Order relation (simple order or linear order)

A relation C on a set A is called an order relation If it has the following properties

(i) Comparability : for every x and y in A for which $x \neq y$, either $x C y$ or $y C x$

(ii) Non reflexivity : for no x in A does the relation $x C x$ holds

In otherwords, $x \not C x$ for any $x \in A$

(iii) Transitivity : If $x C y$ and $y C z$, then $x C z$

Partial Order

If a relation R on A has the following properties

(i) Reflexivity : $a R a \quad \forall a \in A$

(ii) Anti symmetry : $a R b$ & $b R a \Rightarrow a = b$

(iii) Transitivity : $a R b$, $b R c \Rightarrow a R c$

then it is called a partial order relation on A.

Strict partial Order

Given a set A, a relation $<$ on A is called a strict partial order on A If it has the following two properties

(i) Non reflexivity : The relation $a < a$ never holds

(ii) Transitivity : If $a < b$, $b < c$, then $a < c$

Note :

If $<$ is a strict partial order on a set A, it can easily happen that some subset B of A is simply ordered by the relation; all that is needed is for every pair of elements of B to be comparable under $<$

upper bound and Maximal element

let A be a set and $<$ be a strict partial order on A. If B is a subset of A, an upper bound on B is an element c of A such that the every b in B either $b = c$ or $b < c$. A maximal element of A is an element m of A such that for no element a of A does the relation $m < a$ hold.

Zorn's lemma

let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

The Tychonoff Theorem

lemma :-

let x be a set. let $\mathcal{A}$ be a collection of Subsets of X having the finite intersection property. Then there exist a collection $\mathcal{D}$ of Subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, and $\mathcal{D}$ has the finite intersection property and no collection of Subset of X that property contains $\mathcal{D}$ has this property.

proof:

Notation:

Small letter c is an element of X

capital letter C denotes a Subset of X

Script letter $\mathcal{C}$ denotes a collection of subsets of X

$\mathbb{C}$ denotes a superset whose elements are collections of subsets of X

by given, $\mathcal{A}$ is a collection of Subsets of X that has finite intersection property.

let $\mathbb{A}$ denote the superset consisting of all collections $\mathcal{B}$ of subsets of X. such that $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{B}$ has finte intersection property.

let us consider the proper inclusion $\subsetneq$ as our strict partial order on $\mathbb{A}$

claim : $\mathbb{A}$ has a maximal element $\mathcal{D}$

let $\mathbb{B}$ be the sub superset of A that is simply ordered by proper inclusion

First, let us prove this $\mathbb{B}$ has an upper bound in $\mathbb{A}$

The collection $\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$

which is the union of the collections belonging to $\mathbb{B}$, is an element of $\mathbb{A}$ and it is the required upper bound on $\mathbb{B}$

for,

claim $\mathcal{C} \in \mathbb{A}$

certainly $\mathcal{C}$ contains $\mathcal{A}$, since each element of $\mathbb{B}$ contains $\mathcal{A}$.

Next, to show that $\mathcal{C}$ has the finite intersection property.

let $C_1, C_2, \ldots C_n$ be elements of $\mathcal{C}$.

$C_i \in \mathcal{C} \Rightarrow C_i \in \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$

$\Rightarrow C_i \in \mathcal{B}_i$, for some $\mathcal{B}_i \in \mathbb{B}$

that is, for each i, $\exists\ \mathcal{B}_i \in \mathbb{B} \ni$

$C_i \in \mathcal{B}_i$

The superset $\{B_1, B_2, \ldots B_n\} \subseteq B$,
So it is simply ordered, by the relation of proper inclusion. Being finite, it has a largest element. that is $\exists$ an index k Such that

$$B_i \subset B_k \text{ for } i = 1, 2, \ldots n$$

Then $C_i \subset B_k \ \forall \ i = 1, 2, \ldots n$

since B_k has the finite intersection property, $\bigcap_{i=1}^{n} C_i \neq \phi$

That is $\mathcal{C}$ has finte intersection property.

$\therefore \mathcal{C} \in \mathbb{A}$

since $B \subset \mathcal{C} \ \forall \ B \in \mathbb{B}$, $\mathcal{C}$ is the upper bound for B, in $\mathbb{A}$

$\therefore$ By zorn's lemma. $\mathbb{A}$ has the maximal elements say $\mathcal{D}$.

hence the proof.

