UNIT IV

The Countabillity Axioms

DEFINITION:

A space X is said to have a countable basis at x if there is a countable collection $\mathcal{B}$ of neighborhoods of x. Such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the first countability axiom or to be first countable.

THEOREM:

Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points of A. Convering to x, then $x \in \bar{A}$, the converse holds if X is first countable.

(b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X,

the sequence $f(x_n)$ converges to $f(x)$. The converse holds if x is first countable.

PROOF:

Proof is similar to sequence lemma. And theorem following it.

X is metrizable $\Rightarrow$ X is first countable

DEFINTION 2:

If a space X has a countable basis for its topology, then X is said to satisfy the second countability axioms or to be second – countable

Example: 1

The real line $\mathbb{R}$ has a countable basis the collection of all open intervals (a, b) with rational end points. Likewise $\mathbb{R}^n$ has a countable basis the collection of all products of intervals having rational end points. Even $\mathbb{R}^{\omega}$ has a countable basis the collection of all products $\prod_{n \in Z_+} U_n$, where U_n is an open interval with rational end points for finitely many values of n, and $U_n = \mathbb{R}$ for all other values of n.

Example: 2

In the uniform topology $\mathbb{R}^{\omega}$ satisfies the first countability axiom (being metrizable). However, it does not satisfy the second.

THEOREM:

A subspace of a first-countable space is first countable, and a countable product of first-countable space is first-countable. A subspace of a second-countable space is second countable, and a countable product of second countable spaces is second countable.

PROOF:

(i) Any subspaces of a first countable space is first countable

let (x, τ) be a first countable and

let (A, τ_A) be a subspace.

Let there be a countable basis is at each point of x.

Let x$\in$ A be arbitrary then x$\in$X then $\mathcal{B}$ be a countable basis at x.

$\Rightarrow$each neighborhood of x contains atleast one of the element of $\mathcal{B}$

x$\in$ B $\subset U_x$ where U_x is a neighbourhood of x and B$\in$ $\mathcal{B}$

B and U_x are open in x.

$\Rightarrow$ B$\bigcap$ A and $U_x \bigcap$ A are open in A

Also x$\in$ B$\bigcap$ A $\subset U_x \bigcap$ A.

Since x$\in$A is arbitrary, $\mathcal{B}$ is a countable basis at each point of A.

$\Rightarrow$ (A, τ_A) is a first-countable

(ii) Countable product of first-countable spaces is first-countable.

Consider the product $\prod_{n\in\omega} X_n$ where each X_n is first-countable ($\omega \subseteq$ Z+)

Let $\mathcal{B}_n$ be a countable basis at each point of $X_n \forall\ n\in\omega$

Let $x_n \in X_n$.

$\Rightarrow \mathcal{B}_n$ is a countable basis at x_n

$\Rightarrow x_n \in \mathcal{B}_n U_{x_n}$, U_{x_n} is a neighbourhood of x_n

Consider the collection $\prod B_n$, $B_n \in \mathcal{B}$

Then the collection $\{B_n\}$ is countable basis at $\prod X_n$, n=1, 2, . . . (i.e.,) $n\in\omega$

$\Rightarrow \prod X_n$ is first countable.

(iii) Any subspace of a second-countable space is second countable.

Let (x, τ) be second countable

Let (A, τ_A) be a subspace.

Let $\mathcal{B}$ be a countable basis for τ

Consider $\mathcal{B}'$ = {B$\bigcap$ A / B $\in \mathcal{B}$ }

Then $\mathcal{B}'$ is a countable basis for τ_A. The subspace (A, τ_A) is second countable .

(iv) Countable product of second countable spaces is second countable.

Consider the product $\prod_{n \in \omega} X_n$ where X_n is second countable and $\omega \subseteq$ Z+.

Let $\mathcal{B}_n$ be a countable basis for the topology of $X_n \forall\ n \in \omega$

Consider the collection of all products $\prod U_n$ where $U_n \in \mathcal{B}_n$ for finitely many n and $U_n = X_n$ other values of n.

Then the collection $\{\prod U_n\}$ is a countable basis for the topology of $\prod_{n \in \omega} X_n$

$\therefore \prod_{n \in \omega} X_n$ is second countable.

DEFINITION: Dense

A subset A of a space x is said to be dense in X if π=x

THEOREM:

Suppose that x had a countable basis then

a) Every open covering of x contains a countable subcollection covering x.
b) There exists a countable subset of x that is dense in X.

[We know that, $\overline{D} \subseteq$X. To prove: X$\subseteq \overline{D}$, let x∈X. Some $B_n \rightarrow$ x ∈ B_n. Now $B_n \cap D \neq \emptyset$ as $x_n \in B_n \cap D$.]

PROOF:

Let $\{B_n\}$ be a countable basis for x.

a) Let A be an open covering of x. For each positive integer n for which it is possible, choose an element A_n of *A.* Containing the basis element B_n. The collection A' of the sets A_n is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers x. Given a point x ∈X. We can choose an element A of *A* containing x. Since A is open, there is a basis element B_n such that x ∈ $B_n \subset$A . Because B_n lies in an element of *A* that index n belongs to the set J. So A_n is defined, Since A_n contains B_n it contains x. Then A' is a countable sub collection of *A* that overs X.
b) From each non-empty basis element B_n choose a point x_n. Let *D* be the set consisting of the points x_n. Then *D* is dense in x. Given any point x of X every basis element containing x intersects *D,* so x belongs to $\overline{D}$.

