Unit - II

Lebesgue Measure

Definition :- Riemann Integral

Let f be a bounded real valued function defined on an interval $[a,b]$ and let $a = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_n = b$ be subdivision on $[a,b]$ then for each sub division we define the sum as

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i \text{ and}$$

$$s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$$

where $M_i = \sup_{\xi_{i-1} < x \leq \xi_i} f(x)$

$m_i = \inf_{\xi_{i-1} < x \leq \xi_i} f(x)$

Then the upper Riemann integral of f is

$R \overline{\int_a^b} f(x)\,dx = \inf S$

The lower Riemann integral of f is

$R \underline{\int_a^b} f(x)\,dx = \sup s$ where inf and sup taken over all the possible sub division of $[a,b]$

×The upper integral is always atleast as large as the lower integral.×

If $R \overline{\int_a^b} f(x)\,dx = R \underline{\int_a^b} f(x)\,dx$ then f is Riemann integrable and is denoted by $R \int_a^b f(x)\,dx$

Step function :-

A function ψ is said to be a step function on $[a,b]$ if $\psi(x) = c_i$ for every $\xi_{i-1} < x < \xi_i$ for each sub division of $[a,b]$ and some constant c_i.]

The general definition of an integral of the step function is given by

$$\int_a^b \psi(x)\,dx = \sum_{i=1}^{n} c_i (\xi_i - \xi_{i-1})$$

Definition of Riemann Integral interms of step function :-

$$R\overline{\int_a^b} f(x)\,dx = \inf_{f \le \psi} \int_a^b \psi(x)\,dx \text{ and}$$

$$R\underline{\int_a^b} f(x)\,dx = \sup_{\phi \le f} \int_a^b \phi(x)\,dx$$

5 M(S) (i)

Simple function :-

A linear combination $\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is called a simple funtion if the sets E_i are measurable.

[Canonical representation of simple function :-

If ϕ is a simple function and $a_1, a_2, \ldots a_n$ are the non-zero values of ϕ then $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = \{x : \phi(x) = a_i\}$

This representation of ϕ is called canonical representation of ϕ.]

In this representation A_i's are disjoint and a_i's are disjoint and non-zero.

In this case, if ϕ vanishes outside the set of finite measure, the integral of ϕ is defined by

$$\int \phi(x)\,dx = \sum_{i=1}^{n} a_i\, m(A_i)$$

Note :-

If E is any measurable set then $\int_E \phi = \int \phi \cdot \chi_E$

Lemma :-

Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \phi$ for $i \neq j$. Suppose each set E_i is a measurable set of finite measure. Then

$$\int \phi = \sum_{i=1}^{n} a_i\, mE_i$$

Proof :-

Clearly ϕ is defined on union of E_i, $i = 1$ to n

ie) $\phi = \bigcup_{i=1}^{n} E_i$

Let the canonical representation of ϕ be

$\phi = \sum_{i=1}^{n} a_i \chi A_{a_i}$ where $E_i = A_{a_i}$

$$A_{a_i} = \{x : \phi(x) = a_i\}$$

ie) $A_a = \bigcup_{a_i = a} E_i$

$$\text{Now } a\,m(A_a) = a\,m\left(\bigcup_{a_i=a} E_i\right)$$
$$= \sum_{a_i=a} a_i\, m(E_i)$$

$$\int \phi(x)\,dx = \sum a\,m(A_a)$$
$$= \sum_a \sum_{a_i=a} a_i\, m(E_i)$$
$$= \sum_{i=1}^{n} a_i\, m(E_i)$$

M(S) + m(S)

Proposition :- Linearity property

Let ϕ and ψ be simple functions which vanish outside a set of finite measure, then

$$\int (a\phi + b\psi) = a\int\phi + b\int\psi$$

and if $\phi \geq \psi$ almost everywhere then $\int\phi \geq \int\psi$

Proof :-

The set $\{A_i\}$ and $\{B_i\}$ be the sets occurring in canonical representation of ϕ and ψ. Let A_0 and B_0 be the sets where ϕ and ψ are zero then the sets E_k obtained by taking the intersections of $A_i \cap B_i$ form a finite disjoint collection of measurable set.

