Algebra of sets:-

A collection $\mathcal{Q}$ of subsets of X is called an algebra of sets (or) a Boolean algebra if (i) $A \cup B \in \mathcal{Q}$ whenever $A, B \in \mathcal{Q}$ and
(ii) $\tilde{A} \in \mathcal{Q}$ whenever $A \in \mathcal{Q}$.

Note:- 1) If $\mathcal{Q}$ is an algebra, then by De Morgan's law, $A \cap B \in \mathcal{Q}$ whenever $A, B \in \mathcal{Q}$.

(2) If $A_1, A_2, \dots, A_n \in \mathcal{Q}$, then $A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{Q}$ and $A_1 \cap A_2 \cap \dots \cap A_n \in \mathcal{Q}$.

(3) Given any collection $\mathcal{C}$ of subsets of X, there is a smallest algebra $\mathcal{Q}$ which contains $\mathcal{C}$.

(4) Let $\mathcal{Q}$ be an algebra of subsets and $\langle A_n \rangle$ a sequence of sets in $\mathcal{Q}$. Then there is a sequence $\langle B_n \rangle$ of sets in $\mathcal{Q}$ such that $B_n \cap B_m = \phi$ for $n \neq m$ and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

σ - Algebra of sets:-

An algebra $\mathcal{Q}$ of sets is called a σ- Algebra (or) a Borel field, if every union of a countable collection of sets in $\mathcal{Q}$ is again in $\mathcal{Q}$. i.e) if $\langle A_i \rangle$ is a sequence of sets in $\mathcal{Q}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{Q}$.

Note: (1) By De morgan law, the intersection of countable collections of sets in $\mathcal{Q}$ is again in $\mathcal{Q}$.

(2) Given any collection $\mathcal{C}$ of subsets of X, there is a smallest σ-algebra $\mathcal{Q}$ that contains $\mathcal{C}$. i.e) there is a σ- algebra $\mathcal{Q}$ containing $\mathcal{C}$ $\ni$: if $\mathcal{B}$ is any σ-algebra containing $\mathcal{C}$, then $\mathcal{Q} \subset \mathcal{B}$.

The smallest σ-algebra $\mathcal{Q}$ containing $\mathcal{C}$ is called the σ-algebra generated by $\mathcal{C}$.

Remarks:- Every open set of real numbers is the union of a countable collection of disjoint open intervals.

Borel sets:- Let $\mathcal{C}$ be the collection of all open subsets of real numbers. The smallest σ-algebra $\mathcal{B}$ that contains $\mathcal{C}$ is called the collection of Borel sets. This $\mathcal{B}$ contains all the closed subsets of real numbers. Each member of this $\mathcal{B}$ is called a Borel set.

F_σ set:- A set which is a countable union of closed sets is called an F_σ sets. All the F_σ sets are Borel sets.

G_δ set:- A set which is a countable intersections of open sets is called G_δ set. All the G_δ sets are borel sets.

All the F_σ and G_δ sets are relatively simple types of Borel sets.

UNIT-I Lebesgue measure

Introduction:- The length $l(I)$ of an interval I is defined to be the difference of the end points of the interval. Length is an example of a set function that associates an extended real number to each set in some collection of sets. In case of length, the domain is the collection of intervals.

We should like to extend the notion of length to more complecated sets than intervals.

Our aim is to construct a set function m that assigns to each set E in some collection $\mathfrak{M}$ of sets of real numbers a nonnegative extended real numbers mE, called the measure of E, having the following properties:-

(i) mE ~~of~~ is defined for each set E of real numbers. i.e) $\mathfrak{M} = \mathcal{P}(\mathbb{R})$.

(ii) For an interval I, $mI = l(I)$.

(iii) If $\langle E_n \rangle$ is a sequence of disjoint sets, then $m(\cup E_n) = \Sigma\, mE_n$.

(iv) m is translation invariant. i.e) If E is ~~the~~ a set of real numbers and if $E+y$ is the set $\{x+y : x \in E\}$, then $m(E+y) = mE$.

<u>Note</u>: unfortunately, it is impossible to construct a set function having all the four properties.

So we try construct a set function that satisfies the last three properties in a class $\mathfrak{M}$ of sets of real numbers.

Outer measure.

Defn Suppose A is a set of real numbers. The outer measure m^*A is defined by

$$m^*A = \inf \left\{ \Sigma\, l(I_n) : \langle I_n \rangle \text{ is a countable collection of open intervals that covers } A \right\}.$$

<u>Note (1)</u>:- $\phi \subset (0, \frac{1}{n})$ & $\therefore m^*\phi \leq \frac{1}{n}$ $\forall n = 1, 2, \ldots$

$\therefore m^*\phi = 0$.

(2) Suppose $A \subseteq B$. If $\{I_n\}$ covers B, then $\{I_n\}$ also covers A

$\therefore \{\Sigma\, l(I_n) : \{I_n\}$ is countable collection of open intervals that covers $B\}$ is a subset

$\{ \sum l(I_n) : \{I_n\}$ is a countable collection of open intervals that covers $A \}$.

$\therefore \inf\left\{ \sum l(I_n) : \begin{array}{l}\{I_n\} \text{ is countable collection} \\ \text{of open intervals that covers } A\end{array}\right\} \leq \inf\left\{ \sum l(I_n) : \begin{array}{l}\{I_n\} \text{ is countable collection} \\ \text{of open, intervals} \\ \text{that covers } B\end{array}\right\}$

ie) $m^*(A) \leq m^*(B)$.