---

lemma:

let X be a set. let $\mathcal{D}$ be the Collection of subsets of X that is maximal with respect to the finite intersection property. Then

(a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Proof:

(a) let B equal the intersection of finitely many elements of $\mathcal{D}$.

let $\mathcal{E} = \mathcal{D} \cup \{B\}$

Claim $\mathcal{E}$ has finite intersection property

Take finitely many elements of $\mathcal{E}$.
If none of them is B, then their intersection is non-empty. because $\mathcal{D}$ has finite intersection property. If one of them is B then their intersection is of the form

$$D_1 \cap D_2 \cap \dots D_m \cap B$$

Since B equals a finite intersection of elements of $\mathcal{D}$, this set is non-empty. Thus $\mathcal{E}$ has finite intersection property.

Since $\mathcal{D}$ is maximal, w.r.t this property

$$\mathcal{E} = \mathcal{D}$$

$$\Rightarrow B \in \mathcal{D}.$$

hence finite intersection of elements of $\mathcal{D}$ is in $\mathcal{D}$.

(b) let A be a subset of X that intersects every element of $\mathcal{D}$.

Claim $A \in \mathcal{D}$

let $\mathcal{E} = \mathcal{D} \cup \{A\}$

Consider, finitely many elements of $\mathcal{E}$. If none of them is the set A, their intersection is automatically non-empty.

otherwise it is of the form

$D_1 \cap D_2 \cap \dots \cap D_n \cap A$

Now, $D_1 \cap D_2 \cap \dots D_n$ belongs to $\mathcal{D}$ by (a)

let it be $\mathcal{D}_*$. As A intersects every element of $\mathcal{D}$,

$A \cap \mathcal{D}_* \neq \phi$

$\therefore$ $\mathcal{E}$ has finite intersection property

As $\mathcal{D}$ is maximal.

$\mathcal{E} = \mathcal{D}$.

ie $A \in \mathcal{D}$

hence proved the theorem.

Theorem Tychonoff theorem

Box topology :

Let $\{X_\alpha\}_{\alpha \in J}$ let an indexed family of topological spaces. let us take as basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

where U_α open in X_α for each $\alpha \in J$

The topology generated by this basis is called the box topology.

Subbasis formulation

let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$

be a function assigning to each element of the product space its β-th co-ordinate

$\pi_\beta ((x_\alpha)_{\alpha \in J}) = x_\beta$

It is called the projection mapping associated with the index β

Theorem : let X be a topological space. then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is non-empty.

Defn: product topology and product space

let S_β denote the collection

$$S_\beta = \{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \}$$ and

S denote the union of these collections,

$$S = \bigcup_{\beta \in J} S_\beta$$

The topology generated by the subbasis S is called the product topology. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Comparison of the box topology and product topology

The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α open in X_α. for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α open in X_α for each α and U_α equals X_α except for finitely many values of α.

Tychonoff theorem

An arbitrary product of compact space is compact is the product topology.

Proof:

let $X = \prod_{\alpha \in J} X_\alpha$

where each space X_α is compact.

claim : X is compact.

to prove

ie, for every collection of $\mathcal{C}$ of closed subsets of X, with finite intersection property $\bigcap_{C \in \mathcal{C}} C \neq \phi$

let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property

To prove $\bigcap_{A \in \mathcal{A}} \bar{A} \neq \phi$

by lemma 1, $\exists$ a collection $\mathcal{D}$ of subsets of X each $\mathcal{D} \supset \mathcal{A}$ and $\mathcal{D}$ is maximal w.r.t finite intersection property.

It is enough to prove that $\bigcap_{D \in \mathcal{D}} \bar{D} \neq \phi$

Given $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map. consider the collection $\{ \pi_\alpha (D) / D \in \mathcal{D} \}$ of subsets of X_α

This collection has the finite intersection property because $\mathcal{D}$ does. By compactness of X_α,

$$\bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)} \neq \phi$$

let $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$ for each α where $x_\alpha \in X_\alpha$