So $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ and $\psi = \sum_{k=1}^{N} b_k \chi_{E_k}$

$$a\phi + b\psi = a\sum_{k=1}^{N} a_k \chi_{E_k} + b\sum_{k=1}^{N} b_k \chi_{E_k}$$
$$= \sum_{k=1}^{N} (a a_k + b b_k)\chi_{E_k}$$

$$= \sum_{k=1}^{N} (a a_k + b b_k)\, m(E_k) \quad \text{[by previous lemma]}$$

$$\int(a\phi+b\psi) = \sum_{K=1}^{N} a a_K\, m(E_K) + \sum_{K=1}^{N} b b_K\, m(E_K)$$

$$= a\sum_{K=1}^{N} a_K\, m(E_K) + b\sum_{K=1}^{N} b_K\, m(E_K)$$

$$= a\int\phi + b\int\psi$$

Thus linearity property is satisfied.

~~Also note that~~ if a simple function ϕ, ~~such that~~ $\phi \geq 0$ almost everywhere then $\int\phi \geq 0$

Now $\phi \geq \psi$ almost everywhere

$\Rightarrow \phi - \psi \geq 0$ almost everywhere

$\Rightarrow \int(\phi-\psi) \geq 0$

$\Rightarrow \int\phi - \int\psi \geq 0$

$\Rightarrow \int\phi \geq \int\psi$] 5M(S) + 5M(S)

01-19 (Fri)

10M (S)

Proposition :-

Let f be defined and bounded on a measurable set E with $m(E)$ finite in order that $\inf_{f\leq\psi} \int_E \psi(x)\,dx = \sup_{\phi\leq f} \int_E \phi(x)\,dx$. For all simple functions ϕ and ψ it is necessary and sufficient that f is measurable. iff

Proof :-

Let f be measurable.

Since f is bounded $\Rightarrow |f(x)| \leq M \;\; \forall \boxed{x\in E}$

ie) $-M \leq f(x) \leq M$

Divide the interval $[-M, M]$ into $2n$ equal parts

Define $E_K = \left\{x\in E : \frac{M}{n}(K-1) < f(x) \leq \frac{MK}{n}\right\}$

Since f is measurable, E_K is measurable

ie) $E_K = \left\{x : x \in f^{-1}\left((K-1)\frac{M}{n}, \frac{MK}{n}\right)\right\}$

put $K = K+1$

$E_{K+1} = \left\{x : x \in f^{-1}\left(K\frac{M}{n}, \frac{M}{n}(K+1)\right)\right\}$

Thus E_K's are disjoint and $E = \bigcup_{K=-n}^{n} E_K$

$$m(E) = m\left(\bigcup_{K=-n}^{n} E_K\right) = \sum_{K=-n}^{n} m(E_K)$$

For each n, define

$$\psi_n(x) = \frac{M}{n} \sum_{K=-n}^{n} K \chi_{E_K}(x)$$

$$\phi_n(x) = \frac{M}{n} \sum_{K=-n}^{n} (K-1) \chi_{E_K}(x)$$

Thus ψ_n and ϕ_n are simple functions. Also for each K, $\phi_n(x) \le f(x) \le \psi_n(x)$

Since $x \in E_K \Rightarrow \psi_n(x) = \frac{M}{n} K$

$$\phi_n(x) = \frac{M}{n}(K-1)$$

Now $\inf_{f \le \psi} \int_E \psi(x)\,dx \le \int_E \psi_n(x)\,dx$

$$= \frac{M}{n} \sum_{K=-n}^{n} K m(E_K)$$

and $\sup_{\phi \le f} \int_E \phi(x)\,dx \ge \int_E \phi_n(x)\,dx$

$$= \frac{M}{n} \sum_{K=-n}^{n} (K-1) m(E_K)$$

$$\inf_{f \le \psi} \int_E \psi(x)\,dx - \sup_{\phi \le f} \int_E \phi(x)\,dx \le \frac{M}{n} \sum_{K=-n}^{n} K m(E_K) - \frac{M}{n} \sum_{K=-n}^{n} (K-1) m(E_K)$$

