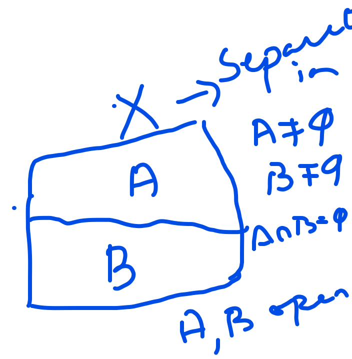


Unit – III

Connected Spaces and Compact spaces

Connected space

Definition: Separation

Let X be a Topological space. Separation of X is a pair U, V of disjoint, non-empty open subsets of X whose union is X.

$$X = U \cup V, \quad \text{U and V are open}$$

$$U \cap V = \emptyset, \quad U \neq \emptyset, \quad V \neq \emptyset$$

The space X is said to be connected if ~~they~~ there does not exist a separation of X.

Result

A Space X is connected iff the only subsets of X that are both open and closed in X are the empty sets and X itself.

$X = \{a, b, c\}$

$\tau = \{\phi, X, \{a\}\}$

cl. sets $\{X, \phi, \{b, c\}\}$

Proof:

Assume X be connected. ~~Let X be connected~~.

Suppose $A \subset X$, is both open and closed

Let $U = A,\ V = X - A$

$X = A \cup (X - A)$

$= U \cup V$

Then U and V form a separation for X, which is a contradiction.

Hence ~~X is connected~~. the only subsets that are both open and closed are ϕ and X.

Conversely,

(Let X is disconnected.) Assume that the only sets of X which are both open and closed are ϕ and X.

Since X is disconnected, there exist a separation, such that

$$X = A \cup B, \quad A \neq \phi, \quad B \neq \phi$$

$$A \cap B = \phi \qquad A \text{ and } B \text{ ~~is~~ are open}$$

Now A is open

$A = X - B$. So A is closed

Therefore the A is both open and closed and A is a proper subset of X which is a contradiction.

Hence X is connected.

Hence the proof.

Note:

Connectedness is a Topological Property. Since it is formulated entirely in terms of the collection open sets of X. If X is connected so is any space homeomorphic to X.

Examples:

1. Let $X = \{a, b\}$

Let $\tau = \{\phi, X\}$. The indiscrete Topology on X. Then there exist no separation of X.

Therefore, X is connected.

2. Consider $X = \{a, b, c\}$. Let $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$

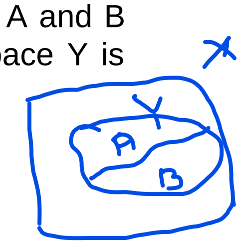
The discrete topology on X. This Topological space is not connected.

Since all subsets of X both open and closed and then there a exist a separation

$X = \{a\} \cup \{b, c\}$.

Lemma 1:

If Y is a subspace of X, a separation of Y is a pair of disjoint non-empty sets A and B whose union is Y neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.



Proof

Suppose A and B form a separation of Y.

$A \cap \bar{B} = \varphi$

$B \cap \bar{A} = \varphi$

A is both open and closed in Y

$cl_Y\ A = \bar{A} \cap Y$

$A = \bar{A} \cap Y \qquad (\because A \text{ is closed in } Y,\ cl_Y\ A = A)$

$\bar{A} \cap B = \phi \quad [A = \bar{A} \cap Y, \qquad A \cap B = \bar{A} \cap (\underline{Y \cap B})$

$A \cap B = \bar{A} \cap B$

$\therefore A \cap B = \phi\ ]$

B contains no limit point of A.

Similarly A contains no limit point of ~~A~~. B

Conversely, Suppose that A and B are disjoint non-empty sets whose union in Y, neither of which contains the limit point of the other.

$\bar{A} \cap B = \phi \quad and \quad A \cap \bar{B} = \phi$

$\bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap B) \cup (A \cap \bar{B})$

$= A \cup \phi = A$

∴ A is closed in Y.

Similarly, $\bar{B} \cap Y = B$

∴ B is closed in Y.

As $Y - B = A$ and $Y - A = B$.

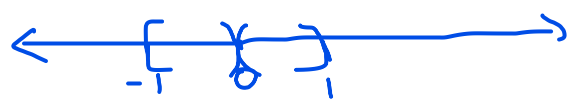
Both A and B are open in Y. A and B Form a separation of Y. ⇒ Y is disconnected

The space Y is connected if there exist no separation of Y.

Example 1:

Let X denote a two pint space in the indiscrete topology. Obviously there is no separation of X, So X is connected.

Example 2:



Let $Y \subseteq R$ the real line such that

$Y = [-1, 0) \cup (0,1]$. Both [-1, 0) and (0, 1] are disjoint non-empty and their union us Y. Both are open in Y [Not in R].

They form a separation of Y.

$\overline{[1, 0)} = [-1, 0]$

$\overline{(0, 1]} = [0,1]$

Note that none of these sets contains the limit points of the other.

Example 3:

Let $X = [-1, 1] \subset R$

$X = [-1, 0] \cup [0,1]$

This does not form separation of X, Since [-1, 0] is not open in X. Note that, here first set contains a limit zero of the second. Indeed there is no separation of the space [-1, 1].

Example 4:

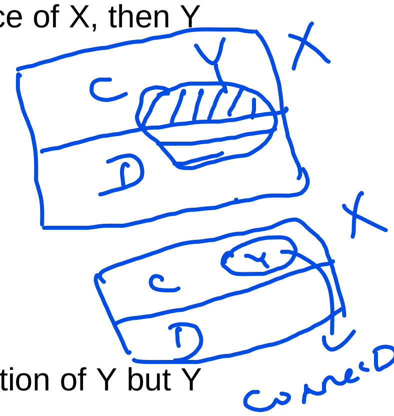
The rationals Q is not connected. The only connected subsets of Q are the 1-pt set. Let Y be a subspace of Q containing two point p and q. In between p and q there always exist a irrational a.

Consider $Y \cap (-\infty, \infty)$ and $Y \cap (a, \infty)$. Both are disjoint non-empty open sets in Y, whose union in Y.

∴ Y has a separation and hence it is not connected.

Lemma 2:

If the sets C and D form a separation of X and if ~~y~~ Y is a connected subspace of X, then Y lies entirely within either C or D.



Proof

Since C and D are both open in X. $C \cap Y$ and D$\cap Y$ are both open in Y.

$(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y = Y$

(C$\cap Y) \cap (D \cap Y) = (C \cap D) \cap Y = \phi$

If both $C \cap Y \neq \phi$ and $D \cap Y \neq \phi$, then $C \cap Y$ and $D \cap Y$ ~~with~~ will form a separation of Y but Y is connected.

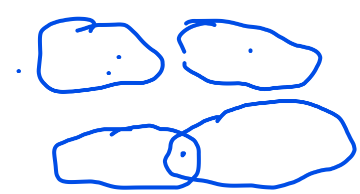
∴ One of these should be empty.

If $C \cap Y = \phi$ then $Y \subset D$

If D$\cap Y = \phi$ then $Y \subset C$

Lemma 3:

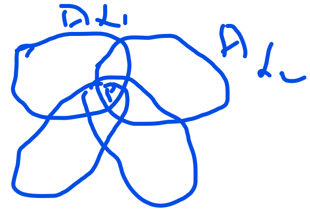
The union of a collection of connected subspaces of X, that have a point in common is connected.



Proof:

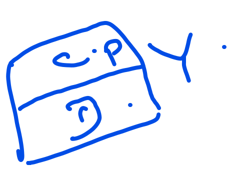
Let $\{A_\alpha\}$ be a collection of connected subspaces of X.

Let $p \in \cap A_\alpha$



Claim: $Y = \cup A_\alpha$ is connected.

Suppose $Y = C \cup D$ is a separation of Y.



Then $p \in C$ or $p \in D$.

Suppose $p \in C$

Since A_α is connected.