Hence the Proof.

DEFINITION: Lindel of Space:

A space for which every open covering contains a countable subcovering is called a Lindel of space.

DEFINITION: Separable:

A space having a countable dense subset is often said to be separable.

SECTION: II The Separation Axioms

DEFINITION: Regular and Normal space:

Suppose that one point sets are closed in x.

a) Then x is said to be regular if for each pair consisting of a point x, and a closed set B, disjoint open sets containing x and B respectively.

b) The space $\underline{X}$ is said to be normal if for each pair A, B of disjoint closed sets of $\underline{X}$

There exists disjoint open sets containing A and B respectively.

LEMMA:

Let X be a topological space. Let one-point sets in x be closed

a) X is regular iff given a point x of X and a neighborhood U of x, there is a neighborhood v of x such that $\overline{V} \subset$U.

b) X is normal iff given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset$U.

PROOF:

a) Suppose that X is regular and suppose that the point x and the neighborhood U of x are given

Let B= X-U;

Then B, is a closed set.

By hypothesis, there exist disjoint open sets, V and W containing x and B, respectively. From $\overline{V}$

$$\overline{V} \cap B = \phi$$

For if y$\in$B. W is a neighborhood of y which is disjoint from V.

y$\in$W, W

$\Rightarrow$y$\notin \overline{V}$ ($\therefore$there exists a neighborhood W of y $\rightarrow$ W$\cap V = \phi$)

$$\overline{V} \cap B = \phi \Rightarrow \overline{V} \subset X - B = U$$

$$\therefore \overline{V} \subset U$$

To prove the converse,

Suppose that the point x and the closed set B not containing x are given

Let U=X-B

By hypothesis, there is a neighborhood v of x. Such that $\overline{V} \subset U$.

The open sets V and X- $\overline{V}$ are disjoint open sets containing x and B, respectively.

Then x is regular.

b) Suppose that X is normal and suppose that the closed set A and the neighborhood U of A are given

Let B= X-U

Then B is a closed set.

By hypothesis, there exists disjoint open sets V and W containing A and B, respectively,

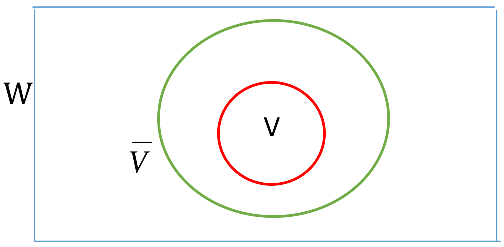
The set $\overline{V}$ is disjoint from B, for if y∈B the set W is a neighborhood of y disjoint from V.

$$\therefore \overline{V} \subset U \text{ as desired}$$

To prove the converse,

Suppose that the closed set A and the closed set B not containing closed set A are given,

Let U= X-B.



By hypothesis, there is a neighborhood V of closed set A such that $\overline{V} \subset U$.

The open sets V and X-$\overline{V}$ are disjoint open sets containing closed sets A and closed set B, respectively.

Thus X is normal.

THEOREM:

a) A subspace of a Hausdorff, a product of Hausdorff space is Hausdorff.

Proof:

i) Let X be Hausdorff

Let x and y be two point of the subspace Y of X.

If U and V are disjoint-neighborhood in X of x and y respectively, then U ∩ Y and V ∩ Y are disjoint neighborhood of x and y in Y.

(ii) Let $\{x_\alpha\}$ be a family of Hausdorff space.

Let X= (x_α) and Y= (y_α) be distinct points of the product space $\prod X_\alpha$. Because x$\neq$ y, there is some index β. Such that $x_\beta \neq y_\beta$

Choose disjoint open sets U and V in x_β containing x_β and y_β respectively,

Then the sets $\prod_\beta^{-1}(U)$ and $\prod_\beta^{-1}(V)$ are disjoint open sets in $\prod x_\alpha$

Containing x and y respectively.

THEOREM:

b) A subspace of a regular space is regular a product of a regular space is regular.

PROOF:

Let Y be a subspace of regular space X. Then one point sets are closed in Y.

Let x be a point of y and let B be a closed subset of y disjoint from x.

Now $\overline{B} \cap Y = B$. Where $\overline{B}$ denotes the closure of B in x.

Therefore, x$\notin \overline{B}$, so using regularity of x. We can choose disjoint open sets U and V of X containing x and $\overline{B}$ respectively.

Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively.

ii) Let $\{x_\alpha\}$ be a family of regular space.

Let X= $\prod X_\alpha$ (By a) X is Hausdorff. So, that one point sets are closed in X. We use the preceding lemma to prove regularity of X.

Let X= (x_α) be a point of X and let U be a neighborhood of x in X. Choose a basis element $\prod U_\alpha$ about x contained in U. Choose, for each α, a neighborhood V_α of x_α

in X_α such that $\overline{V}_\alpha \subset U_\alpha$, if it happens that U_α=X_α, choose V_α= X_α. Then V= $\prod V_\alpha$ is a

neighborhood of x in X. Since $\overline{V} = \prod \overline{V}_\alpha$ by theorem, Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$$

It follows at once that $\overline{V} \subset \prod U_\alpha \subset U$ so that X is regular.

Normal Space:

THEOREM:

Every regular space with a countable basis is normal.