$$= \frac{M}{n} \sum_{K=-n}^{n} m(E_K)$$

$$= \frac{M}{n} m(E)$$

$\because m(E) = m(\cup E_K) = \sum_{K=-n}^{n} m(E_K)$

Since n is arbitrary as $n \to \infty$ $\frac{M}{n} m(E) \to 0$

$$\inf_{\psi \ge f} \int_E \psi(x)\,dx - \sup_{\phi \le f} \int_E \phi(x)\,dx \le 0$$

But $\inf_{\psi \geq f} \int_E \psi\, dx - \sup_{\phi \leq f} \int_E \phi\, dx \geq 0$

[since upper $R\int \geq$ lower $R\int$]

$\inf_{\psi \geq f} \int_E \psi\, dx - \sup_{\phi \leq f} \int_E \phi\, dx = 0$

$\inf_{\psi \geq f} \int_E \psi\, dx = \sup_{\phi \geq f} \int_E \phi\, dx$

Converse part :-

Suppose $\inf_{\psi \geq f} \int_E \psi\, dx = \sup_{\phi \geq f} \int_E \phi\, dx = I$ (say)

Then given 'n' there are simple functions ϕ_n and ψ_n such that $\phi_n(x) \leq f(x) \leq \psi_n(x)$

$\int_E \psi_n(x)\, dx < I + \frac{1}{2^n}$ and $\int_E \phi_n(x)\, dx > I - \frac{1}{2^n}$

Then $\int_E \psi_n(x)\, dx - \int_E \phi_n(x)\, dx < I + \frac{1}{2^n} - \left(I - \frac{1}{2^n}\right)$

$< 2 \frac{1}{2^n}$

$= \frac{1}{2^{n-1}} < \frac{1}{n}$

Let $\psi^* = \inf \psi_n$ and $\phi^* = \sup \phi_n$

Since ψ_n and ϕ_n are measurable, ψ^* and ϕ^* are measurable.

Also $\phi^* \leq f(x) \leq \psi^*$

Let $\Delta = \{x : \phi^*(x) \leq \psi^*(x)\}$ and

$\Delta_V = \{x \in E : \phi^*(x) < \psi^*(x) - \frac{1}{V}\}$

Then $\Delta = \bigcup_{V=1}^{\infty} \Delta V$

ie) If $x \in \Delta V$ then $\phi^*(x) < \psi^*(x) - \frac{1}{V}$

$\phi^*(x) \leq \psi^*(x)$ $\quad \therefore x \in \Delta$

Suppose $x \in \Delta$ $\quad \phi^*(x) \leq \psi^*(x)$

$\phi^*(x) < \psi^*(x) - \frac{1}{V}$

$\therefore x \in \Delta V$

$\therefore \Delta = \bigcup_{V=1}^{\infty} \Delta V$

Define $\Delta_{V,n} = \{x \in E : \phi_n(x) < \psi_n(x) - \frac{1}{V}\}$

For $x \in \Delta v \Rightarrow \phi^*(x) < \psi^*(x) - \frac{1}{v}$

$\Rightarrow \sup \phi_n(x) < \inf \psi_n(x) - \frac{1}{n}$

$\phi_n(x) \leq \sup \phi_n(x) < \inf \psi_n(x) - \frac{1}{v}$

$< \psi_n(x) - \frac{1}{v}$

$\therefore x \in \Delta_{v,n}$

$\therefore \Delta_v \subseteq \Delta_{v,n}$

To prove that $m(\Delta_{v,n}) < \frac{v}{n}$

If $m(\Delta_{v,n}) \geq \frac{v}{n}$

$$\int_{\Delta_{v,n}} \psi_n(x)\,dx - \int_{\Delta_{v,n}} \phi_n(x)\,dx = \int_{\Delta_{v,n}} (\psi_n - \phi_n)\,dx$$

$$> \frac{1}{v} \int_{\Delta_{v,n}} dx$$

$$= \frac{1}{v}\, m(\Delta_{v,n})$$

$$\geq \frac{1}{v} \cdot \frac{v}{n} = \frac{1}{n}$$

ie) $\int_{\Delta_{v,n}} \psi_n(x)\,dx - \int_{\Delta_{v,n}} \phi_n(x)\,dx \geq \frac{1}{n}$

which is a contradiction.