$A_\alpha \subset C$ or $A_\alpha \subset D$ (By Previous lemma)

$A_\alpha \nsubseteq D$ because $p \in A_\alpha$ and $p \in C$.

$\therefore$ $A_\alpha \subset C$ for all A_α

$\Longrightarrow \cup A_\alpha \subset C$ which means $(\cup A_\alpha) \cap D = \phi$

$$Y \cap D = \phi$$

$\Longrightarrow D = \phi$

$\Longrightarrow\Longleftarrow$ $D \neq \phi$

$\therefore Y = \cup A_\alpha$ is connected.

Theorem 4:

Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, B is also connected.

Proof:

Let A is connected.

Given $A \subset B \subset \bar{A}$

Claim: B is connected.

$B = C \cup D$ with $C \neq \phi$, $D \neq \phi$

$C \cap D = \phi$, C and D are open.

Since A is a connected subset of B then by lemma,

"If the sets C and D form a separation of ~~D~~ X. and if Y is a connected subspace of X, then Y lies entirely within either C or D."

$A \subset C$ or $A \subset D$

$A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

Suppose $A \subset C$

Then $\bar{A} \subset \bar{C}$

Then $B \subset \bar{C}$ $\quad \because B \subset \bar{A} \subseteq \bar{C}$

$\bar{C} \cap D = \phi$

Since D contains no limit point of C.

$B \cap D \subseteq \bar{C} \cap D = \phi$

$\therefore D = \phi$

$\Rightarrow\Leftarrow \quad D \neq \phi$

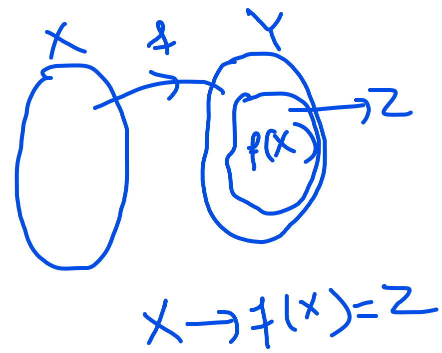
$\therefore$ There exist no separation of B.

In otherwords B is connected.

Theorem 5:

The image of connected space under a continuous map is connected.

Proof



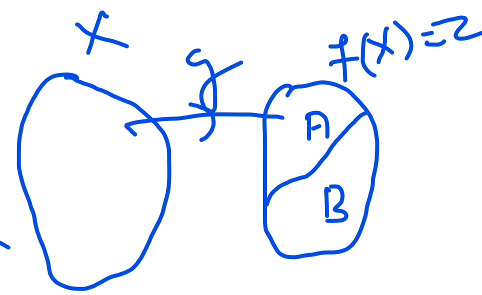
Let $f: X \to Y$ is continuous.

Then $f: X \to Z = (f(X))$ [since restriction to its range is continuous]

Let $g: X \to Z$ is a surjective continuous map.

Claim: Z is connected.

Suppose $Z = A \cup B$ where $A \neq \phi, B \neq \phi$



$A \cap B = \phi$, A and B are open in Z.

$g^{-1}(A) \neq \phi, \;\; g^{-1}(B) \neq \phi$ [$\because$g is surjective]

$g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B) = \phi$. Also $g^{-1}(A)$ and $g^{-1}(B)$ are open in X

$\therefore X = g^{-1}(A) \cup g^{-1}(B)$ for, a separation of X

$\Rightarrow$ X is disconnected, which is a contradiction.

$\Rightarrow$ X has ~~no~~ a separation.

Hence $Z = f(X)$ is connected.

Theorem 6:

A finite Cartesian product of connected spaces is connected.

Proof

Let us first prove for 2-spaces X and Y, first

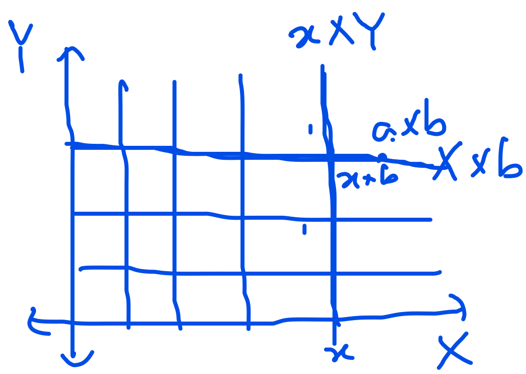
Choose a base point $a \times b$ in $X \times Y$.

The horizontal slice $X \times b$ is connected, being homeomorphic with X, and vertical slice $x \times Y$ is connected being homeomorphic with Y.

Each T- shaped space $T_x = (X \times b) \cup (x \times Y)$ is connected.

Being the union of two connected spaces that have points ~~a~~ x×b is common.

Now, Consider $\bigcup_{x \in X} T_x$ of all these T- shaped spaces.



Since a×b is a common ~~point~~,

$\bigcup_{x \in X} T_x$ is connected.

But $X \times Y = \bigcup_{x \in X} T_x$

∴ $X \times Y$ is connected.

Claim: To prove $X_1 \times X_2 \times \ldots \times X_n$ is connected.

Let us use ind~~ex~~uction on X.

If $n = 2$, the result is true.

Assume that the result for $\cancel{n = k}$ n $- 1$.

$X_1 \times X_2 \times \ldots \times X_{n-1}$ is connected.

If $n \cancel{=} k$, now,

$X_1 \times X_2 \times \ldots \times X_n = (X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$ is connected.

~~Since~~ $(X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$ ~~are~~ is connected ~~and~~ since product of 2-connected spaces are connected.

Hence X_n is connected.

Example 1:

Consider the Cartesian product $\mathbb{R}^\omega$ in the box topology. We can write $\mathbb{R}^\omega$ as the union of the set A. Consisting of all bounded sequence of real numbers and the set B of all unbounded sequence.

These sets are disjoint and each is open in the box topology for if a is a point of $\mathbb{R}^\omega$ the open set.

$U = (a_1 - 1\,, a_1 + 1) \times (a_2 - 1\,, a_2 + 1) \times \ldots$

Consider entirely of bounded sequences if A is bounded and of unbounded sequences if a if unbounded.

Thus, eventhough $\mathbb{R}$ is connected, $\mathbb{R}^{\omega}$ is not connected in the box topology.

Example 2:

Now consider $\mathbb{R}^{\omega}$ in the product topology.

Assuming that $\mathbb{R}$ is connected. We show that $\mathbb{R}^{\omega}$ is connected.

Let $\widetilde{\mathbb{R}^n}$ denote the subspace of $\mathbb{R}^{\omega}$. Consisting of all sequences $x = (x_1, x_2, \dots)$ such that $x_i = 0, \ \ i > n$.

The Space $\widetilde{\mathbb{R}^n}$ is clearly homeomorphic to$\mathbb{R}^n$, so that it is connected.

By the preceeding theorem

It follows that the space $\mathbb{R}^{\infty}$ i.e., the union of the space is $\widetilde{\mathbb{R}^n}$ is connected, for these spaces have the point $0 = (0, 0, \ \dots)$ in common.

We show that the closure of $\mathbb{R}^{\infty}$ equals all of $\mathbb{R}^{\omega}$, from which it follows that $\mathbb{R}^{\omega}$ is connected as well.

Let $a = (a_1, a_2, \dots)$ be a point of $\mathbb{R}^{\omega}$, let $U = \prod U_i$ be a basis element for the product topology that contains a.

We show that U intersects. There is an integer, N there exist $U_i = \mathbb{R}, i > N$ then the point

$x = (a_1, a_2, \dots, a_n, 0, 0, \dots)$ of $\mathbb{R}^{\infty} \in U$. Since $a_i \in U_i$ for all i. $O \in U_i$ for $i > N$.

The argument just given generalizes to show that an arbitrary product of connected spaces is connected in the product topology.