Now consider, $x = (x_\alpha)_{\alpha \in J} \in X$

<u>Claim</u> $x \in \bigcap_{D \in \mathcal{D}} \bar{D}$

ie, $x \in \bar{D} \quad \forall D \in \mathcal{D}$

$x = (x_\alpha)_{\alpha \in J}$. let U_β be a neighbourhood of x_β in X_β. And we know

$$x_\beta \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\beta(D)}$$

ie $x_\beta \in \overline{\pi_\beta(D)} \quad \forall D \in \mathcal{D}$

by definition, $U_\beta \cap \pi_\beta(D) \neq \phi \quad \forall D \in \mathcal{D}$

let $\pi_\beta(y) \in U_\beta \cap \pi_\beta(D)$ where $y \in D$

then $y \in \pi_\beta^{-1}(U_\beta)$ and $y \in D \quad \forall D \in \mathcal{D}$

ie Every sub basis element $\pi_\beta^{-1}(U_\beta)$ containing x intersects every element of $\mathcal{D}$. by lemma 2 (b), every sub basis element belongs to $\mathcal{D}$. since finite intersection of sub basis elements is a bases element, every basis element containging x belongs to $\mathcal{D}$ by lemma 2(a)

Since $\mathcal{D}$ has finite intersection property. every bases element containing x intersects every element of $\mathcal{D}$.

$$x \in \bar{D} \quad \forall D \in \mathcal{D}$$

hence the proof.

Complete Metric spaces

Topological Convergence

In an arbitrary topological space a Sequence $x_1, x_2, \ldots x_n$ of points of the space X converges to a point n of X provided that, corresponding to each neighbourhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$

Def: Cauchy Sequence

let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a cauchy sequence in (X, d)
If it has the property that given $\epsilon > 0$ there is an integer N such that

$d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$

The metric space (X, d) is said to be complete if every cauchy sequence in X converges.

Result :-

A closed subset of a complete metric space is complete in the restricted metric.

proof:

let A be a closed subset of a complete metric space X.

let $\{x_n\}$ be a cauchy sequence in A

$\Rightarrow \{x_n\}$ is a cauchy sequence in X

$\Rightarrow \{x_n\} \rightarrow x$ (say) in X, as X is complete

$\Rightarrow x$ is a limit point of A

and $x \in A$, $\because A$ is closed.

ie, $\{x_n\} \rightarrow x$ in A

$\therefore A$ is complete.

Theorem:-

let X be a metric space with metric d. Define $\bar{d} : X \times X \rightarrow R$ by the equation

$\bar{d}(x, y) = \min \{d(x, y), 1\}$

Then $\bar{d}$ is a metric that induce the same topology as d.

The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Result:

If X is complete under the metric d, then X is complete under the standard bounded metric $\bar{d}$,

$$\bar{d}(x,y) = \min\{d(x,y), 1\}$$

Corresponding to d.

proof:

This follows from the fact that

A sequence $\{x_n\}$ is a cauchy sequence under $\bar{d}$ if and only if it is a cauchy sequence under d. and

A sequence $\{x_n\}$ converges under $\bar{d}$ if and only if it converges under d.

Lemma:

A metric space X is complete if every cauchy sequence in X has a convergent subsequence

proof:

let (x_n) be a cauchy sequence in (X,d):

Given $\varepsilon > 0$, $\exists$ N large enough such that

$d(x_n, x_m) < \varepsilon/2$

for all $n, m \geq N$

let (x_{n_i}) be a subsequence of (x_n) that converges.

Choose an integer i large enough that

$n_i \geq N$ and $d(x_{n_i}, x) < \varepsilon/2$

Now $d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x)$

$< \varepsilon/2 + \varepsilon/2$

$= \varepsilon \quad \forall n \geq N$

$\Rightarrow (x_n) \to x$

and so X is Complete.

Theorem :-

Euclidean space $\mathbb{R}^k$ is complelete in either of its usual metrics, the euclidean metric d or the square metric e

proof :

First, to prove $(\mathbb{R}^k, e)$ is complete (x_n) be a cauchy sequence in $(\mathbb{R}^k, e)$

The set $\{x_n\}$ is a bounded subset of (R^k, e)

For if we choose N so that

$$e(x_n, x_m) \leq 1 \text{ for all } n, m \geq N$$

Consider $M = \max\{e(x_1, 0), e(x_2, 0) \ldots\ldots e(x_{N-1}, 0), e(x_N, 0) + 1\}$

This will be an upper bound for $e(x_n, 0)$. for any n,

Thus the points of the sequence (x_n) all lie in the cube $[-M, M]^k$.

being closed and bounded in $\mathbb{R}^k$, $[-M, M]^k$ is compact and this cube is sequentially compact.

$\therefore$ The sequence $\{x_n\}$ has a convergent. subsequence. Then (R^k, e) is complete.