Since $\int \psi_n(x)\,dx - \int \phi_n(x)\,dx < \frac{1}{n}$

$\therefore m(\Delta_{v,n}) < \frac{v}{n}$

$\because$ This is true for any 'n' as $n \to \infty$

$m(\Delta_{v,n}) < 0$

$m(\Delta v) \leq m(\Delta_{v,n}) < 0$

$\therefore m(\Delta_v) = 0$

$m(\Delta) = 0$

where $\Delta = \{x \in E : \phi^*(x) \leq \psi^*(x)\}$

$\therefore \phi^* \geq \psi^*$ almost everywhere

But $\phi^* \leq \psi^*$

$\therefore \phi^*(x) = \psi^*(x)$ almost everywhere

$\phi^*(x) = f(x) = \psi^*(x)$ almost everywhere

Since ϕ^* and ψ^* are measurable

f is measurable.] 10M(S)

07-01-19 (Mon)

2M(S)

Definition :- Lebesgue Integral of f

If f is a bounded measurable function defined on a measurable set E with mE finite we define the Lebesgue integral of f over E by

$\int_E f(x)\,dx = \inf_{\psi \geq f} \int_E \psi(x)\,dx$ for all simple functions $\psi \geq f$. 2M(S)

Note :-

i) If $E = [a,b]$, we write $\int_{[a,b]} f = \int_a^b f$

ii) If f is a bounded measurable function that vanishes outside a set E of finite measure, we write $\int f$ for $\int_E f$ and $\int_E f = \int f \chi_E$

5M(S) **Proposition :-**

Let f be a bounded function defined on $[a,b]$ and if 'f' is Riemann integrable on $[a,b]$ then it is measurable and $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$

Proof :-

$\phi \leq f \leq \psi$

Let f be Riemann integrable then

$\inf_{\substack{\psi \geq f \\ f \leq \psi}} \int \psi\,dx = R\overline{\int_a^b} f(x)\,dx = R\underline{\int_a^b} f(x)\,dx = \sup_{\phi \leq f} \int \phi\,dx$

where ϕ and ψ are step functions with $\psi \geq f$ and $\phi \leq f$

Since the step function is also a simple function the set $\{\phi : \phi \leq f\} \subseteq \{\phi_1 : \phi_1 \leq f\}$ and $\{\psi : \psi \geq f\} \subseteq \{\psi_1 : \psi_1 \geq f\}$

where ϕ_1 and ψ_1 are simple functions.

Then $\sup\{\phi : \phi \leq f\} \leq \sup\{\phi_1 : \phi_1 \leq f\}$ and

$\inf\{\psi : \psi \geq f\} \geq \inf\{\psi_1 : \psi_1 \geq f\}$

$\therefore$ Lower Riemann integrable

$$R\underline{\int_a^b} f(x)\,dx = \sup_{\phi \le f} \int_a^b \phi \le \sup_{\phi_1 \le f} \int_a^b \phi_1$$

$$\le \inf_{\psi_1 \ge f} \int_a^b \psi_1 \le \inf_{\psi \ge f} \int_a^b f(x)\,dx$$

since f is Riemann integrable

$$R\overline{\int_a^b} f(x)\,dx = R\underline{\int_a^b} f(x)\,dx = \sup_{\phi \le f} \int \phi\,dx$$

$\therefore$ The above inequalities become equalities

$$R\overline{\int_a^b} f(x)\,dx = R\underline{\int_a^b} f(x)\,dx = \sup_{\phi \le f} \int \phi\,dx$$

$$\therefore \sup_{\phi_1 \le f} \int_a^b \phi_1 = \inf_{\psi_1 \ge f} \int \psi_1 = R\int_a^b f(x)\,dx$$

$\Rightarrow f$ is measurable.

$$\therefore \int_a^b f(x)\,dx = R\int_a^b f(x)\,dx$$ 5M(S)

Note :-

The above proposition shows that Lebesgue integral is in fact a generalisation of the Riemann integral.