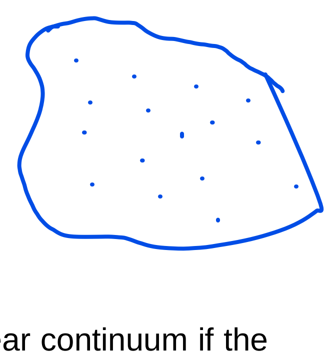
Definition: Totally Disconnected space

A space is totally disconnected if its only connected spaces are one point subsets.

Example:

if X has discrete topology then x is totally disconnected.

The set of Rationals Q is totally disconnected.



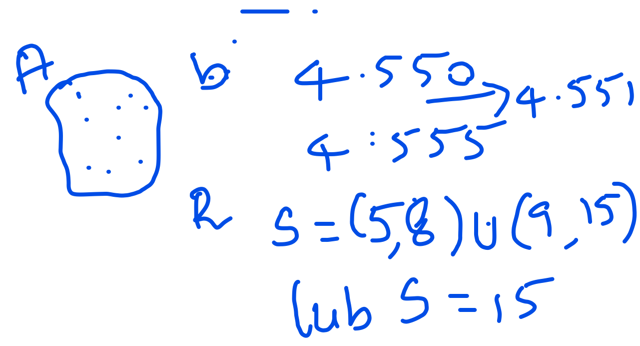
Connected subspaces of the real line

Linear Continuum:

A simply ordered set L having more than one element is called a linear continuum if the following hold.

(i) L has the least upper bound property.

(ii) If $x < y$ there exist z there exist $x < z < y$.



Example:

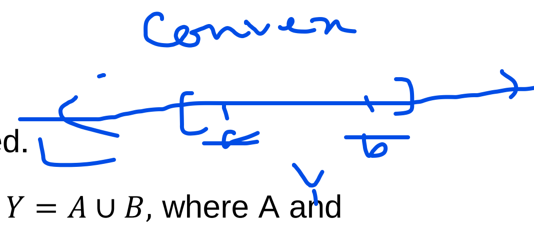
R is a linear continuum.

Z_+ is not a linear continuum.

Theorem 7:

If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof:



Let us prove that if Y is a convex subset of L. then Y is connected.

Let Y be a convex subset of L. Suppose Y is disconnected then $Y = A \cup B$, where A and B are disjoint non-empty sets each of which is open in Y.

Choose $a \in A, b \in B$ say with a<b the interval $[a, b]$ of points of L is contained in Y.

Hence $[a, b]$ is the union of disjoint sets.

$A_0 = A \cap [a, b]$

$B_0 = B \cap [a, b]$

Where A_0 and B_0 are each open in [a,b] in the subspace topology, which is the same as the order topology.

The sets A_0 and B_0 are non-empty because $a \in A_0$ and $b \in B_0$.

Thus A_0 and B_0 constitute a separation of [a,b]. Thus A_0 and B_0 constitute a separation for [a,b]

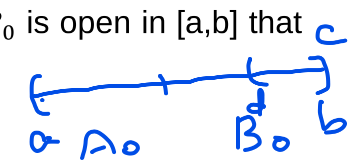
Let $c = \sup A_0$

We show that c belongs neither to A_0 nor B_0 which contradicts the fact that [a,b] is the union of A_0 and B_0.

Case (i)

Suppose that $c \in B_0$ then $c \neq a$.

So either $c = b$ or $a < c < b$ in either case it follows the fact that B_0 is open in [a,b] that there is some intervals of the form $(d, c] \subset B_0$.



If $c = b$, we have contradiction at once for d is a smaller upper bound on A_0, we have contradiction at once for d is a smaller upper bound on A_0 than c. If c<b we note that (c,b] does not intersect A_0 [because c is an upper bound on A_0]

Then $(d, b] = (d, c] \cup (c, b]$ does not intersect A_0.

Again d is a smaller upper bound on A_0 then ~~C~~ c contrary to construction.

Case (ii):

Suppose that $c \in A_0$, then $c \neq b$ so either $c = a$ or $a < c < b$. Because A_0 is open in [a,b]. There must be some interval of the form $[c, e) \subset A_0$.

Because of ordered property-2 of the linear continuum L, we can choose a point z of L such that $c < z < e$.

Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0.

Corollary:

The Real line $\mathbb{R}$ is connected and so are intervals and says in $\mathbb{R}$.

Proof:

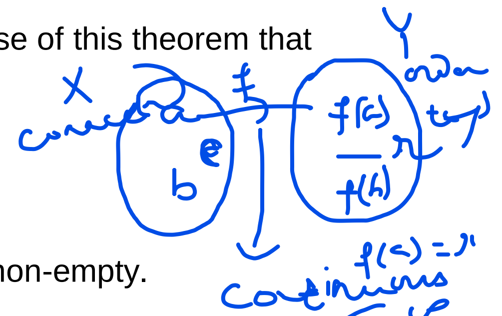
We know that $\mathbb{R}$ is linear continuum then by above theorem,

We can say that $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Theorem 8: (Intermediate Value Theorem)

Let $f: X \rightarrow Y$ be a contin~~uum~~ous map where X is a connected space Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between f(a) and f(b) then there exist a point c of X such that $f(c) = r$.

Note: The Intermediate Value Theorem of calculus is a special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.



Proof

Assume the hypothesis of the theorem the sets,

$A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$ are disjoint ad they are non-empty.

Because one contains f(a) and other contains f(b)

Each is open in f(X), being the intersection of an open ray in Y, with f(X).

If there were no point c of X such that $f(c) = r$. Then f(X) would be the union of the sets A and B.

f(X) = A∪B

Then A and B would constitute a separation of f(~~x~~X), contradicting the fact that the image of a connected space under a continu~~um~~ous map is connected.

Hence the proof.

Example 1:

One example of a linear continuum different from $\mathbb{R}$ is the ordered sequence.

We check the least upper bound property (the 2nd property of the linear continuum is trial to check) the A be a subset of $I \times I$.

Let $b = \sup \prod_1(A)$

If $b \in \prod_1(A)$ then A intersects the subset b×I of I×I.

Because b×I has the order type of I, the set $A \cap (b \times I)$ will have a least upper bound b×c which will be the least upper bound of A.

If $b \notin \prod_1(A)$ then $b \times O$ is the least upper bound of A, No element of the form $b \times c$ with $b' < b$ can be an upper bound for A for then b' would be an upper bound for $\prod_1(A)$.

Example 2:

If X is a well ordered set then $X \times [0\,,1]$ is a linear continuum in the dictionary order. This set can be thought of an having been constructed by "fitting in" a set of the order type of [0 , 1] immediately following each element of x.

Note:

Connectedness of intervals in $\mathbb{R}$ given rise to an especially useful criterion for showing that a space X is connected, namely the condition that every pair of points of X can be joined by a path in X.

Definition 4:

Given points x and y of the space X, a path in X from x to y is a continu~~um~~ous map

$f: [a, b] \to X$ of some closed interval in the real line into X such that $f(a) = x$ and

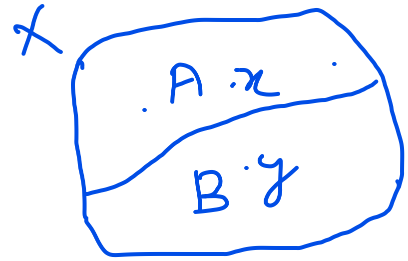
$f(b) = y$.

A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

$f : [a,b] \to X$

$f(a) = x$

$f(b) = y$



Result:

Path connected $\Longrightarrow$ connected.

Proof:

It is easy to see that a path connected space X is connected. Suppose $X = A \cup B$ is a separation of X.

Let $f: [a, b] \to X$ be any path in X. Being the continu~~um~~ous image of a connected set, the set $f([a\, , b])$ is connected.

So that, it lies entirely in either A or B.

Therefore, there is no path in X joining a point of A to a point of B, contrary to the assumption that X is path connected.

Example 1: But connectedness $\not\Rightarrow$ path connectedness.