Consider (R^k, d)

We know

$$e(x, y) \leq d(x, y) \leq \sqrt{n}\ e(x, y)$$

A sequence is a cauchy sequence relative to $d \Leftrightarrow$ it is a cauchy sequence relation to e and a sequence converges.

Sequentially Compact:

The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

relative to d if and only if converges relative to e

$\therefore$ since $(\mathbb{R}^k, e)$ is complete

$(\mathbb{R}^k, d)$ is also complete.

product space $\mathbb{R}^\omega$.

Any element of $\mathbb{R}^\omega$ will be of the form $(x_1, x_2, \dots)$

lemma:-

~~let X be the product~~

Theorem:-

let X be a matrizable space. Then the following are equivalent.

(i) X is compact

(ii) X is limit point compact

(iii) X is a sequentially compact.

lemma:-

let X be the product space $X = \prod X_\alpha$; let x_n be a sequence of points of X. Then $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α.

proof:

The projection mapping

$\pi_\alpha : X \to X_\alpha$ is continuous.

$\therefore$ whenever $x_n \to x$ in X,

$\pi_\alpha(x_n) \to \pi_\alpha(x) \quad \forall \alpha \in J$

to prove the converse part,

Suppose $\pi_\alpha(x_n) \to \pi_\alpha(x) \quad \forall \alpha \in J$

to prove $x_n \to x$ in X

let $U = \prod U_\alpha$ be a basis element for X that contains x.

Then for each α for which U_α does not equal to the entire X_α,

U_α is a neighbourhood of $\pi_\alpha(x)$

Choose $N_\alpha \ni \pi_\alpha(x_n) \in U_\alpha \quad \forall n \geq N_\alpha$

let N be the largest of the numbers N_α

Then for all $n \geq N$,

we have $x_n \in U$

ie, $(x_n) \to x$

hence the proved.

---

limit point compact:

A space X is said to be limit point compact if every infinite subset of X has a limit point.

Theorem :-

There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is complete

proof :-

let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard metric on $\mathbb{R}$. let D be the metric on $\mathbb{R}^\omega$ defined by

$$D(x,y) = \sup\left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

Then D induces the product topology on $\mathbb{R}^\omega$.

Claim: $\mathbb{R}^\omega$ is complete under D.

Let $\{x_n\}$ be a cauchy sequence in $(\mathbb{R}^\omega, D)$.

by defn, $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x,y)$

$$\Rightarrow \frac{\bar{d}(\pi_i(x), \pi_i(y))}{i} \leq D(x,y)$$

$\because \pi_i(x) = x_i$ & $\pi_i(y) = y_i$

ie, $\bar{d}(\pi_i(x), \pi_i(y)) \leq i\, D(x,y)$.

for a field i, $\pi_i(x_n)$ is a cauchy sequence in $\mathbb{R}$. So it converges say to a_i ($\mathbb{R}$ is complete)

then the sequence x_n converges to the point $a = (a_1, a_2, \ldots)$ of $\mathbb{R}^\omega$.

hence $\mathbb{R}^\omega$ is complete.

Example for non - complete metric spaces :-

Example 1 :

consider the space $\mathbb{Q}$ of rational numbers in the usual metric

$$d(x,y) = |x-y|$$

The sequence

1.4, 1.41, 1.414, 1.4142, 1.41421, ... of finite decimals converging (in $\mathbb{R}$) to $\sqrt{2}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge (in $\mathbb{Q}$)

Example 2.

consider the open interval $(-1,1)$ in $\mathbb{R}$ in the metric $d(x,y) = |x-y|$

In this space the sequence (x_n) defined by $x_n = 1 - 1/n$ is a cauchy sequence that does not converge.

here $(-1,1)$ is homomorphic to the real line $\mathbb{R}$ and $\mathbb{R}$ is complete in its usual metric. But $(-1,1)$ is not complete.

Thus completeness is not preserved by homeomorphisms.

Completeness is not a topological property.

Defn: uniform metric.

let (Y,d) be a metric space; let $\bar{d}(a,b) = \min\{d(a,b),1\}$ be the standard bounded metric on Y derived from d.