PROOF:

Let X be regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of x. Each point x of A has a neighborhood U not intersecting B using regularity, choose a neighborhood V of x. Whose closure lies in U. Finally, choose an element of $\mathcal{B}$ containing x and contained in V. By choosing such a basis element for each x in A, we construct a countable covering of A by open sets. Whose closure do not intersect B. Since this covering of A is countable. We can index it with the positive integers.

Let us denote it by $\{U_n\}$ similarly, choose a countable collection $\{V_n\}$ of open sets covering B. Such that each set $\overline{V_n}$ is disjoint from A. The sets U= $\bigcup U_n$ and V= U V_n are open sets containing A and B, respectively but they need not be disjoint. We perform the following simple, tricks to construct two open sets that are disjoint.

NOTE:

Regular with countable basis ⇒ Normal

Normal ⇒ Regular ⇒ Hausdorff.

Given n, define

$U_n{}'= U_n - \bigcup_{i=1}^{n} \overline{V_i}$ and $V_n{}'= V_n - \bigcup_{i=1}^{n} \overline{U_i}$ note that each set $U_n{}'$ is open being the difference of open set U_n and a closed set $\bigcup_{i=1}^{n} \overline{V_i}$. Similarly, each set $V_n{}'$ is open.

The collection $\{ U_n{}'\}$ covers A, because each x in A belongs to U_n for some n, and x belongs to none of the sets $\overline{V_i}$.

Similarly, the collection $\{V_n{}'\}$ covers B.

Diagram

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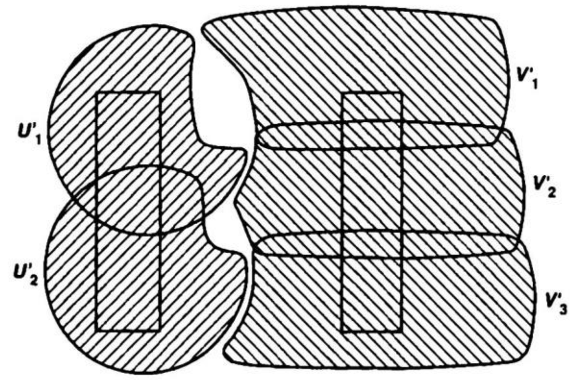

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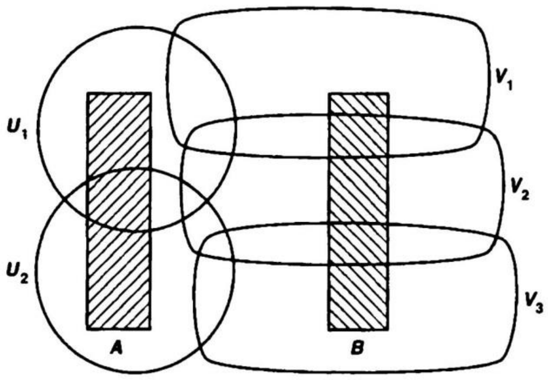

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Finally the open sets

$U'= \bigcup_{n \in Z_+} U_n{}'$ and $V'= \bigcup_{n \in Z_+} V_n{}'$ are disjoint. For if x$\in U' \cap V'$, then

x$\in U'_j \cap V'_k$ for some j and k.

Suppose that j$\leq$k.

If follows from the definition of U'_j that x$\in U_j$ and since j$\leq$k it follows from the definition of V'_k that x$\notin \overline{U_j}$.

A similar contradiction arises if $j \geq$ k.

THEOREM:

Every metrizable space is normal.

PROOF:

Let X be metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each a$\in$A,

Choose ε_a so that the ball B (a, ε_a) does not intersect B.

Similarly, for each b$\in$B. choose ε_b so that the ball B(b, ε_b) does not intersect.

Define

$$U= \bigcup_{a \in A} B\ (a, \varepsilon_a/2) \quad \text{and}$$

$$V= \bigcup_{b \in B} B\ (a, \varepsilon_b/2)$$

Then U and V are open sets A and B respectively, we assert they are disjoint for if z∈U∩V, then

z∈B(a, $\varepsilon_a/2$) ∩ B (a, $\varepsilon_b/2$) for some a∈A and some b∈B. The triangle inequality applies to show that

$$d\ (a, b) < (\varepsilon_a + \varepsilon_b)/2$$

If $\varepsilon_a \leq \varepsilon_b$, then d(a, b)< ε_b so that the ball B(b, ε_b) contains the point a. If $\varepsilon_b \leq \varepsilon_a$, then d(a, b)< ε_a so that the ball B (a, ε_a) contains that b. Neither situation is possible.

THEOREM:

Every well ordered set X is normal in the order topology.

PROOF:

Let X be a well ordered set

To prove: X is normal

First, we need to prove that, every interval of the form.

(x, y] is open in X.

If X has the largest element and y is that element.

⇒(x, y] is just a basis element about y. If y is not largest element of X.

Then (x, y] = (x, y'] where y' is the immediate success of y.

∴ The interval of the form (x, y] is open in X.

Let A and B be disjoint closed sets in X. Assume that neither A nor B containing the smallest element a_0 of X for each a∈A, there exist a basis element about 'a' disjoint from B.

⇒ a∈U⊂A and U∩B = ∅ for each b∈B, ∃ a basis element about 'b' disjoint from A.

⇒ b∈U⊂B and V∩A = ∅. Here, U= $\bigcup_{a\in A}(x_a, a]$ and V= $\bigcup_{b\in B}(y_b, b]$ i.e,) U and V are open sets containg A and B respectively.