Define the unit ball $\mathbb{R}^n$ in $\mathbb{R}^n$ by equation $B^n = \{x/||x|| \leq 1\}$ where

$||x|| = \left||x_1, x_2, \ldots, x_n|\right| = (x_1^2, x_2^2, x_3^2, \ldots, x_n^2)^{1/2},$

The unit ball is path connected given any two points x and y of B^n, the straight line path.

$f: [0\, , 1] \to \mathbb{R}^n$ defined by $f(t) = (1-t)x + ty$ lies in B^n for if x and y are in B^n and t is in [0 , 1].

$\left||f(t)|\right| \leq (1-t)\left||x|\right| + t||y|| \leq 1$

A similar argument show that every open ball $B_d(x, \epsilon)$ and every closed ball $\overline{B_d}(x, \epsilon)$ in $\mathbb{R}^n$ is path connected.

Example 2:

Define punctured Euclidean space to be the space $\mathbb{R}^n - \{0\}$ where 0 is the origin $\mathbb{R}^n$.

If $n > 1$ this space is path connected given x and y different from O.

We can join x and y by the straight line path between them. If that path does not go through the origin otherwise we can choose a point z, not on the line joining x and y and take the broken line path from x to z and then z to y.

A connected space need not be a path connected.

Example 3:

The ordered square I_0^2 is connected but not path connected.

Being a linear continuum the order square is connected.

Let $P = O \times O$ and $q = 1 \times 1$

We suppose there is a path $f: [a\ ,\ b] \to I_0^2$ joining p and q and derive a contradiction.

The image set $f([a, b])$ must contain every point $x \times Y$ of I_0^2, by the intermediate value theorem.

∴ for each $x \in I,$ the set $U_x = f^{-1}(x \times (0\ ,1))$ is non-empty subset of [a,b] by continuity it is open in [a,b].

Choose for each $x \in I$ a rational number q_x belonging to U_x. Since the sets U_x are disjoint the map $x \to q_x$ is an injective mapping of I into Q. This contradicts the fact that the interval I is countable.

Example 4:

Let S denote the following subset of the plane

$S = \{x \times \sin(1/x) \mid 0 < x \leq 1\}$

Because S is the image of the connected set (0 ,1] under a continuum map, S is connected.

∴ Its closure $\bar{S}$ in $\mathbb{R}^2$ is also connected.

The Set $\bar{S}$ is a classical example in Topology called the Topology's sine curve

It equals the union of s and the vertical interval $0 \times [-1\ ,1]$

We show that $\bar{S}$ is not path connected. Suppose there is a path $f: [a\ , c] \to \bar{S}$ beginning at the origin and ending at the point of S.

The Set of those t for which $f(t) \in 0 \times [-1\ ,1]$ is closed, so it has a largest element b.

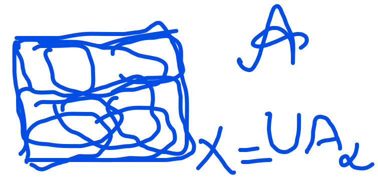
Then $f: [b\ , c] \to \bar{S}$ is a path that maps b into the vertical interval $0 \times [-1\ ,1]$ and maps the other points of $[b\ , c]$ to points of S.

Replace [b ,c] by [0 ,1] for convenience let $f(t) = [x(t), y(t)]$. Then $x(0) = 0$, while $x(t) > 0$ and $y(t) = \sin\left[\frac{1}{x(t)}\right]$ for t>0. We show that there is a sequence of points $t_n \to 0$. Such that $y(t_n) = (-1)^n$.

Then the sequence $y(t_n)$ does not converge, contradicting continuity of f. To find t_n, we proceed as follows given n, choose u with $0 < u < x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u}\right) = (-1)^n$. Then use the intermediate value theorem to find t_n with $0 < t_n < \frac{1}{n}$ such that $x(t_n) = u$.

Compact Spaces

A collection $\mathcal{A}$ of subsets of a space X is said to cover X if the union of elements of $\mathcal{A}$ is equals to X.



Here A is called the covering of X.

$\mathcal{A} = \{A_1, A_2, \ A_3, \ldots, \}$ where $A_i \subseteq$ X for all I and $X = \bigcup_{i \in J} A_i$

Open Covering

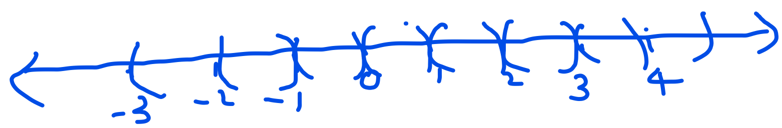
The covering of X is called an open covering of X if its elements are open subsets of X.

Compact Space

A Space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite sub collection that also covers X.

Example 1:

The real line $\mathbb{R}$ is not compact



Consider the open covering of $\mathbb{R}$. $\mathcal{A} = \{(n\,, n+2)/n \in Z\}$. This contains no finite collection.

Example 2:

Consider the subspace $X = \{0\} \cup \left\{\frac{1}{n}/n \in Z_+\right\}$ of $\mathbb{R}$. This is compact in $\mathbb{R}$. Given an open covering $\mathcal{A}$ of X there is an element U of $\mathcal{A}$ containing zero The Set U contains all but finitely many points 1/n.

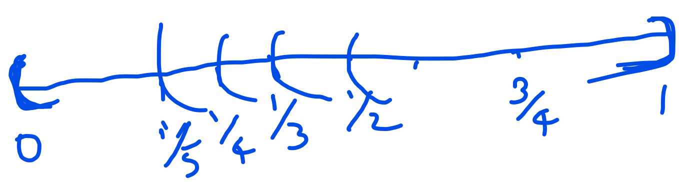
Choose for each point of X not in U and element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$ along with the element U is a finite sub collection of $\mathcal{A}$ that covers X.

Example 3:

Any space X containing only finitely many points is necessarily compact $B'(o)$ every covering of X in this case is finite.

Example 4:

The interval (0 ,1] is not compact.



Soln:

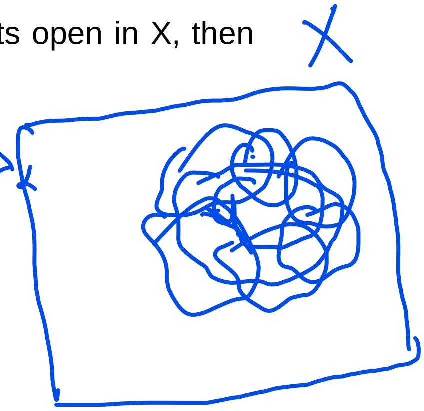
The open covering $\mathcal{A} = \left\{\left(\frac{1}{n}, 1\right]/n \in Z_+\right\}$ contains no finite sub collection that covers (0,1]

Lemma 9:

Let Y be a subspace of X then Y is compact iff every covering of Y by sets open in X contains a finite sub collection covering of Y.

Proof:

Suppose Y is compact. Let $\mathcal{A} = \{A_\alpha / \alpha \in J\}$ is a covering of Y. By sets open in X, then the collection,



$\{A_\alpha \cap Y / \alpha \in J\}$ is a covering of Y By sets open in Y.

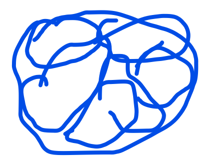
$Y \subseteq \cup A_\alpha$

Since Y is compact there exist a finite sub collection

$\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$ which covers Y.

$$Y \subseteq \bigcup_{i=1}^{n} A_{\alpha_i} \cap Y$$

$$= \left(\bigcup_{i=1}^{n} A_{\alpha_i} \right) \cap Y$$



$Y \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}$ and $Y \subseteq Y$

Y=Y

Then there exist a finite sub collection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ which covers Y.

Conversely, Suppose the given condition holds. Let $\{A_\alpha' / \alpha \in J\}$ be an open covering of Y by sets open in Y.

For each A_α' there exist an A_α which is open in X.