(10 M(S)) **Proposition :-**

If f and g are bounded measurable functions defined on a set E of finite measure, then

i) $\int_E (af+bg) = a\int_E f + b\int_E g$

ii) If $f = g$ almost everywhere $\int_E f = \int_E g$

iii) If $f \le g$ almost everywhere $\int_E f \le \int_E g$

Also $\left|\int f\right| \le \int |f|$

iv) If $A \le f(x) \le B$, then $A \cdot mE \le \int_E f(x) \le B \cdot mE$

v) If A and B are disjoint measurable sets of finite measure then $\int_{A\cup B} f = \int_A f + \int_B f$

Proof :-

i) If ψ is a simple function, $a\psi$ is also a simple function, first let us prove

$\int_E f(x) = \inf \int_E \psi(x)\,dx$

$\int_E af = a\int_E f$

a) If $a=0$ then $a\psi = 0$

$\Rightarrow \int a\psi = 0 \Rightarrow \inf\limits_{a\psi \geq af} \int a\psi = 0$ $\qquad \psi \geq f$

$\Rightarrow \int af = 0 \Rightarrow \int_E af = a\int_E f = 0$

b) If $a>0$, then $\int_E af = \inf\limits_{a\psi \geq af} \int_E a\psi = a \inf\limits_{\psi \geq f} \int_E \psi = a\int_E f$

c) If $a<0$, then $\int_E af = \inf\limits_{a\psi \geq af} \int_E a\psi = \inf\limits_{\psi \leq f} \int_E -(-a)\psi$

$= \inf\limits_{\psi \leq f} -\int_E (-a)\psi$

$= -a \inf\limits_{\psi \leq f} -\int_E \psi$

$\because -\inf(-f) = \sup f$ $\qquad = a \sup\limits_{\psi \leq f} \int_E \psi$

$= a\int_E f$

$\therefore \forall a \int_E af = a\int_E f$

To prove :- $\int_E f+g = \int_E f + \int_E g$

Consider all simple functions ϕ and ψ such that $\psi \geq f$ and $\phi \geq g$

$\int_E (\psi+\phi) \geq \int_E (f+g)$

$\int_E \psi + \int_E \phi \geq \int_E (f+g)$

$\inf\limits_{\psi \geq f} \int_E \psi + \inf\limits_{\phi \geq g} \int_E \phi \geq \int_E (f+g)$

$\int_E f + \int_E g \geq \int_E (f+g)$ —— (1)

Now consider all simple functions ϕ_1 and ψ_1 such that $\psi_1 \leq f$ and $\phi_1 \leq g$

Then $\psi_1 + \phi_1 \leq f+g$

$\int_E (\psi_1+\phi_1) \leq \int_E (f+g)$

$\int_E \psi_1 + \int_E \phi_1 \leq \int_E (f+g)$

$$\sup_{\psi_1 \le f} \int_E \psi_1 + \sup_{\phi_1 \le g} \int_E \phi_1 \le \int_E (f+g)$$

$$\int_E f + \int_E g \le \int_E (f+g) \quad \text{—②}$$

From ① and ② $\int_E (f+g) = \int_E f + \int_E g$

Hence proved.

Also $\int_E (af + bg) = a\int_E f + b\int_E g$

Hence proved.

ii) Let $f = g$ almost everywhere then

$f - g = 0$ almost everywhere.