$\therefore \quad m^* A \leq m^* B$ when $A \subseteq B$.

Proposition 1 :- The outer measure of an interval is its length.

proof :- case i consider closed interval $[a,b]$ of finite length.

Since $(a-\epsilon, a+\epsilon) \supset [a,b]$ for every $\epsilon > 0$,

$m^*([a,b]) \leq l(a-\epsilon, b+\epsilon) = b-a+2\epsilon$.

$\therefore \; m^*([a,b]) \leq b-a \qquad (\because \epsilon > 0$ is arbitrary).

Next To prove that $m^*([a,b]) \geq b-a$.

ie) to prove that for any countable collection $\{I_n\}$ of open intervals covering $[a,b]$, $\sum l(I_n) \geq b-a \rightarrow ①$

By Heine-Borel theorem, any collection of open intervals covering $[a,b]$ contains a finite subcollection that also covers $[a,b]$ and since the sum of lengths of the finite subcollection is less than or equal to the sum of lengths of the original collection, it is enough to inequality ① for finite collections $\{I_n\}$ that covers $[a,b]$.

Suppose $\{I_n\}$ is a finite collection of open intervals that covers $[a,b]$.

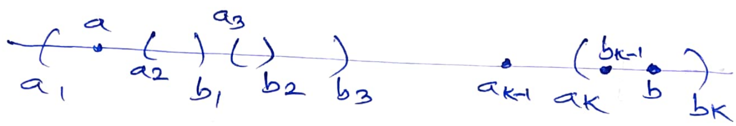
Since $[a,b] \subseteq \cup I_n$, there must be one of the I_n's that contains a & let this interval be (a_1, b_1). ie) $a_1 < a < b_1$.

If $b_1 \le a$, then $b_1 \in [a, b]$ & $b_1 \notin (a_1, b_1)$, there must be an interval (a_2, b_2) in the collection $\{I_n\}$ such that $b_1 \in (a_2, b_2)$. i.e) $a_2 < b_1 < b_2$.

Continue this process, we obtain a sequence $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ from the collection $\{I_n\}$ such that $a_i < b_{i-1} < b_i$.

Since $\{I_n\}$ is finite collection, our process must terminate with some interval (a_k, b_k).

But it terminates only if $b \in (a_k, b_k)$.



$$\therefore \sum l(I_n) \ge \sum l(a_i, b_i)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1)$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots - (a_2 - b_1) - a_1$$

$$> b_k - a_1 > b - a$$

i.e) $\sum l(I_n) \ge b - a$

$\therefore m^*([a, b]) = b - a$.

Case (ii) I is any finite interval

For a given $\epsilon > 0$, there is a closed interval $J \subset I$ $\ni$: $l(J) > l(I) - \epsilon$.

Now $l(I) - \epsilon < l(J) = m^*(J) \le m^*(I) \le m^*(\bar{I})$
$\le l(\bar{I}) = l(I)$

i.e) $l(I) - \epsilon < m^*(I) \le l(I)$

$\therefore m^*(I) = l(I)$.

Case (iii) I is an infinite interval

For any given real number Δ, there is a closed interval $J \subseteq I$ with $l(J) = \Delta$.

$\therefore m^*(I) \ge m^*(J) = l(J) = \Delta$.

$m^*(I) \ge \Delta$ for every real number Δ.

i.e) $m^*(I) = \infty = l(I)$.

(Countable Subadditive property of m^*)

Proposition 2:- If $\{A_n\}$ is a countable collection of sets of real numbers, then $m^*(\cup A_n) \le \sum m^*(A_n)$

Proof: If one of the sets A_n has infinite outer measure, then the inequality holds trivially.

If $m^*(A_n) < \infty$, then given $\epsilon > 0$, There is a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A_n \subset \bigcup_i I_{n,i}$ and $\sum_i l(I_{n,i}) < m^* A_n + \frac{\epsilon}{2^n}$.

The collection $\{I_{n,i}\}_{n,i}$ is countable and covers ~~$\{A_n\}$~~ $\bigcup_n A_n$.

Thus $m^*(\bigcup_n A_n) \le \sum_{n,i} l(I_{n,i}) = \sum_n \sum_i l(I_{n,i})$

$$< \sum_n \left\{ m^*(A_n) + \frac{\epsilon}{2^n} \right\}$$

$$= \sum_n m^*(A_n) + \epsilon$$

i.e. $m^*(\cup A_n) \le \sum m^*(A_n) + \epsilon$.

Since $\epsilon > 0$ is arbitrary, $m^*(\cup A_n) \le \sum m^*(A_n)$.

Corollary 1:- If A is a countable set, then $m^*(A) = 0$.

Proof: First we prove $m^*(\{a\}) = 0$.

For any $\epsilon > 0$, $\{a\} \subset (a-\epsilon, a+\epsilon)$

& $\therefore m^*(\{a\}) \le m^*(a-\epsilon, a+\epsilon) = 2\epsilon$.

Since $\epsilon > 0$ is arbitrary number, $m^*(\{a\}) = 0$.

Since A is a countable set, so let $A = \{a_1, a_2, \dots\}$

$A = \bigcup_i \{a_i\}$.