Such that $A_\alpha' = A_\alpha \cap Y$

Ie, $Y \subseteq \bigcup_{\alpha \in J} A_\alpha'$

$$Y \subseteq \bigcup_{\alpha \in J} (A_\alpha \cap Y)$$

$$Y \subseteq \left(\bigcup_{\alpha \in J} A_\alpha \right) \cap Y$$

$$Y \subseteq \bigcup_{\alpha \in J} A_\alpha$$

By hypothesis there exist a finite sub collection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ that covers Y

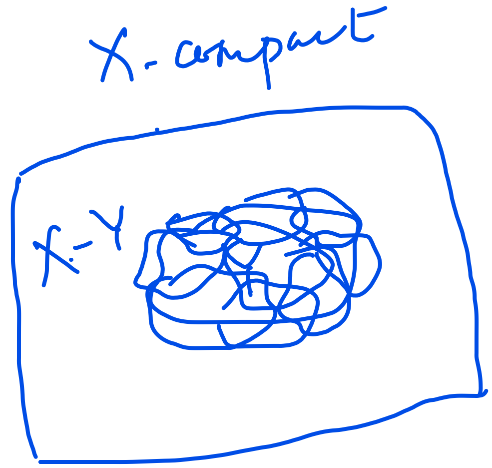
$\{A_{\alpha_1}', A_{\alpha_2}', \ldots, A_{\alpha_n}'\}$ is a sub-collection of $\{A_\alpha' / \alpha \in J\}$ which covers Y.

∴ Y is compact.

Hence the proof.

Theorem 10:

Every closed subspace of a compact space is compact.



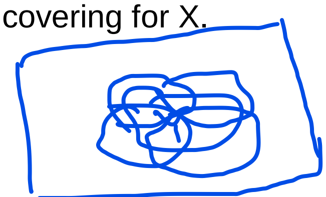
Proof:

Let Y be a closed subspace of the compact space X.

Claim: Y is compact.

Let $\{A_\alpha / \alpha \in J\}$ be an open covering of Y, by sets open in X.

Now, since Y is closed, X-Y is open and $\mathcal{B} = \{\cup A_\alpha\} \cup \{X - Y\}$ is an open covering for X.



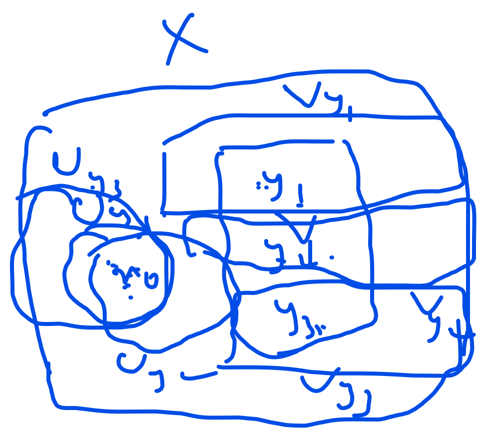
Since X is compact, a finite sub-collection of $\mathcal{B}$ covers X.

If this sub-collection contains the set X-Y, discard X-Y.

Otherwise leave the sub-collection alone. The resulting sub-collection is a finite sub collection of $\{A_\alpha \mid \alpha \in J\}$ which covers Y.

∴ Y is compact.

Hence the proof.



Theorem 11:

Every compact subspace of a Hausdorff space is closed.

Proof:

Let Y be a compact subspace of the Hausdorff space X.

Claim: Y is closed.

i.e, X-Y is open.

Let $x_0 \in$ ~~U~~ $X - Y$. For each point y of Y,

Let us choose disjoint neighbourhood U_y and V_y of the points x_0 and y respectively. This is possible because the space X is Hausdorff.

Now the collection $\{V_y \mid y \in Y\}$ is an open covering of Y, by sets open in X. Since Y is compact, a finite sub collection $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ covers $Y \subseteq V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n} = V$

Note that V is an open set containing Y.

Considering the intersection of the corresponding neighbourhood of x_0, $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}$ where U is an open set containing x_0.

$U \cap \overset{V}{\cancel{Y}} = \phi$ for $z \in V$

$V_{y_i} \cap U_{y_i} = \varphi$

$\Longrightarrow z \in V_{y_i}$ for some i

$\Longrightarrow z \notin U_{y_i}$ for some i

$\Longrightarrow z \notin U$

Then U is a neighbourhood of x_0 disjoint from ~~Y~~ V

Then $Y \subseteq V$

$x_0 \in \dot{U} \subset X - Y$

Hence the proof.

Lemma 12:

If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, there exist disjoint open sets U and V of X, containing x_0 and Y respectively.

Proof:

This proof follows from the previous theorem.

Example:

The intervals of the type (a , b), [a, b), [a, b] are not compact in $\mathbb{R}$. Because they are not closed in the Hausdorff space $\mathbb{R}$.

Theorem 13:

X: The image of compact space under a continuous map is compact.

Proof:

Let $f: X \rightarrow Y$ be continuous and X be compact.

Claim: f(~~x~~ X) is compact.

Let $\{A_\alpha \mid \alpha \in J\}$ be an open covering by sets open in Y.

$\Longrightarrow \{f^{-1}(A_\alpha) | \alpha \in J\}$ is an open covering of X, by sets open in X.

Since f is continuous.

i.e., $X \subseteq \bigcup_{\alpha \in J} f^{-1}(A_\alpha)$

$$\Rightarrow X \subseteq \bigcup_{i=1}^{n} f^{-1}(A_{\alpha_i}) \qquad [\because X \text{ is compact }]$$

$$f(\overset{X}{\cancel{x}}) \subseteq \bigcup_{i=1}^{n} f\left(f^{-1}(A_{\alpha_i})\right)$$

$$f(\overset{X}{\cancel{x}}) \subseteq \bigcup_{i=1}^{n} A_{\alpha_i} \qquad \left[\because f\left(f^{-1}(A_{\alpha_i})\right) \subseteq A_{\alpha_i}\right]$$

$\Rightarrow f(\overset{X}{\cancel{x}})$ is compact.

Theorem 14:

Let $f: X \to Y$ be a bijective continuous function. If X is compact and Y is a Hausdorff then f is a homeomorphism.

Proof:

$f^{-1}: Y \to X$

C

$(f^{-1})^{-1}(C) = f(C)$

Given that f is one-one and onto and continuous.

To prove that f is a homeomorphism.

It remains to prove that f is an open map (or) f^{-1} is continuous for this it is enough to prove that the image under f of each closed set is closed.

Let A be a closed subset of X, which is compact.

∴ A is compact.

∴ $f(A)$ is compact.

Since f is continu~~um~~ous [∵ closed subset of compact set is compact and continuum image of a compact set is compact]

f(A) is compact subset of Y.

∴ $f(A)$ is closed.

Since compact subset of a Hausdorff space is closed.

∴ f is a homeomorphism.

Theorem 15:

The product of finitely many compact spaces is compact.

Proof:

We prove this theorem for 2-spaces. Then it follows for finitely many spaces.

By induction hypothesis

Step 1:

The tube lemma

Consider the product space X×Y, where Y is compact. Let $x_0 \in X$.

If N is an open set of X×Y, containing the slice $x_0 \times Y$ of $X \times Y$. Then N contains some tube W×Y about $x_0 \times Y$ where W is an neighbourhood of x_0 in X.

Proof:

X and Y are ~~tow~~ topological spaces where Y is compact. Let x_0 be a point of X and N be an open subset of X×Y containing the slice $x_0 \times Y$.

To prove there exist a neighbourhood W of x_0 in X, such that N contains W×Y.

W×Y is called a Tube about $x_0 \times Y$. First, let us cover $x_0 \times Y$ by a basis elements U×V (for the topologies of X×Y) lying in N.

Given Y is compact, the space $x_0 \times Y$ is homeomorphic to Y and therefore it is compact.