Next, we have to prove

U∩V=∅ (i.e, they are disjoint) Suppose z ∈ U∩V

⇒ z∈ $(x_a, a] \cap (y_b, b]$ for some a∈A and b∈B.

Assume that, a<b intervals if a$\leq y_b$ ⇒(The two intervals are disjoint) $(x_a, a]$ and $(y_b, b]$ are disjoint while a>y_b ⇒ a ∈$(y_b, b]$ ⇒ a ∈B which is a contradiction (∵ a∈A)

Similarly for b< a ⇒contradiction.

Finally, ∴ U∩V=∅

Assume that A & B are disjoints sets in X and A containing the smallest element of $a_0$of X.

The set $\{a_0\}$ is both open and closed in X.

∴∃ disjoint open sets U and V containing the closed sets. A-$\{a_0\}$ and B respectively then, U ∪ $\{a_0\}$ & V are disjoint open sets containing A and B respectively.

THEOREM:

Every compact Hausdorff space is normal.

PROOF:

Let X be a compact Hausdorff space.

We have already essentially proved that X is regular.

For if x is a point of X and B is closed set in X not containing x, then B is compact.

By theorem,

Every closed subspace of a compact space is compact.

Applying lemma,

"If Y is compact subspace of a Hausdorff space X and x_0 is a point not in Y then there exist open sets U and V containing x_0 and Y respectively."

∴There exist open sets containing x and B respectively.

∴X is regular.

Essentially the same argument as given in that lemma can be used to show that

X is normal.

Given disjoint closed sets A and B in X.

Choose for each point "a" of A, disjoint open sets U_a and B containing a and B respectively.

The collection { U_a} covers A, because A is compact.

A may be covered by finitely many sets $U_{a_1}, U_{a_2}, \ldots U_{a_n}$

Then U=$U_{a_1} \cup \ldots \ldots \ldots \cup U_{a_n}$ and

V= $V_{a_1} \cap \ldots \ldots \ldots . . \cap V_{a_m}$ are disjoint open sets containing A and B respectively.

Hence X is normal.

SECTION: III URYSOHN lemma:

THEOREM: 10

Let X be a normal space, let A and B be disjoint closed subsets of X. Let [a, b] be a closed intervals in the real line. Then there exist a continuous map.

f: X → $[a, b]$ such that

f(x) = a for every x in A and f(x)=b for every x in B.

PROOF:

We consider the particular interval [0, 1] to prove the lemma,

First we construct a certain family U_p of open sets of X indexed by rational numbers. We can use the sets to define the continuous function f.

STEP 1: Let P be the set of all rational numbers in the interval [0,1]. We shall define, for each p in P, an open set U_p of x, in such a way that whenever p<q, we have $\overline{U_p} \subset U_q$.

Thus, the sets U_p will be simply ordered by inclusion in the same way their subscripts are ordered by the usual ordering in the real line.

Because P is countable, we can use induction to define the sets U_p (or rather, the principle of reccursive definition. U_1= X-B (including α)

Arrange the elements of the sequence infinite sequence in some way for convenience. Let us suppose that the numbers 1 and 0 are the first two elements of the sequence.

Now define the sets U_p first define $U_1 = X - B$ second, because A is a closed set contained in the open set U_1 we may be normality of x choose an open set U_0 such that

A ⊂ U_0 and $\overline{U_0} \subset U_1$

In general, let P_n denote the set consisting of the first n rational numbers in the sequence. Suppose that U_p is defined for all rational numbers p belonging to the set P_n satisfying the condition

$$P<q \Rightarrow \overline{U_p} \subset U_q \longrightarrow (*)$$

Let r denote the next rational number in the sequence; we wish to define U_r.

Consider the set $P_{n+1}= P_n \cup\{r\}$. It is a finite subset of the interval [0, 1] and as such it has a simple ordering derived from the usual order relation < on the real line. In a finite simply ordered set, every element (other than the smallest predecessor and the largest) has an immediate predecessor and an immediate successor.

“ **Theorem 10.1.** Every nonempty finite ordered set has the order type of a section $\{1, \ldots, n\}$ of Z+, so it is well-ordered.”

(see theorem 10.1). The number 0 is the smallest element, and 1 is the largest element of the simply ordered set P_{n+1}, and r is neither 0 nor 1. So r has an immediate predecessor p in P_{n+1} and an immediate successor q in P_{n+1}. The sets U_p and U_q are already defined and $\overline{U_p} \subset U_q$ by the induction hypothesis usinf normality of X. We can find an open set U_r of X such that

P < r < q

$$\overline{U_p} \subset U_r \text{ and } \overline{U_r} \subset U_q$$

We assert that $(*)$ now holds for every pair of elements of P_{n+1}. If both elements lie in P_n, $(*)$ holds by the induction hypothesis. If one of them is r and the other is a point s of P_n, then either s $\leq p$, in which case,

$$\overline{U_s} \subset \overline{U_p} \subset U_r,$$

or s ≥ q, in which case

$$\overline{U_r} \subset U_q \subset U_s .$$

Thus for every pair of elements of P_{n+1}, relation $(*)$ holds.

By induction, we have U_p defined for all p∈P.