$m^*(A) = m^*(\cup \{a_i\}) \le \sum m^*(\{a_i\}) = 0$.

$\therefore m^*(A) = 0$.

Corollary 2:- The set $[0,1]$ is not countable.

proof:- $m^*([0,1]) = 1 \ne 0 \implies [0,1]$ is not countable.

problem:- Prove that m^* is translation invariant.

proof: to prove $m^*(A) = m^*(A+x)$.

where $A+x = \{a+x : a \in A\}$.

For given $\epsilon > 0$, $\exists$ a countable collection $\{I_n\}$ of open intervals $\ni$: $\{I_n\}$ covers A and $\sum l(I_n) < m^*(A)+\epsilon$.

Here $\{I_n + x\}$ covers $A+x$,

$\therefore m^*(A+x) \leq \sum l(I_n+x) = \sum(l_n) \leq m^*(A)+\epsilon$

ie) $m^*(A+x) \leq m^*(A)+\epsilon$.

Since ϵ is arbitrary & $\therefore m^*(A+x) \leq m^*(A)$.

Also $A = (A+x)-x$ & $\therefore m^*\{(A+x)-x\} \leq m^*(A+x)$.

ie) $m^*(A) \leq m^*(A+x)$.

Hence $m^*(A+x) = m^*(A)$.

$\therefore m^*$ is translation invariant.

Measurable sets and Lebesgue measure

Defn:- A set E of real numbers is said to be measurable if, for each set A, we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$.

Example If $E = \phi$ or $\mathbb{R}$, then $m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$

& $\therefore$ ϕ & $\mathbb{R}$ are measurable sets.

Note: 1) Since $A = (A \cap E) \cup (A \cap \tilde{E})$ &

$\therefore m^*(A) = m^*\{(A \cap E) \cup (A \cap \tilde{E})\} \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$.

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$ is always true.

So, for proving E to be measurable, it is enough to prove only $m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$.

2) From the defn, if E is measurable set, then $\tilde{E}$ is also measurable set.

Lemma 1:- If $m^*(E) = 0$, then E is measurable set.

proof: For any set A, $A \cap E \subseteq E$

& $\therefore m^*(A \cap E) \leq m^*(E) = 0 \Rightarrow m^*(A \cap E) = 0$.

Now $A \cap \tilde{E} \subseteq A \Rightarrow m^*(A \cap \tilde{E}) \leq m^*(A)$

ie) $m^*(A) \geq m^*(A \cap \tilde{E}) + m^*(A \cap E) \Rightarrow E$ is measurable.

Proposition 15:- Let E be a given set. Then the following 5 statements are equivalent:

(i) E is measurable set (ii) Given $\epsilon > 0$, $\exists$ an open set $O \supset E$ with $m^*(O \sim E) < \epsilon$ (iii) Given $\epsilon > 0$, $\exists$ a closed set $F \subset E$ with $m^*(E - F) < \epsilon$ (iv) There is a G in G_δ with $E \subset G$ and $m^*(G \sim E) = 0$ (v) There is a F in F_σ with $F \subset E$ and $m^*(E \sim F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent to:
(vi) Given $\epsilon > 0$, there is a finite union U of open intervals $\ni$: $m^*(U \Delta E) < \epsilon$.

Proof: First we prove that (ii) $\Leftrightarrow$ (vi) when $m^*(E) < \infty$.

Assume that $m^*(E) < \infty$ and for a given $\epsilon > 0$, $\exists$ an open set $O \supset E \ni$: $m^*(O \sim E) < \epsilon$. $\rightarrow$ (1)

Since $m^*(E) < \infty$ & $\therefore$ $m^*(O) < \infty$.

Also $O = \bigcup_1^\infty I_n$ where $\{I_n\}$ is sequence of disjoint open intervals and $\exists$ a +ve integer $N \ni$:

$\sum_{N+1}^\infty l(I_n) < \epsilon$. $\rightarrow$ (2)

Write $U = \bigcup_1^N I_n$. Then $E \Delta U = (E \sim U) \cup (U \sim E)$.

$$\subseteq (O - U) \cup (O \sim E)$$

$$\therefore m^*(E \Delta U) \le m^*(O - U) + m^*(O \sim E)$$
$$\le m^*\left(\bigcup_{N+1}^\infty I_n\right) + m^*(O \sim E)$$
$$= \sum_{N+1}^\infty m^*(I_n) + m^*(O \sim E)$$
$$= \sum_{N+1}^\infty l(I_n) + m^*(O \sim E)$$
$$< \epsilon + \epsilon = 2\epsilon, \text{ by (1) \& (2)}.$$

$m^*(E \Delta U) < 2\epsilon$.

$\therefore$ (ii) $\Rightarrow$ (vi).

Conversely assume that $\exists$ open intervals $I_1, I_2, \ldots, I_n$ $\ni$: $m^*\left(E \Delta \left(\bigcup_1^n I_i\right)\right) < \epsilon$.

Take $J = \bigcup_1^n I_i$. & so $m^*(E \Delta J) < \epsilon$ $\rightarrow$ (3)

By defn, $\exists$ an open set $O \supset E \ni$: $m^*(O) < m^*(E) + \epsilon$ $\rightarrow$ (4)

Take $U = O \cap J$.