If $x=(x_\alpha)_{\alpha\in J}$ and $y=(y_\alpha)_{\alpha\in J}$ all points of the cartesian product Y^J, let

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha,y_\alpha) \mid \alpha\in J\}$$

$\bar{\rho}(x,y)$ is a metric

(i) Since $\bar{d}(x_\alpha,y_\alpha)$ is a metric,

$$\bar{d}(x_\alpha,y_\alpha)\geq 0$$

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha,y_\alpha) \mid \alpha\in J\}$$
$$\geq 0$$

$$\rho(x,y)=0 \Leftrightarrow \sup\{\bar{d}(x_\alpha,y_\alpha) \mid \alpha\in J\} = 0$$
$$\Leftrightarrow \bar{d}(x_\alpha,y_\alpha) = 0 \ \forall \alpha\in J$$
$$\Leftrightarrow x_\alpha = y_\alpha \ \forall\alpha\in J \quad \because \bar{d} \text{ is a metric}$$
$$\Leftrightarrow x=y$$

(ii) $$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha,y_\alpha) \mid \alpha\in J\}$$
$$= \sup\{\bar{d}(y_\alpha,x_\alpha) \mid \alpha\in J\}$$
$$= \bar{\rho}(y,x)$$

(iii) $\bar{d}(x_\alpha, z_\alpha) \le \bar{d}(x_\alpha, y_\alpha) + \bar{d}(y_\alpha, z_\alpha) \quad \forall \alpha \in J$

$$\le \sup\{\bar{d}(x_\alpha, y_\alpha) / \alpha \in J\}$$

$$+ \sup\{\bar{d}(y_\alpha, z_\alpha) / \alpha \in J\}$$

$$= \bar{\rho}(x, y) + \bar{\rho}(y, z)$$

$\bar{d}(x_\alpha, z_\alpha) \le \bar{\rho}(x, y) + \bar{\rho}(y, z)$

$\Rightarrow \sup\{\bar{d}(x_\alpha, z_\alpha) / \alpha \in J\} \le \bar{\rho}(x, y) + \bar{\rho}(y, z)$

$\Rightarrow \bar{\rho}(x, z) \le \bar{\rho}(x, y) + \bar{\rho}(y, z)$

$\therefore \bar{\rho}$ is a metric on Y^J and it is called the uniform metric on Y^J, corresponding to the metric d on Y.

Note:

since the elements of Y^J are simply functions from J to Y. we could use functional notation for them

In this notation, the definition of uniform metric takes the following form: If $f, g : J \to Y$ then

$$\bar{\rho}(f, g) = \sup\{\bar{d}(f(\alpha), g(\alpha)) / \alpha \in J\}$$

Theorem :

If the space Y is complete in the metric d, then the space Y^J is complete in the metric $\bar{e}$ corresponding to d.

proof:

Since the metric d and its standard bounded metric $\bar{d}$ have the same topology on the given topological space d.

(Y,d) is complete $\Leftrightarrow$ $(Y,\bar{d})$ is complete

let $f_1, f_2 \ldots$ be a cauchy sequence in Y^J, relative to the metric $\bar{e}$.

Then given $\varepsilon > 0$, $\exists N \ni$

$$\bar{e}(f_n, f_m) < \varepsilon/2 \quad \forall n, m \geq N \rightarrow (1)$$

Given α in J, the fact that

$$\bar{d}(f_n(\alpha), f_m(\alpha)) \leq \bar{e}(f_n, f_m) \rightarrow (2)$$

for all n, m means that the sequence $f_1(\alpha), f_2(\alpha) \ldots$ is a cauchy sequence in $(Y, \bar{d})$. Since $(Y,\bar{d})$ is complete, this sequence converges, say to y_α.

let $f: J \rightarrow Y$ be the function defined by $f(\alpha) = y_\alpha$.

Claim $f_n \to f$ in the metric $\bar{e}$

Now, for $\alpha \in J$, $n, m \geq M$

$\bar{d}(f_n(\alpha), f_m(\alpha)) < \epsilon/2$ by ①, ②

Let n and α be fixed. As m becomes very large, $\bar{d}(f_n(\alpha), f(\alpha)) < \epsilon/2$

This holds, for all α in J and $n \geq N$

Therefore,

$$\sup\{\bar{d}(f_n(\alpha), f(\alpha)) / \alpha \in J\} < \epsilon/2$$

ie $\bar{e}(f_n, f) < \epsilon/2 < \epsilon \quad \forall n \geq N$

The sequence $\{f_n\}$ converges relative to $\bar{e}$

$\therefore (Y, \bar{e})$ is complete.

hence the proof.

Definition : Bounded function :

A function $f : X \to Y$ is said to be bounded if its image $f(X)$ is a bounded subset of the metric space (Y, d).