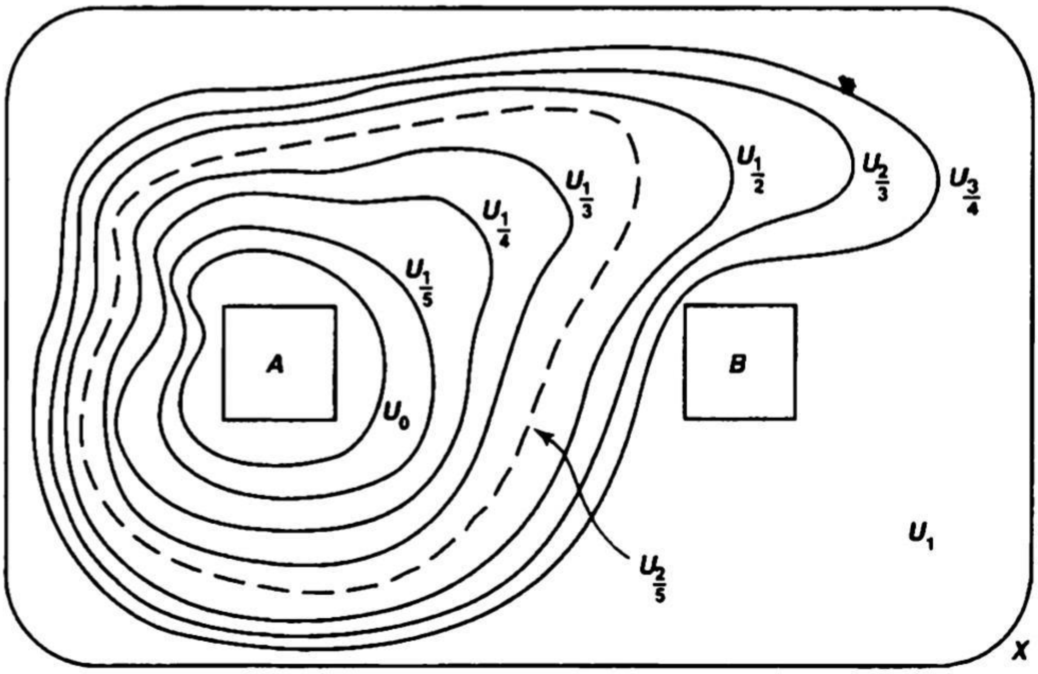
To illustrate let us suppose we started with the standard way of arranging the elements of P in an infinite sequence.

P= {1, 0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5,…………..}.

After definition U_0 and U_1 we would define $U_{1/2}$ so that $\overline{U_0} \subset U_{1/2}$ and $\overline{U_{1/2}} \subset U_1$.

Thus we would fit in $U_{1/3}$ between U_0 *and* $U_{1/2}$ and $U_{1/3}$ between $U_{1/2}$ and U_1. And so on.

At the eight step of the proof we would have the situation pictured in figure. And the ninth step would consist of choosing an open set $U_{2/5}$ to fit in between $U_{1/3}$ and $U_{1/2}$. And so on.



STEP 2: Now we have defined U_p for all rational numbers P in the interval [0, 1] we extend this definition to all rational numbers P in R, by defining,

$U_p = \emptyset$ if p<0

$U_p = X$ if p<1

It is still true (as you can check) that for any pair of rational numbers p and q.

P < q $\Rightarrow$ $\overline{U_p} \subset U_q$

STEP3: Given a point x of X. Let us define $\mathbb{Q}(x)$ be the set of those rational numbers p. Such that the corresponding open sets U_p contains x.

$\mathbb{Q}(x) = \{p/ x \in U_p\}$

This set contains no number less than 0, since no x is in U_p for p < 0. And it contains every number greater than 1. Since every x is in U_p for p>1.

$\therefore$ $\mathbb{Q}(x)$ is bounded below, and greatest lower bound is a point of is [0,1]. Define

$$f(x)= \inf \mathbb{Q}(x)$$

$$= \inf \{p/x \in U_p\}$$

<u>STEP4:</u> We show that f is the desired. If x∈A, then $x \in U_p$ for every p≥0. So that $\mathbb{Q}(x)$ equals the set of all non-negative rational, and f(x)= inf $\mathbb{Q}(x)$. Similarly, if x∈B, then $x \in U_p$ for no p≤1, so that $\mathbb{Q}(x)$ consists of all rational numbers greater than 1 and f(x)=1.

All this is easy. The only hard part is to show that f is continuous.

For this purpose, we first prove the following elementary facts:

(1) $x \in \overline{U_r} \Rightarrow f(x) \leq r$

(2) $x \notin U_r \Rightarrow f(x) \geq r$

To prove: (1) note that if $x \in \overline{U_r}$ then $x \in U_s$ for every s>r

$\therefore$ $\mathbb{Q}(x)$ contains all rational numbers greater than r, so that by definition we have f(x)= inf $\mathbb{Q}(x) \leq r$

To prove (2), note that $x \notin U_r$ then x is not in U_s for any s<r.

$\therefore$ $\mathbb{Q}(x)$ contains no rational numbers less than r, so that $x \in \overline{U_r}$, r<s,

$\overline{U_r} \leq U_s$.