Since $O \Delta E \subset (O \Delta U) \cup (U \Delta E)$,

$m^*(O \Delta E) \le m^*(O \Delta U) + m^*(U \Delta E) \rightarrow (5)$

Since $U \subset J$, we have $U - E \subseteq J \sim E$.

Since $E \subset O$, we have $E - U = E \sim (O \cap J) = E \cap \overline{(O \cap J)}$

$= E \cap (\bar{O} \cup \bar{J})$

$= (E \cap \bar{O}) \cup (E \cap \bar{J})$

$= \phi \cup (E \cap \bar{J})$

$E - U = E - J$

So $U \Delta E = (U - E) \cup (E \sim U) \subseteq (J \sim E) \cup (E - J) = (J \Delta E)$

$\therefore m^*(U \Delta E) \le m^*(J \Delta E) < \epsilon$ by (3) $\rightarrow (6)$

Since $E \subseteq U \cup (U \Delta E)$

$\therefore m^*(E) \le m^*(U) + m^*(U \Delta E) < m^*(U) + \epsilon \quad (\because (6))$

ie) $m^*(E) < m^*(U) + \epsilon \rightarrow (7)$

Now $m^*(O \Delta U) = m^*(O \sim U) \quad (\because U \sim O = \phi)$

$= m^*(O) - m^*(U)$

$< m^*(E) + \epsilon - m^*(U)$

$< m^*(U) + \epsilon + \epsilon - m^*(U) = 2\epsilon \quad (\because (7))$

$m^*(O \Delta U) < 2\epsilon \rightarrow (8)$

Now $m^*(O \sim E) = m^*(O \Delta E) \quad (\because E \sim O = \phi)$

$\le m^*(O \Delta U) + m^*(U \Delta E) < 3\epsilon \quad (\because (5), (6), (8))$

$m^*(O \sim E) < 3\epsilon$.

$\therefore$ (vi) $\Rightarrow$ (ii).

Hence (ii) $\Leftrightarrow$ (vi).

Next we prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Assume (i). By defn. of $m^*(E)$ & proposition (5), $\exists$ an open set $O \supset E$ with $m^*(O) < m^*(E) + \epsilon$.

Now $m^*(O \sim E) = m^*(O) - m^*(E) \quad (\because O \,\&\, E$ are measurable$)$

$m^*(O - E) < \epsilon$.

Hence (i) $\Rightarrow$ (ii).

Assume (ii). For each $n = 1, 2, \cdots$, $\exists$ an open set $O_n \supset E$ with $m^*(O_n - E) < \frac{1}{n}$.

Take $G = \bigcap_1^\infty O_n$. Then G is a G_δ set and $G \supset E$.

$m^*(G \sim E) \leq m^*(O_n - E) < \frac{1}{n}$

$m^*(G - E) < \frac{1}{n} \quad \forall n = 1, 2, \cdots$

$\therefore m^*(G - E) = 0.$

Hence (ii) $\Rightarrow$ (iv).

Assume (iv). $\therefore$ $G - E$ is measurable set

Since $E = G - (G - E)$ and both G & $G - E$ are measurable

$\therefore$ $E = G \sim (G \sim E)$ is measurable.

Finally we prove that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).

Assume (i).

since $\bar{E}$ is measurable, by (ii), & given $\epsilon > 0$ $\exists$ an open set $O \supset \bar{E}$ with $m^*(O \sim \bar{E}) < \epsilon$.

Take $F = \bar{O}$. Then $F = \bar{O} \subset E$ & F is closed set.

Also $O \sim \bar{E} = O \cap E = E \cap \bar{\bar{O}} = E \cap \bar{F} = E \sim F$.

$\therefore m^*(E - F) = m^*(O - \bar{E}) < \epsilon$.

Hence (i) $\Rightarrow$ (iii).

Assume (iii). For each $n = 1, 2, \cdots$, $\exists$ a closed set $F_n \subset E$ with $m^*(E - F_n) < \frac{1}{n}$.

Take $F = \bigcup_1^\infty F_n$, then $F \subset E$ & F is a F_σ set

and $m^*(E - F) \leq m^*(E - F_n)$ $\quad$ ($\because F_n \subset F$ & $E - F_n \supset E - F$)

$< \frac{1}{n} \quad \forall n$

$\therefore m^*(E - F) = 0.$

Hence (iii) $\Rightarrow$ (v).

Assume (v). Then $E - F$ is measurable

$\therefore$ $E = F \cup (E - F)$ is also measurable

because both F & $E - F$ are measurable.

Hence (v) $\Rightarrow$ (i).

Hence the proposition.

Note: Here (i) $\Leftrightarrow$ (vi) is Littlewood's first principle. ie) Every measurable set is nearly a finite union of open intervals.

Problems :-

5M(S) 1. If E is measurable, then prove that each translate $E+y$ is also measurable.

Solution :-

Since E is measurable. By Littlewood's first principle for all $\epsilon > 0$ there exists an open set U such that $E \subset U$ such that $m^*(U-E) < \epsilon$

Since $E \subset U \Rightarrow U \supset E$

$$U+y \supset E+y$$

Since U is open, $U+y$ is also open.

Also $(U+y)-(E+y) = (U-E)+y$

$$\therefore m^*(U+y-E+y) = m^*((U-E)+y)$$
$$= m^*(U-E)$$
$$< \epsilon$$

$$\therefore m^*(U+y-E+y) < \epsilon$$

Hence $\forall \epsilon > 0$, there exists an open set $U+y \supset E+y$ such that $m^*(U+y-E+y) < \epsilon$

$\therefore E+y$ is measurable.