Consider the simple function ψ such that

$\psi \ge f - g$

$\therefore$ $\psi \ge 0$ almost everywhere

$\therefore$ $\int_E \psi \ge 0$

$$\Rightarrow \inf_{\psi \ge f-g} \int_E \psi \ge 0$$

$$\Rightarrow \int_E (f-g) \ge 0$$

$$\Rightarrow \int_E f - \int_E g \ge 0 \Rightarrow \int_E f \ge \int_E g \quad \text{—①}$$

Also since $f = g$ almost everywhere

$\Rightarrow g - f = 0$ almost everywhere

we have $\int_E g \ge \int_E f$ —②

By ① and ② $\int_E f = \int_E g$

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rue)

iii) Let $f \le g$ almost everywhere then

$f - g \le 0$ almost everywhere

Let $\psi \le f - g$

$\Rightarrow \psi \le 0$ almost everywhere

$\Rightarrow \int_E \psi \le 0$

$$\Rightarrow \sup_{\psi \le f-g} \int_E \psi \le 0$$

$$\Rightarrow \int_E f - g \leq 0$$

$$\Rightarrow \int_E f - \int_E g \leq 0$$

$$\Rightarrow \int_E f \leq \int_E g$$

Also since $|f| \geq f$

$|f| - f \geq 0$

$\int_E (|f| - f) \geq 0$

$\int_E |f| - \int_E f \geq 0$

$$\int_E |f| \geq \int_E f$$

Also,

$$\int_E |f| \geq -\int_E f$$

$$\int_E |f| \geq \left|\int_E f\right|$$

ie) $\left|\int_E f\right| \leq \int_E |f|$

iv) Let $A \leq f(x) \leq B$

__To prove :-__ $A \cdot mE \leq \int_E f \leq B \cdot mE$

__given :-__ $A \leq f(x) \leq B$

$$\int_E A \leq \int_E f(x) \leq \int_E B$$

$$\int_E A\,dx \Rightarrow \int_E A \chi_E\,dx \Rightarrow A \int \chi_E\,dx = A\,mE$$

Similarly $\int_E B\,dx \Rightarrow \int_E B \chi_E\,dx \Rightarrow B \int_E \chi_E\,dx \Rightarrow B\,mE$

$\therefore A\,mE \leq \int_E f \leq B\,mE$

Hence proved.

v) $\int_{A \cup B} f = \int_A f + \int_B f$ (To prove)

If A and B are disjoint measurable sets.

$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B} = \chi_A + \chi_B$

Since A and B are disjoint sets

$A \cap B = \phi$ and $\chi_{A\cap B} = 0$

$$\int_{A\cup B} f\,dx = \int f \chi_{A\cup B}\,dx$$
$$= \int f(\chi_A + \chi_B)\,dx$$
$$= \int f(\chi_A)\,dx + \int f\chi_B\,dx$$
$$\int_{A\cup B} f\,dx = \int_A f\,dx + \int_B f\,dx$$

Hence proved.

5M(S) **Proposition :- Bounded convergence theorem**

Let $\{f_n\}$ be a sequence of measur~~f_n~~able functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \le M$ $\forall n$ and $\forall x$

If $f(x) = \lim f_n(x)$ for each x in E then $\int_E f = \lim \int_E f_n$

Proof :-

Since f_n is measurable ~~and~~ for every n and $\underline{\lim}\, f_n$ and $\overline{\lim}\, f_n$ are measurable.

Also $\lim f_n = f$ is measurable.

$$|f(x)| = \left|\lim_{n\to\infty} f_n(x)\right|$$
$$\le \lim_{n\to\infty} |f_n(x)|$$
$$\le M \quad (\because f \text{ is bounded})$$

Thus f_n is a sequence of bounded measurable function converging pointwise to the bounded measurable function f on E.

So, by Littlewood's third principle given $\varepsilon > 0$ there exist $\boxed{A \subset E}$ such that $mA < \delta$ $\left(\text{taking } \delta = \frac{\varepsilon}{4M}\right)$ and positive integer N such that

for $n \ge N$ and $x \in E - A$.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2ME}$$