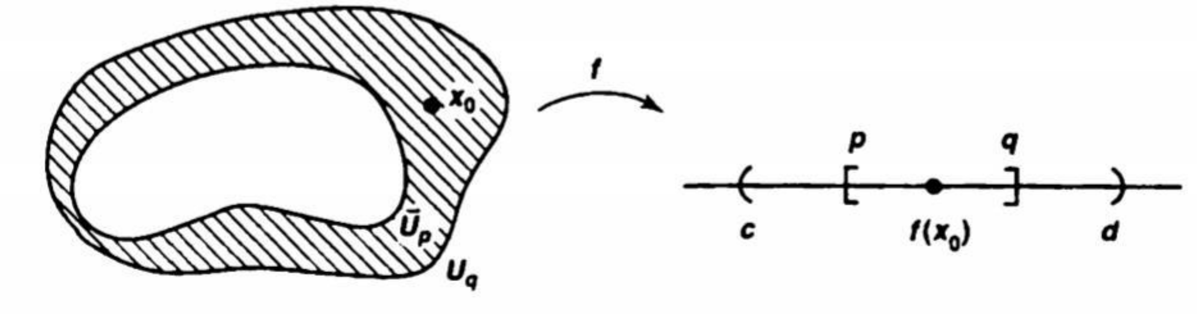
$$f(x)= \inf \mathbb{Q}(x) \geq r$$

Now we prove continuity of f given a point x_0 of X and an open intervals (c, d) in $\mathbb{R}$ containing the point f(x_0). We wish to neighborhood U of x_0 such that f(x) ≤ r

$$f(U) \subset (c, d)$$

Choose rational numbers p and q such that $c < p < f(x_0) < q < d$.

We assert that the open set U = $U_q - \overline{U_p}$ is desired neighborhood of x_0



First, we note $x_0 \in$ U. For the fact that f(x_0) <q implies by condition (2) that $x_0 \in U_q$, while the fact that f(x_0)>p implies by (1) that $x_0 \notin \overline{U_p}$

Second, we show that f(U) ⊂ (c, d). Let x∈U. Then x∈ $U_q \subset \overline{U_q}$. So that f(x) ≤q by (1)

And x $\notin \overline{U_p}$. So that x $\notin$ U_q and f(x) ≥ p, by (2)

Thus, f(x)∈[p, q] ⊂(c, d) as desired

Separated by a continuous function

If A and B are two subsets of the topological space X, and if there is a continuous function f: X→[0, 1] such that f(A)= {0} and f(B)={1}. We say that A and B can be separated by a continuous function.

Completely regular:

A space X is completely regular if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function f: X→ [0, 1] such that f(x_0)=1 and f(A)=0.

THEOREM:

(i) A subspace of a completely regular space is completely regular.

PROOF:

Let X be completely regular, let Y be a subspace of X. Let x_0 be a point of Y, and let A be a closed set of Y disjoint from x_0.

Now A= $\overline{A}$ ∩Y

When $\overline{A}$ is the closure of A in X.

$\therefore x_0 \in \overline{A}$ Since x is completely regular we can choose the continuous function.

f: X→ [0, 1] such that

f(x_0)=1 and f($\overline{A}$)={0}

The restriction of f to Y i.e, f_0i which s the continuous function Y→[0, 1]

$$f_0\text{i} = \text{Y} \rightarrow [0, 1]$$

(f_0i) (x_0) = f(x_0)= 1 and

(f_0i)(A)= -f(A)= f($\overline{A}$)= {0}   (A⊂ $\overline{A}$)

Y is completely regular space.

(ii) A product of completely regular space is completely regular.

PROOF:

Let X= $\prod_{\alpha \in \tau} X_\alpha$ be the product of completely regular space.

Let b= $(b_\alpha)_{\alpha \in \tau}$ be a point of X, disjoint from b, choose a basis element.

$\prod U_\alpha \subset b$ that does not intersect A, then $U_\alpha = X_\alpha$ except for finitely many α.

Say α= $\alpha_1, \alpha_2, \ldots, \alpha_n$ given i =1, 2 . . . n

Choose a continuous function

$f_i: X_{\alpha_i} \rightarrow [0, 1]$ such that $f_i\left(b_{\alpha_i}\right)$ =1 and $f_i(X_\alpha - U_{\alpha_i})$={0}

Let φ_i(x) = $f_i\ (\pi_{\alpha_i}(x))$

Then $\varphi_i: X \rightarrow R$ and vanishes outside

$\pi_{\alpha_i}{}^{-1}\ (U_{\alpha_i}) = \pi U_\alpha$

[∵φ_i(X- πU_α)= $f_i(\pi_{\alpha_i}\ (\text{X} - \pi U_\alpha))$

$= f_i\ (X_{\alpha_i} - U_{\alpha_i}) = \{0\}$]

i.e., φ_1=0 if $\overline{x} \in \pi U_\alpha$

φ_2=0 outside πU_α

φ_i : X→ R vanishes outside $\prod U_\alpha$.

Let f($\overline{x}$)= $\varphi_1(\overline{x})\ \varphi_2(\overline{x}) \ldots \varphi_n(\overline{x})$

This product is continuous on X and f ($\overline{b}$) = $\varphi_1(\overline{b})\ \varphi_2(\overline{b}) \ldots \varphi_n(\overline{b})$

$$= f_1\left(\pi_{\alpha_1}(\overline{b})\right) f_2\left(\pi_{\alpha_2}(\overline{b})\right) \ldots f_n(\pi_{\alpha_n}(\overline{b}))$$

$= f_1\ (b\ \alpha_1)\ f_2(b\alpha_2)\ \ldots\ f_n(b\alpha_n)$

$= 1.1.\ \ .\ .\ .\ 1$

$f\ (\overline{b}) = 1$

$f(X - \prod U_\alpha) = \varphi_1(X - \prod U_\alpha)\ \varphi_2(X - \prod U_\alpha) .\ .\ .\ \varphi_n(X - \prod U_\alpha)$

$= f_1\left(x_{\alpha_1} - U_{\alpha_1}\right) f_2\left(\pi_{\alpha_2} - U_{\alpha_2}\right) .\ .\ .\ f_n(\pi_{\alpha_n} - U_{\alpha_n})$

$= 0.\ 0\ .\ \ .\ \ .\ \ .\ 0$

$f(X - \pi U_\alpha) = 0$ i.e., $f(A)=0$ $\quad (\because \subset (X - \pi U_\alpha))$.