Notation :

Y^X denote the set of all functions $f : X \to Y$ (X topological space)

$\mathcal{C}(X,Y) \subseteq Y^X$ is the set of all continuous functions from $X \to Y$.

Theorem :-

let X be a topological space and (Y,d) be a metric space. The set $\mathcal{C}(X,Y)$ of continuous function is closed in Y^X, under the uniform metric. so is the set $\mathcal{B}(X,Y)$ of bounded functions. Therefore, If Y is complete, these spaces are complete in the uniform metric

proof:

First to show that if a sequence of elements (f_n) of Y^X converges to the element f of Y^X relative to the metric $\bar{\rho}$ on Y^X, then it converges uniformly to f, relative to the metric $\bar{d}$ on Y.

Def: uniform convergence: let $f_n : X \to Y$ be a sequence of functions from the set X to be metric space Y. let d be the metric for Y. we say that the sequence (f_n) converges uniformly to the function $f : X \to Y$ is given $\varepsilon > 0$ there exist an integer N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and x in X.

Note: uniform convergence depends not only on the topology of Y but also on its metric.

Given $\varepsilon > 0$, choose an integer N such that

$\bar{e}(f, f_n) < \varepsilon$

for all $n > N$. Then for all x and all $n \geq N$

$d(f_n(x), f(x)) \leq \bar{e}(f_n, f) < \varepsilon$

Thus (f_n) converges uniformly to f.

Since closed subspace of a complete metric space is complete. It is enough to show that $e(X, Y)$ and $B(X, Y)$ are closed in Y^X.

$e(X, Y)$ is closed in Y^X relative to the metric $\bar{e}$.

let f be an element of Y^X that is a limit point of $e(X, Y)$.

let f be an element of Y^X that is a limit point of $e(X, Y)$. Then there is a sequence (f_n) of elements of $e(X, Y)$, converging to f in the metric $\bar{e}$. by the uniform limit theorem, f is continuous, So that $f \in e(X, Y)$.

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uniform limit theorem

let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f then f is continuous.

$B(X,Y)$ is closed in Y^X

If f is a limit point of $B(X,Y)$ there is a sequence of elements f_n of $B(X,Y)$ converging to f. Choose N so large that $\bar{\rho}(f_N, f) < 1/2$

Then for $x \in X$, we have $\bar{d}(f_N(x), f(x)) < 1/2$

$$\Rightarrow d(f_N(x), f(x)) < 1/2$$

let M be the diameter of $f_N(X)$.

Then $d(f_N(x), f_N(y)) \leq M \quad \forall x, y \in X$

Now $\forall x, y \in X$

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y))$$

$$< \frac{1}{2} + M + \frac{1}{2}$$

$$= M+1$$

$\sup\{d(f(x), f(y)) / x, y \in X\}$ is atmost $M+1$ is the diameter of $f(X)$ is atmost $M+1$

$\Rightarrow f(X)$ is bounded in (Y, d)

$\Rightarrow f: X \to X$ is bounded

ie, $f \in B(X, X)$.

hence the proof.

Definition: let X be a metric space with metric d. A subset A of X is said to be bounded if $\exists M \ni d(a_1, a_2) \leq M \ \forall a_1, a_2 \in A$. If A is bounded and non-empty, the diameter of A is defined as

diameter $A = \sup\{d(a_1, a_2) / a_1, a_2 \in A\}$

Definition:-

If (Y,d) is a metric space, another metric can be defined on the set $\mathcal{B}(X,Y)$ of bounded functions from X to Y

by $\rho(f,g) = \sup\{d(f(x), g(x)) / x \in X\}$

The metric ρ is called the sup metric

ρ is well defined, for the set $f(X) \cup g(X)$ is bounded if both $f(X)$ and $g(X)$ are.

Relation between sup metric and uniform metric.

If $f, g \in \mathcal{B}(X,Y)$ then

$\bar{\rho}(f,g) = \min\{\rho(f,g), 1\}$

On $\mathcal{B}(X,Y)$, the metric $\bar{\rho}$ is just the standard bounded metric derived from the metric ρ.

Topological imbedding or imbedding

let $f: X \to Y$ be the injective continuous map, where X and Y are topological spaces. Then the let $Z = f(X)$, a subspace of Y.

restriction of the range, $f' : X \to Z$ is bijection

If f' happens to be a ~~homeomo~~ homeomorphism of X with Z, then the map $f : X \to Y$ is a topological imbedding on X in Y.

Every metric space can be imbedded isometrically in a complete metric space.