Now, $|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)|$ $\quad (\because |f_n(x)| \leq M$
$x \in A$ $\quad \leq 2M$ $\quad \forall x \in A)$

$$\left|\int_E f_n(x) - \int_E f\right| = \left|\int_E f_n - f\right|$$

$$\leq \int_E |f_n - f|$$

$$= \int_{E-A} |f_n - f| + \int_A |f_n - f|$$

$$< \int_{E-A} \frac{\varepsilon_1}{2mE} + \int_A 2M\,dx$$

$$\leq \frac{\varepsilon_1}{2mE} \int_{E-A} \chi_{E-A} + 2M \int_A \chi_A$$

$$= \frac{\varepsilon_1}{2mE} m(E-A) + 2M\, mA$$

$$\leq \frac{\varepsilon_1/2}{\cancel{mE}} \cancel{m(E)} + 2M\delta$$

$$\cancel{= \frac{\varepsilon_1}{2mE} mE - \frac{\varepsilon_1}{2mE} m(A) + 2M\, m(A)}$$

$$\cancel{= \frac{\varepsilon_1}{2} - \left(\frac{\varepsilon_1}{2mE} - 2m\right) m(A)}$$

$$\leq \varepsilon_1/2 + \cancel{2M}\left(\frac{\varepsilon_1}{\cancel{4M}_{2}}\right)$$

$$= \frac{\varepsilon_1}{2} + \varepsilon_1/2$$

$\because E - A \subset E$

$\boxed{m(E-A) \leq mE}$

$$= \varepsilon_1$$

$$\therefore \left|\int_E |f_n - f|\right| < \varepsilon_1$$

$$\Rightarrow \int f = \lim_{n \to \infty} \int f_n$$

Hence the proof.] 5M(S)

09-01-19 (Wed)

<u>Integral of non-negative function</u> :-

If f is a non-negative measurable function defined on a measurable set E, we define $\int_E f = \sup_{h \leq f} \int_E h$ where h is a bounded measurable function such that $m\{x : h(x) \neq 0\}$ is finite.

Proposition :-

If f and g are non-negative measurable function, then

(i) $\int_E cf = c\int_E f \quad c>0$

(ii) $\int_E f+g = \int_E f + \int_E g$

(iii) If $f \le g$ almost everywhere $\int_E f \le \int_E g$

Proof :-

i) To prove :-

$$\int_E cf = c\int_E f \quad c>0$$

Let h be a bounded measurable function which vanishes outside the finite set and $h \le f$

Now $\int_E cf = \sup_{ch \le cf} \int_E ch$

$= c \sup_{h \le f} \int_E h$

$= c\int_E f$

ii) To prove :-

$$\int_E (f+g) = \int_E f + \int_E g$$

Let h and K be two bounded measurable functions vanishes outside a set of finite measure

And $h \le f$ and $K \le g$

Also $m\{x : h(x) \ne 0\} < \infty$ and

$m\{x : K(x) \ne 0\} < \infty$

$\therefore h+K \le f+g$

$\int_E (h+K) \le \int_E (f+g)$

$\left(\int_E h + \int_E K\right) \le \int_E (f+g)$

$\sup_{h \le f} \int_E h + \sup_{K \le g} \int_E K \le \int_E (f+g)$

$\int_E f + \int_E g \le \int_E (f+g) \quad \text{—} 1$

Let l be a bounded measurable function which vanishes outside the set of finite measure and $l \le (f+g)$

Also $m\{x : l(x) \neq 0\} < \infty$

Define $h(x) = \min\{l(x), f(x)\}$

$$k(x) = l(x) - h(x) \quad \text{—} (2)$$

h and k are measurable functions

Also $h(x) \le f(x)$ and $k(x) \le g(x)$

$$h(x) + k(x) \le f(x) + g(x)$$

Also from (2) $h(x) + k(x) = l(x)$

If $h(x) = f(x)$, $k(x) \le g(x)$

If $h(x) = l(x)$ then $k(x) = 0$

ie) $k(x) \le g(x)$

Since g is non negative we have $k(x) \le g(x)$

$$\int_E l(x) = \int_E (h(x) + k(x))$$

$$= \int_E h(x) + \int_E k(x)$$

$$\sup_{l \le (f+g)} \int_E l = \sup_{h \le f} \int_E h + \sup_{k \le g} \int_E k$$

$$\int_E (f+g) \le \int_E f + \int_E g$$

$$\int_E f + \int_E g \ge \int_E (f+g) \quad \text{—} (3)$$

From (1) and (3), $\int_E (f+g) = \int_E f + \int_E g$

iii) Since $f \le g$ almost everywhere and the integral over a set of measure zero is zero. we can assume $f \le g$ everywhere ※.