THEOREM:

URYSOHN METRIZATION THEOREM:

Every regular space X with a countable basis is meterizable.

PROOF:

It is enough to prove that X is metrizable by imbedding.

X in a metrizable space Y.

(i.e.,). By showing X homomorphic with a subspace of Y.

Two version:

First version:

Y is $\mathbb{R}^\omega$ which is metrizable in the product topology (refer theorem 20.5)

Theorem 20.5. *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points of $\mathbb{R}^\omega$, define*

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Second version:

Y is $\mathbb{R}^\omega$ which is metrizable but the topology induces by uniform metric ….. (Uniform topology)

i.e., Our construction imbeds X in the subspace [0, 1] of $\mathbb{R}^\omega$

STEP: 1

We prove that there exist a countable collection of continuous function f_n : X→ [0, 1] having the property that given any point x_0 of X and any neighborhood U of x_0. ∃ index n, show that f_n is $(+ve)$ of x_0 and vanishes outside U given x_0 and U there exist such a function (f_n). If we choose one such function for each pair (x_0, U). The resulting collection will not in general be countable

Claim: f_n is countable.

Let $\{B_n\}$ be a countable basis for X, given $x_0 \in$X, ∃ a neighborhood U of x_0 and for each pair n, m of indices.

We can choose B_m (basis element) containing x_0 and that is contained in U.

Using regularity we can choose B_n such that $x_0 \in B_n$ and $\overline{B_n} \subset B_m$.

Applying Urysohn lemma we choose an continuous function.

$g_{n,m}$: X → [0, 1]

$g_{n,m}(\overline{B_n}) = \{1\}$

$g_{n,m}(X - B_m) = \{0\}$

Since [If X is normal iff it can be separated by a continuous function.]

Then n, m is a pair of indices for X which the function $g_{n,m}$ is defined which is positive at x_0 and vanishes outside U.

By the result $(*)$

(Any subset of $Z_+ \times Z_+$is countable)

The collection of $\{ g_{n,m}\}$ is indexed with the subset of $Z_+ \times Z_+$

By this result $(*)$. It is countable.

It can be reindexed with the +ve integers giving the desired collection {fn}

STEP 2:

Assume eqn (1) and take $\mathbb{R}^{\omega}$ in the product topology and define

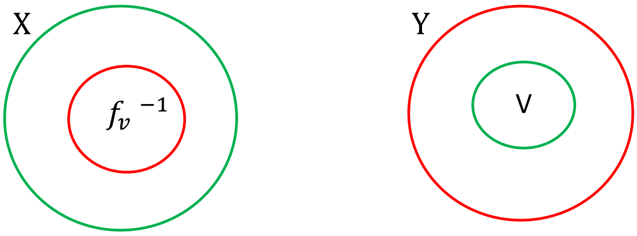
F: X → $\mathbb{R}^{\omega}$ such that F(x)= $(f_1$(x), f_2(x), . . . , f_n(x))

[$f_n : X \rightarrow [0, 1]$; f_n(x)$\in \mathbb{R}$]

CLAIM:

F is imbedding (injective continuous map and Homeomorphisms).

We will first show that F is continuous, since $\mathbb{R}^{\omega}$ has the product topology and each f_n is continuous. [Basis elements generates topology basis element are open].



$\therefore$ F is continuous.

Next we will show that F is injective (1, -1)

Consider 2 distinct points x ≠ y

To prove F(x)≠ F(y)

For an index n, $f_n\ (x)$ >0 and $f_n\ (y) = 0$

$\therefore$ F(x)≠ F(y)