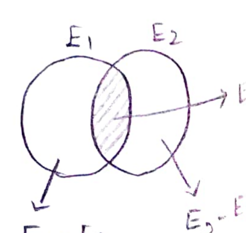
2. If E_1 and E_2 are measurable, then
$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$ (or)
$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

Proof :-

Consider two sets E_1 and E_2

$E_1 \cup E_2 = (E_1 - E_2) \cup (E_2 - E_1) \cup (E_1 \cap E_2)$

$m(E_1 \cup E_2) = m(E_1 - E_2) + m(E_2 - E_1) + m(E_1 \cap E_2)$



Consider $E_1 = (E_1 - E_2) \cup (E_1 \cap E_2)$ — (1)

$mE_1 = m\,($

consider $E_2 = (E_2 - E_1) \cup (E_1 \cap E_2)$

$$mE_2 = m(E_2 - E_1) + m(E_1 \cap E_2)$$

$$\Rightarrow m(E_2 - E_1) = mE_2 - m(E_1 \cap E_2) \quad \text{--- (3)}$$

Substituting (2) and (3) in (1)

$$m(E_1 \cup E_2) = mE_1 \cancel{- m(E_1 \cap E_2)} + mE_2 - m(E_1 \cap E_2) \cancel{+ m(E_1 \cap E_2)}$$

$$m(E_1 \cup E_2) = mE_1 + mE_2 - m(E_1 \cap E_2)$$

Hence proved.

17-12-18 (Mon) 3. If E_1 and E_2 are measurable then $E_1 \cap E_2$ is also measurable and hence finite intersection is also measurable.

Proof:-

If E_1 and E_2 are measurable. (complement of a measurable set is measurable)

Then $\bar{E}_1$ and $\bar{E}_2$ are measurable.

$\Rightarrow \bar{E}_1 \cup \bar{E}_2$ is measurable. [If E_1 and E_2 are measurable $E_1 \cup E_2$ is measurable)

$\Rightarrow \overline{E_1 \cap E_2}$ is measurable.

$\Rightarrow E_1 \cap E_2$ is measurable.

Hence finite intersection of measurable sets is measurable.

4. The sets ϕ and R are measurable.

Proof:-

Let A be any set.

consider $m^*(A \cap \phi) + m^*(A \cap \bar{\phi}) = m^*(\phi) + m^*(A \cap R)$

$$= 0 + m^* A$$

$$= m^* A$$

$\therefore m^* A = m^*(A \cap \phi) + m^*(A \cap \bar{\phi})$

$\therefore \phi$ is measurable.

Consider $m^*(A \cap R) + m^*(A \cap \bar{R}) = m^* A + m^*(A \cap \phi)$

$$= m^* A + m^*(\phi)$$

$$= m^* A + 0$$

$\therefore m^*A = m^*(A\cap R) + m^*(A\cap \tilde{R})$

$\therefore$ R is measurable.

Lec 5

Measurable functions :-

Proposition :-

Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:

i) For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

ii) For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.

iii) For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

iv) For each real number α the set $\{x : f(x) \leq \alpha\}$ is measurable

v) For each extended real number α, the set $\{x : f(x) = \alpha\}$ is measurable.

Proof :-

Let f be a extended real-valued function with domain D and D is measurable.

i) To prove :- (i) $\Rightarrow$ (iv)

Let $\{x : f(x) > \alpha\}$ is measurable.

$D - \{x : f(x) > \alpha\} = \{x : f(x) \leq \alpha\}$

$D - \{x : f(x) > \alpha\}$ is measurable.

$\therefore \{x : f(x) \leq \alpha\}$ is measurable. $\therefore$ (i) $\Rightarrow$ (iv)

ii) To prove :- (iv) $\Rightarrow$ (i)

Let $\{x : f(x) \leq \alpha\}$ is measurable.

$D - \{x : f(x) \leq \alpha\} = \{x : f(x) > \alpha\}$

$D - \{x : f(x) \leq \alpha\}$ is measurable.

$\therefore \{x : f(x) > \alpha\}$ is measurable. $\therefore$ (iv) $\Rightarrow$ (i)

iii) To prove :- (ii) $\Rightarrow$ (iii)

$\{x : f(x) \geq \alpha\}$ is measurable.

$D - \{x : f(x) \geq \alpha\} = \{x : f(x) < \alpha\}$

$D - \{x : f(x) \geq \alpha\}$ is measurable.

$\therefore \{x : f(x) > \alpha\}$ is measurable.

$\therefore$ (ii) $\Rightarrow$ (iii)

Similarly (iii) $\Rightarrow$ (ii) is true.

iv) To prove :- (i) $\Rightarrow$ (ii)

$\{x : f(x) > \alpha\}$ is measurable.

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\}$$

We know that intersection of measurable sets is measurable.

$\therefore \{x : f(x) \geq \alpha\}$ is measurable.

v) To prove :- (ii) $\Rightarrow$ (i)

$\{x : f(x) \geq \alpha\}$ is measurable

$$\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq \alpha + \frac{1}{n}\}$$

We know that union of measurable sets is measurable.

