence f_2 is continuous.

Similarly,

$\because \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $(x,y) \to (x,-y)$ is continuous.

$\mathbb{R}\times(\mathbb{R}-\{0\}) \to \mathbb{R}$ is given by $(x,y) \to \frac{x}{y}$ is continuous.

Theorem:

If X is a topological space and if $f \circ g\colon \mathrm{X} \to \mathbb{R}$ are continuous functions, then

$f + g$, $f - g$ and $f \cdot g$ are continuous. If $g(x) \neq 0\ \forall x$ then $\frac{f}{g}$ is continuous.

Proof:

The map $h\colon \mathrm{X} \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ is continuous by the theorem, "maps into products", 'Let $f\colon \mathrm{A} \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a))$$

Then f is continuous iff if the function $f_1\colon A \to X$ and $f_2\colon A \to Y$ are continuous. The maps f_1 and f_2 are called the co-ordinate functions of f'.

The function $f + g = f_1 \circ h$ where $h\colon X \to \mathbb{R} \times \mathbb{R}$, $f_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$x \to (f(x), g(x))$, $(f(x), g(x) \to f(x) + g(x)$

$(f + g)x = f(x) + g(x)$, $\forall x \in X$

is continuous.

Since f_1 is continuous (by previous lemma), h is continuous and composition of continuous function is continuous.

The function $fg = f \circ h$ defined by

$$fg(x) = (f_2 \circ h)(x) = f_2\big(h(x)\big)$$
$$= f_2(f(x), g(x))$$
$$= f(x)g(x)$$

is continuous.

Since f_1 is continuous, composition of continuous function is continuous.

Similarly, functions $f - g$ and $\frac{f}{g}$ $(g \neq 0)$ are continuous.

Definition:

Let $f_n\colon X \to Y$ be a sequence of functions from the set X to the metric space Y.

We say that the sequence $\{f_n\}$ converges uniformly to the function $f\colon X \to \mathbb{R}$ if given $\varepsilon > 0$, there exists an integer $\mathbb{N}$ such that

$$d\big(f_n(x), f(x)\big) < \varepsilon \,, \qquad \forall n > \mathbb{N} \text{ and for all } x \text{ in X.}$$

Uniform limit theorem

Let $f_n: X \to Y$ be a sequence of continuous functions from the topological space X into the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.

Proof:

Let V be open in Y.

Let x_0 be a point of $f^{-1}(V)$. We wish to find a neighborhood U of x_0 such that $f(U) \subset V$.

Let $y_0 = f(x_0)$ first choose ε so that the ε-ball $\mathrm{B}(y_0, E)$ is contained in V.

Then using the uniform convergence

Choose $\mathbb{N}$ so that for all $n \geq \mathbb{N}$ and all $x \in \mathrm{X}$

$$d\big(f_n(x), f(x)\big) < \tfrac{\varepsilon}{3} \qquad \text{--- (1)}$$

Finally, using continuity of f_N , choose a neighborhood U of x_0 such that

$$f_N(U) \subseteq \mathrm{B}(f_N(x_0), \tfrac{\varepsilon}{3}) \qquad \text{--- (2)}$$

Claim:

$$f(V) \subset B(y_0, \varepsilon) \subset V$$

If $x \in U$ then

$$d\big(f(x), f_N(x)\big) < \tfrac{\varepsilon}{3} \qquad \text{--- (3)} \qquad \text{(by the choice of } N\text{)}$$

by (1)

$$d\big(f_N(x), f_N(x_0)\big) < \tfrac{\varepsilon}{3} \qquad \text{--- (4)} \qquad \text{(by the choice of } V\text{) by (2)}$$

$$d\big(f_N(x_0), f(x_0)\big) < \tfrac{\varepsilon}{3} \qquad \text{--- (5)} \qquad \text{(by the choice of } N\text{)}$$

by (1)

Adding and using the triangle inequality at x_0 we see that

$$d\big(f(x), f(x_0)\big) < \tfrac{\varepsilon}{3}$$

as desired.

$$\Rightarrow f(x) \in B(f(x_0), \varepsilon) = B(y_0, \varepsilon)$$

$$f(U) \subset B(y_0, \varepsilon) \subseteq V$$

Hence f is continuous.

Hence the theorem.