∴ ~~Each open~~ We can cover ~~of~~ $x_0 \times Y$ by finitely many such basis elements (U×V)

$U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n$

We assume that each basis elements $U_i \times V_i$ intersects $x_0 \times Y$.

Define $W = U_1 \cap U_2 \cap \ldots \cap U_n$

Then W is open since $U_i's$ are open and also $x_0 \epsilon$ W

Since $U_i \times V_i$ intersects $x_0 \times Y$.

We claim that the sets $U_i \times V_i$ which cover the slice $x_0 \times Y$ actually covers the tube W×Y.

For let $x \times y \in W \times Y$

Consider the point $x_0 \times y$ of $x_0 \times Y$.

Now $x_0 \times y \in U_i \times V_i$ for some i.

$\therefore y \in V_i$

But $x \in U_j$ for all j [$\because x \in W$]

$\therefore x \times y \in U_i \times V_i$

Since all the sets $U_i \times V_i$ *lie in N and* they cover W×Y. We have $W \times Y \subset N$.

Step 2:

Let X and Y be compact spaces to prove $x \times Y$ is compact.

Let $\mathcal{A}$ be an open covering of $X \times Y$. Let $x_0 \in X$. Then the slice $x_0 \times Y$ is homeomorphic to Y, Y is compact.

$\therefore$ Each open covers of $x_0 \times Y$ has a finite number of elements of $\mathcal{A}$ say ($A_1, A_2, \dots, A_m$)

If $N = A_1 \cup A_2 \cup \dots \cup A_m$

Then $x_0 \times Y \subset N$

By step -1 the open set N contains a tube W×Y about $x_0 \times Y$ where W is open in X then W×Y is covered by finitely many elements.

$(A_1, A_2, \dots, A_m)$ of $\mathcal{A}$ thus for each x in X.

We can choose a neighborhood W_x of x, such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

The collection of all such neighbourhood W_x is an open covering of X.

Since X is compact, there exist a finite sub cover for X.

Namely $\{W_1, W_2, \dots, W_n\}$ the union of the tube $\{W_1 \times Y, W_2 \times Y, \dots, W_n \times Y\}$ equals X×Y ie., there exist a finite such cover for X×Y.

$\therefore$ X×Y is compact.

We can extent this result to a finite number of spaces.

$(X_1 \times X_2 \times \dots \times X_n)$ by using induction hypothesis.

Suppose $X_1, X_2, \dots, X_n$ are compact. To Prove that $X_1 \times X_2 \times \dots \times X_n$is compact.

The result is true for n=2.

$\Longrightarrow X_1 \times X_2$ is compact.

Assume the result to be true for $n - 1$

$(X_1 \times X_2 \times \dots \times X_{n-1})$ is compact.

Now $(X_1 \times X_2 \times \dots \times X_n)$ is compact, and X_n is compact.

$(X_1 \times X_2 \times \dots \times X_n) \times X_n$ is compact.

$(X_1 \times X_2 \times \ldots \times X_n)$ is compact.

Definition:

A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite sub collection $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{C}$.

The intersection $C_1 \cap C_2 \cap \ldots \cap C_n$ is non empty.

Theorem 16:

Let X be a topological space then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is non-empty.

Proof:

Given a collection $\mathcal{A}$ of subsets of X.

Let $\mathcal{C} = \{X - A | A \in \mathcal{A}\}$ be the collection of their compliments then the following statements hold.

(1) $\mathcal{A}$ is a collection of open sets iff $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X iff $\bigcap_{C \in \mathcal{C}} C = \phi$.

$\mathcal{A}$ covers $X \Longrightarrow X = \bigcup_{A \in \mathcal{A}} A$

Take complements on both sides,

$X - X = X - \bigcup_{A \in \mathcal{A}} A$

$= \bigcap_{A \in \mathcal{A}} (X - A)$ [By Demorgan's Law]

$\phi = \bigcap_{C \in \mathcal{C}} C$

(3) The finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$ covers X iff $C_1 \cap C_2 \cap \ldots \cap C_n = \phi$.

$X = \bigcup_{i=1}^{n} A_i$ if the finite sub collection covers X.

$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$

$\phi = \bigcap_{i=1}^{n} C_i$

Suppose X is compact, that

$\Longrightarrow$ Each open cover of X has a finite sub cover.

$\Longrightarrow$ If $\mathcal{A}$ is a family of open sets in X such that $\mathcal{A}$ covers X then some finite sub collection of $\mathcal{A}$ covers X.

$\Longrightarrow$If no finite sub-collection of $\mathcal{A}$ covers X then $\mathcal{A}$ also does not cover X.

Let $\mathcal{C} = \{X - A | A \in \mathcal{A}\}$

Then by (1) $\mathcal{C}$ is a family of closed sets.

By (3), No finite sub collection of $\mathcal{A}$ covers X, means

$\{C_1 \cap C_2 \cap \ldots . \cap C_n\}$ of $\mathcal{C}$.

By (2) $\mathcal{A}$ does not cover X means

$\bigcap_{C \epsilon \mathcal{C}} C \neq \phi$. $\mathcal{C}$ is a collection of closed sets satisfying the finite intersection property then $\bigcap_{C \epsilon \mathcal{C}} C \neq \phi$.

Refracting the steps we can prove this converse.

Note:

A special case of the above theorem occurs when we have a nested sequence.

$C_1 \supset C_2 \supset \ldots \supset C_n \supset C_{n+1}$ of closed sets in a compact space X. If each of the sets C_n is non-empty then the collection $\mathcal{C} = \{C_n\}_{n \in Z_+}$

Automatically has the finite intersection property then the intersection is non-empty.

$\bigcap_{n \in Z_+} C_n$

Compact Subspaces of Real line:

Theorem 17:

Let X be a Simply ordered set having the Least upper bound property in the ordered topology, each closed interval in X is compact.

Proof:

Let $a < b, \quad a, b \in X$

Consider the interval $[a, b]$ in X. Let $\mathcal{A}$ be an open covering of $[a, b]$ by sets openin $[a, b]$ in the subspace topology which is the sane as the ordered topology.

To prove that there exist a finite sub-collection of $\mathcal{A}$ which covers (a,b] will be compact.

Step 1:

To prove that "If x is a point of [a,b], $x \neq b$. Then there exist a point $y > x$ in [a,b] such that [x,y] can be covered by atmost two elements of A".

Suppose x has an immediate successor in X, then [x,y] consists of the two points x and y.

[x,y] can be covered by atmost two elements of $\mathcal{A}$.

Suppose x has a no-immediate successor in X. Then choose an element $A \in \mathcal{A}$ containing x.

Now $x \neq b$ and A is open.

∴ A contains an interval of the form $[x, c)$ for some $c \in [a, b]$. Choose a pint y in $[x, c)$. Then $[x, y]$ is covered by $A \in \mathcal{A}$.

Step 2:

Let $\mathcal{C}$ be the set of all points $y > a$ of [a,b] such that [a,y] can be covered by finitely many elements of $\mathcal{A}$.

$i.e., \quad \mathcal{C} = \{y \epsilon [a, b] | (a, y] \ can\ be\ covered\ by\ finitely\ many\ elements\}$

Apply step 1, by taking $x = a$. Then there exist atleast one such y.

$\mathcal{C}$ is not empty. Let C be the least upper bound of $\mathcal{C}$. Then $a < c \leq b$.

Step 3:

To prove that $c \in \mathcal{C}$

To prove that [a,c] can be covered by finitely many elements of $\mathcal{A}$. Choose an element $A \in \mathcal{A}$ containing C.

∵ A is open A contains an interval of the form (d , c] for some d in [a,b].

Suppose $c \notin \mathcal{C}$.

Then there is a point $z \in \mathcal{C}$ such that $z \in (d\,, c]$ (for otherwise d would be a smaller upper bound of $\mathcal{C}$ than ⟹⟸.

Since $z \in \mathcal{C}$, [a , z] can be covered by finitely many elements of $\mathcal{A}$ (By definition of $\mathcal{C}$).