$\therefore \{x : f(x) > \alpha\}$ is measurable.

vi) To prove :- (iii) $\Rightarrow$ (iv)

$\{x : f(x) < \alpha\}$ is measurable.

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha - \frac{1}{n}\}$$

We know that intersection of measurable sets is measurable.

$\therefore \{x : f(x) \leq \alpha\}$ is measurable

$\therefore$ (iii) $\Rightarrow$ (iv)

Similarly (iv) $\Rightarrow$ (iii) is true.

vii) To prove :- (v)

If α is a real number then

$$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$$

Hence $\{x : f(x) = \alpha\}$ is measurable

Since $\{x : f(x) = \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\}$

(ii) $\Rightarrow$ (v) for $\alpha = \infty$

Similarly (iv) $\Rightarrow$ (v) for $\alpha = -\infty$

Also (iii) and (iv) $\Rightarrow$ (v)

Hence proved.

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2)

Measurable functions :-

Definition :-

An extended real valued function 'f' is said to be measurable, if its domain is measurable and it satisfies any one of the four statements :

i) For all real number α, the set $\{x : f(x) > \alpha\}$ is measurable.

ii) For all real number α, the set $\{x : f(x) \geq \alpha\}$ is measurable.

iii) For all real number α, the set $\{x : f(x) < \alpha\}$ is measurable.

iv) For all real number α, the set $\{x : f(x) \leq \alpha\}$ is measurable.

Note :-

* The constant function is measurable.

Proof :-

Let $f : D \to R \cup (-\infty, \infty)$ is defined by $f(x) = c$

Consider $\{x : f(x) > \alpha\}$ for all real number α.

$$\{x : f(x) > \alpha\} = \begin{cases} D & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases}$$

$c > c-1$

$\phi \Leftarrow c > c$

Since D is measurable and ϕ is measurable

$\therefore \{x : f(x) > \alpha\}$ is measurable

$\therefore f(x) = c$ is measurable.

MIS)

* [Characteristic function ψ_A χ_A is measurable.

Proof :-

If A is measurable set of R, then

characteristic function of ψ_A is defined as

$$\psi_A(x) = 1 \quad \text{if } x \in A$$
$$\psi_A(x) = 0 \quad \text{if } x \notin A$$

Consider $\{x : \underset{(0,1)}{\psi_A(x)} > \alpha\} = \begin{cases} R & \text{if } \alpha < 0 \\ A & \text{if } 0 \le \alpha < 1 \\ \phi & \text{if } \alpha \ge 1 \end{cases}$

we know that R and ϕ are measurable. Also A is measurable set of R

$\therefore \{x : \psi_A(x) > \alpha\}$ is measurable.

$\therefore \psi_A(x)$ is measurable if A is measurable.

If A is not a measurable set then ψ_A is also not measurable.] 2M(S)

* **Continuous function is measurable.**

Proof :-

Suppose $f : D \to R$ is continuous $\{x : f(x) > \alpha\} = f^{-1}(\alpha, \infty)$

Now (α, ∞) is an open interval in R and f is continuous.

$\therefore f^{-1}(\alpha, \infty)$ is open

Hence $\{x : f(x) > \alpha\}$ is open and continuous.

$\therefore \{x : f(x) > \alpha\}$ is measurable.

$\therefore$ Continuous function is measurable.

10M(S) 5M(S) **Theorem :-**

If f and g are measurable functions defined on the same domain D and c be constant, then

i) $f+c$ ii) cf iii) $f+g$ iv) $f-g$ v) f^2 vi) fg

are also measurable.

Proof :-

(i) To prove :- $f+c$ is measurable.

$$\{x : (f+c)(x) > \alpha\} = \{x : (f(x)+c) > \alpha\}$$
$$= \{x : f(x) > \alpha - c\}$$

Since f is measurable.

$\Rightarrow \{x : f(x) > \alpha - c\}$ is measurable.

$\Rightarrow \{x : (f+c)(x) > \alpha\}$ is measurable.

$\Rightarrow (f+c)$ is measurable.

(ii) <u>To prove :-</u> cf is measurable.

If $c > 0$

consider $\{x : (cf)(x) > \alpha\}$

$\{x : (cf)(x) > \alpha\} = \{x : c(f(x)) > \alpha\}$

$= \{x : f(x) > \frac{\alpha}{c}\}$

Since $f(x)$ is measurable.

$\Rightarrow \{x : f(x) > \frac{\alpha}{c}\}$ is measurable.

$\Rightarrow \{x : (cf)(x) > \alpha\}$ is measurable.

If $c < 0$

consider $\{x : (cf)(x) > \alpha\}$

$\{x : (cf)(x) > \alpha\} = \{x : cf(x) > \alpha\}$

$= \{x : f(x) < \alpha/c\}$

Since $f(x)$ is measurable.

$\Rightarrow \{x : f(x) < \alpha/c\}$ is measurable.

$\Rightarrow \{x : cf(x) > \alpha\}$ is measurable.

$\Rightarrow (cf)$ is measurable.

If $c = 0$

consider $\{x : (cf)(x) > \alpha\}$

$cf = 0$ then cf is constant function.

We know that constant function is measurable.

$\therefore$ cf is measurable.

Hence cf is measurable $\forall c$.