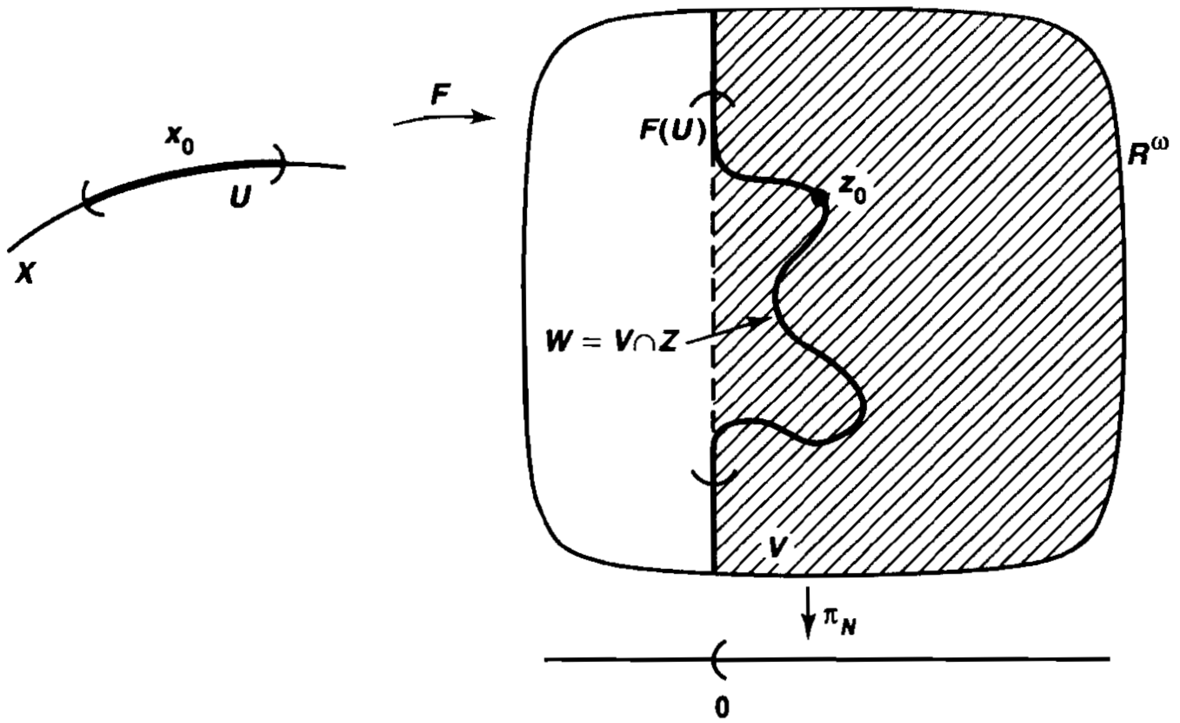
Finally we will show that F is homeomorphism. We must prove that F is a homeomorphism F X onto its image, the subspace Z = F(x) of $\mathbb{R}^{\omega}$. We know that F define a continuous bijection of X with Z. so we need only show that for each open set U in X, the set F(U) is open in Z. let z_0 be a point of F(U). We shall find an open set W of Z such that

$$z_0 \in W \subset F(U).$$

Let x_0 be the point of U such that $F(x_0) = z_0$. Choose an index N for which $f_N(x_0) > 0$ and $f_N(X - U) = \{0\}$. Take the open ray (0,+∞) in R, and let V be the open set

$$V = \pi_N^{-1}((0, +\infty))$$

of $\mathbb{R}^{\omega}$. Let $W = V \cap Z$; then W is open in Z, by definition of the subspace topology.



By the fig. We assert that $z_0 \in W \subset F(U)$. First, $z_0 \in W$ because $\pi_N(z_0) = \pi_N(F(x_0)) = f_N(z_0) > 0$.

Second, $W \subset F(U)$. For if $z \in W$, then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0,+\infty)$. Since $\pi_N(z) = \pi_N(F(x)) = f_N(x)$, and f_N vanishes outside U, the point x must be in U. Then $z = F(x)$ is in $F(U)$, as desired. Thus F is an imbedding of X in $\mathbb{R}^\omega$.

Step 3 *(Second version of the proof)*.

In this version, we imbed X in the metric space $(\mathbb{R}^\omega, \bar{\rho})$. Actually, we imbed X in the subspace $[0, 1]^\omega$, on which $\bar{\rho}$ equals the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i|\}.$$

We use the countable collection of functions $f_n : X \to [0, 1]$ constructed in Step 1. But now we impose the additional condition that $f_n(x) \leq 1/n$ for all x. (This condition is easy to satisfy; we can just divide each function f_n by n.)

Define $F : X \to [0, 1]^\omega$ by the equation

$$F(x) = (f_1(x), f_2(x), \dots)$$

as before. We assert that F is now an imbedding relative to the metric ρ on $[0, 1]^\omega$. We know from Step 2 that F is injective. Furthermore, we know that if we use the product topology on $[0, 1]^\omega$, the map F carries open sets of X onto open sets of the subspace

Z = F(X). This statement remains true if one passes to the finer (larger) topology on $[0, 1]^{\omega}$ induced by the metric ρ.

The one thing left to do is to prove that F is continuous. This does not follow from the fact that each component function is continuous, for we are not using the product topology on $\mathbb{R}^{\omega}$. now. Here is where the assumption f_n (x) ≤ 1/n comes in.

Let $x_0$be a point of X, and let $\varepsilon > 0$. To prove continuity, we need to find a neighborhood U of x_0 such that

$$x \in U \Rightarrow \rho(F(x), F(x_0)) < \varepsilon.$$

First choose *N* large enough that $1/N \le \varepsilon/2$. Then for each *n* = 1, . . . , *N* use the continuity of f_n to choose a neighborhood *Un* of *x*0 such that

$$| f_n (x) - f_n (x0)| \le \varepsilon/2$$

for $x \in f_n$. Let $= U_1 \cap \dots \dots . . \cap U_N$; we show that *U* is the desired neighborhood of x_0. Let $x \in U$. If $n \le N$,

$$| f_n (x) - f_n (x0)| \le \varepsilon/2$$

by choice of *U*. And if *n* > *N*, then

$$| f_n (x) - f_n (x0)| \le {}^1/_N \le \varepsilon/2$$

because f_n maps *X* into [0, 1/*n*]. Therefore for all *x* ∈ *U*,

$$\rho(F(x), F(x0)) \le \varepsilon/2 < \varepsilon,$$

as desired.