Say n elements $A_1, A_2, \dots, A_n$.

Now $[z, c]$ lies in the single element $A \in \mathcal{A}$.

∴ $[a, c] = [a, z] \cup [z, c]$

∴ [a,c] can be covered by finitely many elements (n+1 elements of $\mathcal{A}$)

∴ $C \in \mathcal{C}$

⟹⟸ to our assumption that $C \notin \mathcal{C}$.

∴ $C \in \mathcal{C}$.

Step 4:

Claim: $c = b$

Suppose $c < b$

Apply step 1 by taking $x = c$

Then there exist a point $y > c$ of $[a, b]$.

Such that [c,y] can be covered by finitely many elements of $\mathcal{A}$.

By step 3, $C \in \mathcal{C}$.

∴ $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$.

∴ $[a, y] = [a, c] \cup [c, y]$

∴ $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$.

∴ y$\in \mathcal{C}$.

This is a contradiction to the fact that C is the least upper bound of $\mathcal{C}$.

∴ $c = b$.

∴ $[a, b]$ can be covered by finitely many elements of $\mathcal{A}$.

∴ $[a, b]$ is compact.

Corollary:

Prove that closed interval in $\mathbb{R}$ is compact.

Proof:

The $\mathbb{R} = X$, $\mathbb{R}$ is linearly ordered set with least upper bound property.

Every closed interval in $\mathbb{R}$ is compact.

Theorem 18: Characterization of Compact subset of $\mathbb{R}^n$

A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the Euclidean metric d or the square metric ρ.

Proof:

We know that $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\, \rho(x, y)$.

A is bounded under ρ iff it is bounded under d.

∴ Let us consider the metric ρ

(i) Suppose A is compact.

To prove that A is closed and bounded $\mathbb{R}^n$ is Hausdorff.

∴ A is closed.

[∵ Compact subset of Hausdorff space is closed.]

It remains to prove that A is bounded.

Consider the collection of open sets $\{B_\rho(0, m) | m \in Z_+\}$ whose union is all $\mathbb{R}^n$.

i.e., This is an open covers for $\mathbb{R}^n$.

A is a subset of $\mathbb{R}^n$.

We can consider this is an open covers for A, also A is compact.

There exist a finite subcover for A.

∴ $A \subset B_\rho(0, m)$ for some M.

Suppose $x, y \in A$. Then $x , y \in B_\rho(0, m)$

$\Longrightarrow$ $\rho(0 , x) < M$ and

$\rho(0 , y) < M$

$\Longrightarrow$ $\rho(x , y) \leq \rho(0 , x) + \rho(0 , y) \leq 2M$

∴ A is bounded under the metric ρ. Then A is closed and bounded.

(Ii) Suppose A is closed and bounded under ρ

To prove that A is compact

A is bounded, let us assume that $\rho(x, y) \leq N$ for all pair $x, y \in A$

Choose a point x_0 of A and let $\rho(x_0, 0) = b$

$\rho(x_0, 0) \leq \rho(x, x_0) + \rho(x_0, 0)$

$\leq N + b$

Let $P = N + b$

Consider $[-P, P]^n$. Then A is a subset of $[-P, P]^n$.

$\therefore$ $[-P, P]^n$ is a compact.

So A is closed compact subset of a compact space is compact.

Theorem 19: (Extreme value theorem)

Let $f: X \to Y$ be continuous where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that

$f(c) \leq f(x) \leq f(d)$ for every$x \in X$.

Proof:

Given $f: X \to Y$ is continuous and X is compact.

$f(x)$ is compact.

Let $f(x)$ is compact.

Let $f(x) = A$

To prove that A has a largest element M and a smallest element m.

Suppose A has no larger element ,

Then the collection

$\{(-\infty, a) | a \in A\}$ is an open covering for A.

A is compact. Therefore there exist a finite sub cover.

ie., the finite subcollection $\{ (-\infty, a_1), (-\infty, a_2), .., (-\infty, a_n)\}$ covers A.

Suppose $a_i = \max\{a_1, a_2, \ldots, a_n\}$. Then a_i must belong to A.

But A has no larger element which is a contradiction.

Therefore, A has a largest element.

Similarly we can prove that A has a smallest element.

Hence A has a larger element M and a smallest element m.

i.e., $m, M \in A$

We have $f(c) = m$ and $f(d) = M$ for some $c, d \in X$

$\therefore f(c) \leq (x) \leq f(d),$ for all $x \in X$

Hence the proof.

Definition: The distance from x to A

Let (x, d) be a metric space. Let A be a nonempty subset X. For each $x \in X$, We define the distance from x to A by the equation.

$d(x, A) = \inf\{d(x, a) | a \in A\}$

Definition:

The diameter of a bounded subset A of a metric space (x, d) is the number

$\sup\{d(a_1, a_2) | a_1, a_2 \in A\}$.

Lemma 20:

The lebesque number lemma.

Let $\mathcal{A}$ be an open covering of the metric space (x, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exist an element of $\mathcal{A}$ containing it. The number δ us called a lebesgue number for the covering $\mathcal{A}$.

Proof:

Let $\mathcal{A}$ be an open covering of X. Suppose $x \in \mathcal{A}$. Then any positive number is a lebesgue number for $\mathcal{A}$. So assume $x \notin \mathcal{A}$.

As X is compact, there exist a finite Sub-collection $\{A_1, A_2, \dots, A_n\}$ of $\mathcal{A}$ that covers X.

Let $C_i = X - A_i$, $i = 1,2, \dots, n$

Define $f: X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(c, x_i)$$

Then $f(x) > 0$ for all x, for if $x \in X$. Choose i so that $x \in A_i$. Then choose the ϵ neighbourhood of x that lies in $\mathcal{A}$.

Then $d(x_i, c) \geq \epsilon$

$f(x) \geq \frac{\epsilon}{n}$

Since f is continuous it has a minimum value δ.

Claim: δ is one required lebesque number. Let B be a subset of X of diameter less than δ.

Let $x_0 \in B$.

Now B lies in the Neighbourhood of x_0. Now,

$\delta \leq f(x_0) = \dfrac{1}{n}[d(x_o, c_1), d(x_1, c_2), \ldots, d(x_0, c_n)]$

$\leq d(x_0, c_m)$

where $d(x_0, c_m)$ is the largest of the numbers $d(x_0, c_i)$.

Now the δ –neighbourhood of x_0 is contained in $A_m = X - C_m$ of the covering $\mathcal{A}$.

Definition: Uniformly Continuous

A function f from the metric space $(X, d_x)$is said to be uniformly continuous if given $\varepsilon > 0$ there is a $\delta > 0$ such that for every pair of points x_0, $x_1$of X.

$d_x(x_0, x_1) < \delta \Longrightarrow d_x\big(f(x_0), f(x_1)\big) < \varepsilon$.

Theorem 21:

Uniform Continuity Theorem

Let $f: X \to Y$ be a continuous map of the compact metric space (X, d_x) to the metric space $\big(Y, d_y\big)$. Then f is uniformly continuous.

Proof:

Given $\varepsilon > 0$ take the open covering of Y. By balls $B(y, \varepsilon/2)$ of radius $\varepsilon/2$.

Let $\mathcal{A}$ be the open covering of X by

$\big\{f^{-1}\big(B(y, \varepsilon/2)\big)/y \in Y\big\}$. Let r be the lebesgue number for the covering $\mathcal{A}$. To prove f is uniformly continuous. Let $x_1, x_2 \in X$ such that $d_x(x_1, x_2) < \delta$. Then the set $\{x_1, x_2\}$ has diameter less than δ.