1-12-18 (Thurs) (iii) <u>To prove :-</u> $(f+g)$ is measurable

Consider $\{x : (f+g)(x) < \alpha\} = \{x : f(x) + g(x) < \alpha\}$

$= \{x : f(x) < \alpha - g(x)\}$

By Archimedes principle between any two real numbers $\exists$ infinite number of rational numbers.

$\therefore$ $\exists$ a rational number 'r' such that

$f(x) < r < \alpha - g(x)$

i.e) $f(x) < r$

$\Rightarrow \alpha > r + g(x)$

$\alpha - g(x) > r \quad \Rightarrow r + g(x) < \alpha$

$\Rightarrow g(x) < \alpha - r$

Thus for all rational numbers 'r'

$\{x : (f+g)(x) < \alpha\} = \{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}$

$= \bigcup_r [\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}]$ — (1)

Since f and g are measurable

$\Rightarrow \{x : f(x) < r\}$ and $\{x : g(x) < \alpha - r\}$ are measurable.

Therefore R.H.S of (1) is measurable.

$\therefore (f+g)$ is measurable.

(iv) Jo prove :- $f-g$ is measurable.

Since g is measurable, by statement (ii)

cf is measurable. where c is constant if $\boxed{c = -1}$

Then $(-1)g = -g$ is measurable.

Also f is measurable.

By statement (iii)

$f + (-g)$ is measurable.

ie) $f - g$ is measurable.

(v) Jo prove :- f^2 is measurable

If D is the domain of f, then

$\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha} \text{ or } f(x) < -\sqrt{\alpha}\}$

$= \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$ if $\alpha \geq 0$

$= 0$ if $\alpha < 0$

Thus RHS is measurable since f is measurable.

$\therefore f^2$ is measurable.

(vi) Jo prove :- fg is measurable.

$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

Since f and g are measurable.

$(f+g)$ and $(f-g)$ are measurable.

Also by (v), $(f+g)^2$ and $(f-g)^2$ are measurable.

by (iv), $(f+g)^2 - (f-g)^2$ is measurable.

by (ii) $\frac{1}{4}\left[(f+g)^2-(f-g)^2\right]$ is measurable

Then fg is measurable.] 10 M (S)

Theorem :-

Let $\{f_n\}$ be a sequence of measurable functions (with same domain) then the functions $\sup\{f_1, f_2, \dots f_n\}$, $\inf\{f_1, f_2, \dots f_n\}$, $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are all measurable.

Proof :-

Let $g(x) = \sup\limits_n f_n(x)$

If $g(x) > \alpha$ then $\sup f_n > \alpha$

For a 'n', $f_n(x) > \alpha$

Then atleast one $f_n(x) > \alpha$

Hence $\{x : g(x) > \alpha\} = \bigcup\limits_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$ — (1)

Since f_n's are measurable.

RHS of (1) is measurable.

$\therefore g(x)$ is measurable.

ie) $\sup\limits_n f_n(x)$ is measurable.

To prove :- $\inf f_n(x)$ is measurable.

Let $h(x) = \inf\limits_n f_n(x)$

Consider $\{x : h(x) < \alpha\} = \bigcup\limits_{n=1}^{\infty} \{x : f_n(x) < \alpha\}$

Since f_n's are measurable.

RHS is measurable $\Rightarrow h(x)$ is measurable

ie) $\inf\limits_n f_n(x)$ is measurable.

To prove :- $\overline{\lim} f_n$ is measurable.

ie) $\overline{\lim} f_n = \inf\limits_n \sup\limits_{K \geq n} \boxed{f_K}$

Let $g_n = \sup\limits_{K \geq n} f_K$

By (1) g_n is measurable

$\overline{\lim} f_n = \inf\limits_n g_n$

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$\therefore \overline{\lim} f_n$ is measurable

To prove :- $\underline{\lim} f_n$ is measurable.

$$\underline{\lim} f_n = \sup \inf_{K \geq n} f_K$$

Let $h_n = \inf_{K \geq n} f_K$

Since infimum of a measurable sequences is measurable.

h_n is measurable.

$\underline{\lim} f_n = \sup_n h_n$

$\therefore \sup_n h_n$ is measurable.

$\Rightarrow \underline{\lim} f_n$ is measurable.

29-12-18 (Sat)

Almost Everywhere (a.e) :-

Definition :-

A property is said to hold almost everywhere if the set of points where it fails to hold is a set of measure zero]

In particular $f = g$ almost everywhere if f and g have same domain and $m\{x : f(x) \neq g(x)\} = 0$.

$\{f_n\}$ converges to g almost everywhere if $m\{x : f_n(x) \not\to g(x)\} = 0$ .ie If $E = \{x : f_n(x) \not\to g(x)\}$ then $m(E) = 0$

ie) $f_n(x) \to g(x)$ except on E. & $m(E) = 0$.

5M(S) 2M(S)

Proposition :-

If f is a measurable function and $f = g$ almost everywhere then g is measurable.

Proof :-

given f is measurable and $f = g$ almost everywhere

ie) Let $E_1 = \{x : f(x) = g(x)\}$

$E_2 = \{x : f(x) \neq g(x)\}$ since $f = g$ almost everywhere $\Rightarrow mE_2 = 0$

Let $A = \{x : g(x) > \alpha\}$

We know that $A \cap E_2 \subset E_2$

$$m(A \cap E_2) \leq mE_2 = 0$$

$$m(A \cap E_2) = 0$$

$\therefore A \cap E_2$ is measurable.

Now $A \cap E_1 = \{x : g(x) > \alpha\} \cap E_1$

$A \cap E_1 = \{x : f(x) > \alpha\} \cap E_1$ (since $f(x) = g(x)$ in E_1)

Since $f(x)$ is measurable

$A \cap E_1$ is measurable.

Now $A = (A \cap E_1) \cup (A \cap E_2)$

The union of any 2 measurable sets is measurable.

$\therefore A = \{x : g(x) > \alpha\}$ is measurable

$\therefore g$ is measurable.] 5M (S)

2M (S)

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10 M (S) ii) [**Proposition : Littlewood's second principle :-**

Let 'f' be a measurable function defined on an interval $[a, b]$ and assume that f takes the values $\pm\infty$ only on a set of measure zero. Then given $\varepsilon > 0$ we can find a step function 'g' and a continuous function 'h' such that

$$|f - g| < \varepsilon \text{ and } |f - h| < \varepsilon$$

except on a set of measure less than ε.

i.e) $m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$

$m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$

In addition if $m \leq f \leq M$, then we may choose the function g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.]

2M (S) 5M (S) (i) [**Simple function :-**

A real valued function ϕ is called simple function if it is measurable and assume only a finite number of values.

ie) If ϕ is simple and has the value $\alpha_1, \alpha_2, \dots \alpha_n$ then $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ when $A_i = \{x : \phi(x) = \alpha_i\}$] 2M(S)

5M(S) (ii)

Characteristic function :-

If A is any set, we define the characteristic function χ_A of the set A to be the function given by $\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$]

The function χ_A is measurable if and only if A is measurable. Thus the existance of a non-measurable set implies the existance of non-measurable function.

5M(S) 10M(S) (iii) Sec 6

Littlewood's Third principle :-

Let E be a measurable set of finite measur~~able~~e and $\{f_n\}$ be a sequence of measurable functions defined on E. Let 'f' be a real valued function such that for each 'x' in E ($x \in E$) we have $f_n(x) \to f(x)$ then given $\varepsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and for $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

Proof :-

Let $G_n = \{x \in E \,/\, |f_n(x) - f(x)| \geq \varepsilon\}$ and

let $E_N = \bigcup_{n=N}^{\infty} G_n$

$E_{N+1} = \bigcup_{n=N+1}^{\infty} G_n$

Therefore $E_{N+1} \subset E_N$

Hence $\{E_n\}$ is a decreasing sequence of measurable sets — (1)

Since $f_n(x) \to f(x)$ $\forall x \in E$ and for all $\varepsilon > 0$ there exists a integer N such that $|f_n(x) - f(x)| <$

$n \geq N$

$\therefore x \notin G_n$ for some $n \geq N$

Hence $x \notin E_N$ for some N

$\therefore \cap E_N = \phi$ since if $x \in E_N$ then $x \in \cap E_N$ then $x \in G_N$ — (2)

Since f_n and f are measurable $f_n - f$ is measurable.

$\therefore G_n = \{x \in E / |f_n(x) - f(x)| \geq \epsilon\}$ is measurable.

Hence $\boxed{E_N = \bigcup_{n=N}^{\infty} G_n}$ is also measurable — (3)

From (1) and (3) $\{E_N\}$ is an infinite decreasing sequence of measurable sets.

Hence $\lim\limits_{N \to \infty} mE_N = m(\cap E_N)$

$= m\phi$

$= 0$

$\therefore$ For given $\epsilon > 0$ $\exists$ $\delta > 0$ such that $m(E_n) < \delta$

ie) $m\{x \in E : |f_n(x) - f(x)| \geq \epsilon\} < \epsilon$

If $E_N = A$

$\bar{A} = \{x \in E : |f_n(x) - f(x)| < \epsilon\}$ $\forall n \geq N$

Thus there is a measurable set A such that $m(A) < \delta$ and an integer N such that $\forall x \notin A$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.] 5M(S)

Pointwise convergence :-

A $\{f_n\}$ on E is said to converge pointwise to f on E, if $f_n(x) \to f(x)$ $\forall x \in E$]

If there is a subset B of E with $m(B) = 0$ and $f_n \to f$ pointwise on $E \setminus B$, then we say that $f_n \to f$ pointwise almost everywhere on E.

2-18 (on)

Proposition :-

If E be a measurable set of finite measurable and $\{f_n\}$ is a sequence of measurable function that converge to a real valued function 'f' almost everywhere on E. Then given $\epsilon > 0$ and $\delta > 0$ there is a set $A \subset E$ with $m(A) < \delta$ and a N such that for all $x \notin A$ and all $n \geq N$

$$|f_n(x) - f(x)| < \epsilon.$$

Proof :-

Since $f_n \to f$ almost everywhere on E. There exists a set $B \subseteq E$ with $m(B) = 0$. $f_n \to f$ pointwise on $E-B$ so by previous theorem (little wood's third principle) there exists a set $A \subseteq E-B \subseteq E$ with $m(A) < \delta$ and an integer N $\forall x \notin A$ and $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

1. If A is countable then $m^*(A) = 0$

Since A is countable we can write A as an infinite sequence $A \subseteq \bigcup I_i$ $m^* A \leq \sum l(I_i)$

$m^*(A) > 0$ always

$\Rightarrow m^*(A) = 0$