So that there exist some element in $\big\{f^{-1}\big(B(y, \varepsilon/2)\big) | y \in Y\big\}$ such that

$\{x_1, x_2\} \subset f^{-1}\big(B(y, \varepsilon/2)\big)$

$\Longrightarrow f(\{x_1, x_2\}) \subseteq B\left(y, \frac{\varepsilon}{2}\right)$ $for\ some\ y.$

$\Longrightarrow f(x_1), f(x_2) \subseteq B\left(y, \frac{\varepsilon}{2}\right)$ $for\ some\ y.$

$\Longrightarrow d\big(y, f(x_1)\big) < \frac{\varepsilon}{2}.$

Now

$d_y\big(f(x_1), f(x_2)\big) < d(f(x_1), y) + d\big(y, f(x_2)\big)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$d_x(x_1, x_2) < \delta \Longrightarrow d_y\big(f(x_1), f(x_2)\big) < \varepsilon$

$x_1, x_2 \in X$ are arbitrary.

∴ f is uniformly continuous.

Definition: Isolated Point

If X is a space a point x of X is said to be an isolated point of X. If the one point set $\{x\}$ is open in X.

Theorem 22:

Let X be a non-empty compact Hausdorff space. If X has no isolated points then X is uncountable.

Proof:

Step 1:

We show first that given any non-empty openset U of X and any point x of X there exist a non-empty open set $V \subset U$ such that $x \notin \bar{V}$.

Choose a point of y of U different from x.

If $x \in U$, then U≠ $\{x\}$ as x is not an isolated point of X.There exist y∈ U.

If $x \notin U$, then as U≠ ϕ, there exist y∈ U.

Now $x \neq y$ and X is Hausdorff. ∴ disjoint non-empty open sets W_1 and W_2 about x and y respectively.

Let $V = U \cap W_2$. Then V is open (being intersection of open sets).

$V \neq \phi$ as (y∈ V).

$x \notin \bar{V}$ (as there exist a neighbourhood W_1 of x $W_1 \cap Y = \phi$).

Step 2:

To Prove: X is uncountable.

To Prove $f: Z_+ \to X$ is not surjection.

Let $f(n) = x_n$.

Now $x_1 \in X$ and take $U = X$.

By step 1, there exist an open set $V_1 = \phi$

$V_1 \subset X, x_1 \notin \overline{V_1}$.

Now $x_2 \in X$ and V_1 is open in X, there exist $V_2 \subset V_1$ such that $x_2 \notin X$ there exist an oopen set $V_n \neq \phi$.

$V_n \subset V_{n-1}$ and $x_n \notin \overline{V_n}$

Consider the nested sequence.

$\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \supset \cdots \supset \overline{V_n}$ of non-empty closed sets.

Therefore they satisfy finite intersection property $\overline{V_n} \neq \phi$.

By theorem.

Let $x \in \cap \overline{V_n}$

$x \in \cap \overline{V_n}$ for all n and $x_n \notin \overline{V_n}$ for all n.

$x \neq x_n$ for any n.

There exist no principle for x under f.

$\therefore$ f is not surjective.

f is not bijective and so.

X is uncountable.

<u>Limit Point Compactness</u>

<u>Definition:</u>

A space X is said to be limit point compact, if every infinite subset of X has a limit point.

Theorem 23:

Compactness implies limit point compactness but not conversely.

Proof:

Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point.

We prove the contrapositive if A has no limit point, then A must be finite.

So, suppose A has no limit point. Then A contains all its limit points so that A is closed.

Furthermore, for each $a \in A$. We can choose a neighbourhood U_a of a such that U_a intersects A in the point a alone.

The space X is covered by the open sets $X - A$ and the open sets U_a being compact, it can be covered by finitely many of there sets.

Since x-a does not intersect A and each set U_a contains only one point of A, the set a must be finite.

Example 1:

Let Y consists of two points give Y the topology consisting of T and the empty set then the space, $X = Z_+ \times Y$ is limit point compact, for every non-empty subset of X has limit point. It is not compact for the covering of X by the open sets,

$U_n = \{n\} \times Y$ has no finite subcollection covering of X.

Definition

Let X be a topological space. If (x_n) is a sequence of points of X, and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of positive integers then the sequence (y_i) defined by setting $y_i = x_{ni}$ is called a subsequence of the sequence (x_n).

The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

Theorem 24:

Let X be a metrizable space. Then the following are equivalent.

(i) X is compact.

(ii) X is limit point compact.

(iii) X is sequentially compact.

Proof:

<u>To Prove: (i) ⟹(ii)</u>

Refer Theorem-23.

<u>To Prove: (ii) ⟹(iii)</u>

Assume X is a limit point compact.

To prove that X is sequentially compact. Given a sequence (x_n) of points of X

Consider the set $A = \{x_n | n \in Z_+\}$. If the set A is finite then there is a point x such that $x = x_n$.

For finitely many values of n.

In this case, the sequence (x_n) has a subsequence that is constant.

∴ Converges trivially.

If A is infinite, then A has limit point x.

We define a subsequence of (x_n). Converging to x.

First Choose n_1, $x_{n_1} \in (x, 1)$

Suppose that the positive integers n_{i-1} is given. Because the ball $B\left(x, \frac{1}{i}\right)$ intersects A in infinitely many points.

We can choose $n_i > n_{i-1}$ such that

$x_{n_i} \in \left(x, \frac{1}{i}\right)$

U contains $x_{n_i}$for all $i \geq N$.Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converge to x.

<u>To Prove: (iii) ⟹ (i)</u>

First we show that X is sequentially compact then the lebesgue number lemma holds for X. Let A be an open covering of X. Assume that there is a δ>0 such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it and derive a contradiction our assumption.

In Particular that for each positive integers n there exist a set of diameter less than1/n that is not contained in any element of $\mathcal{A}$.

Let C_n be such a set. Choose a point $x_n \in C_n$ for each n. By hypothesis,

Some subsequence (x_{n1}) of the sequence (x_n) converges say to the point a.

Now a belongs to some element A of the collection $\mathcal{A}$, because A is open.

We may choose an $\varepsilon > 0$ such that $B(a, \varepsilon) \subset A$.

It is large enough that $\frac{1}{n_i} < \frac{\varepsilon_0}{2}$. Then the set C_{ni} lies in the $\frac{\varepsilon}{2}$- neighbourhood of x_{n_i}.

If i is also chosen large enough that

$$d(x_{n_i}, a) < \frac{\varepsilon}{2}$$

Then C_{n_i} lies in the ε-neighbouhood of A.

$C_{n_i} \subset A$

Contrary to hypothesis

Second we show that if X is sequentially compact given $\varepsilon > 0$ there exist a finite covring of X by open ε-balls.

Once again we proceed by contradiction, Assume that there exist an $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls.

Construct a sequence of points x_n of X follows

First we choose x to be any point of X Nothing that the ball $B(x, \varepsilon)$ is not all of X/

Otherwise X could be covered by a single ε –ball.

Choose x_2 be a point off X not in $B(x, \varepsilon)$.

In general, given $x_1, x_2, \dots, x_n$. Choose x_{n+1} *to be* point not in the union.

$B(x_1, \varepsilon), B(x_2, \varepsilon), \dots, B(x_n, \varepsilon)$. Using the fact that these balls do not cover X.

Note by construction $d(x_{n+1}, x_i) \geq \varepsilon$ for i=1,2,…,n.

The sequence (x_n) can have no convergent subsequence.

In fact any ball of radius $\varepsilon/2$ can contain x_n for at least one value n which is a contradiction.

Finally, we show that if X is sequentially compact, then X is compact.

Let A be an open covering X because ~~x~~ X is sequentially compact. Then the open covering A has a lebesgue number δ. Let $\varepsilon = \delta/3$. Use sequential compactness of X to find a finite converging of X by open $\varepsilon -$ balls.

Each of these balls has diameter at most $\frac{2\delta}{3}$ so lies in an element of $\mathcal{A}$.

Choosing one such element of $\mathcal{A}$ for each of those $\varepsilon -$ balls; we obtain a finite sub collection of $\mathcal{A}$ covers X.

∴ X is compact.

Hence the proof.

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