Let h be a bounded measurable function which vanishes outside a set of measure zero

Also $h \le f$ and $f \le g \Rightarrow h \le g$

$$\therefore \sup_{h \le f} \int_E h \le \sup_{h \le g} \int_E h \Rightarrow \int_E f \le \int_E g$$

Hence proved.

Theorem :- Fatou's lemma

[If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \leq \underline{\lim} \int_E f_n]$$

Proof:-

Since the integral over a set of measure zero is zero, we may assume that $f_n \to f$ everywhere

$\therefore m\{x : f_n(x) \not\to f(x)\} = 0$

Let h be a bounded measurable function defined on E which is not greater than f ($h \leq f$) and which vanishes outside a set E of finite measure.

i.e) $h \leq f$

and $mE = m\{x \in E : h(x) \neq 0\} < \infty$

Define $h_n(x) = \min\{h(x), f_n(x)\}$

clearly $h_n(x)$ is measurable and $h_n(x) \leq h(x)$ for every n.

Also 'h_n' is bounded and vanishes outside E.

Also $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min\{h(x), f_n(x)\}$

$$= \min\{h(x), \lim_{n\to\infty} f_n(x)\}$$
$$= \min\{h(x), f(x)\}$$
$$= h(x)$$

$\therefore h_n(x) \to h(x)$

Thus the sequence $\{h_n\}$ is a sequence of bounded measurable functions converging to the bounded measurable function 'h' and vanishes outside E.

$\therefore$ By Bounded convergence theorem

$$\int_E h = \lim_{n\to\infty} \int_E h_n$$

$$\int_E h = \lim_{n\to\infty} \int_E h_n \leq \underline{\lim}_{n\to\infty} \int_E f_n \quad \because h_n \leq f_n$$

and outside E, h vanishes

Thus $\int_E h \leq \underline{\lim}_{n\to\infty} \int_E f_n$

$$\sup_{h\leq f} \int_E h \leq \underline{\lim}_{n\to\infty} \int_E f_n \Rightarrow \int_E f \leq \underline{\lim}_{n\to\infty} \int_E f_n$$

Hence proved.]

5M(S)
2M(S)

[Monotone convergence Theorem :-

Let $\{f_n\}$ be an increasing sequence of non-negative measurables functions and let $f = \lim_{n\to\infty} f_n$ almost everywhere, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$] 2M(S)

Proof :-

Since $\{f_n\}$ is a sequence of non-negative measurable function converging to f, we have by Fatou's lemma

$$\int_E f \leq \underline{\lim}_{n\to\infty} \int_E f_n \quad \text{—①}$$

As $\{f_n\}$ is a increasing sequence of non-negative measurable functions converging to f, we have $f_n \leq f \ \forall n$

$$\int_E f_n \leq \int_E f$$

$$\therefore \overline{\lim} \int_E f_n \leq \int_E f \quad \text{—②}$$

From ① and ② $\overline{\lim} \int_E f_n \leq \int_E f \leq \underline{\lim}_{n\to\infty} \int_E f_n$

But $\underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n$

$$\therefore \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n$$

$$\therefore \lim_{n\to\infty} \int_E f_n = \int_E f$$] 5M(S)

Corollary :-

Let $\{u_n\}$ be a sequence of non-negative measurable functions and let $f = \sum_{n=1}^{\infty} u_n$, then $\int_E f = \sum_{n=1}^{\infty} \int_E u_n$

Proof :-

Let $f_n = \sum_{i=1}^{n} u_i$

Since u_i's are measurable, f_n is measurable.

Also f_n is non-negative since each u_i is non-negative

Also $f_n \leq f_{n+1}$

i.e) $f_n = u_1 + u_2 + \dots + u_n \leq u_1 + u_2 + \dots + u_n + u_{n+1} = f_{n+1}$

$\therefore \{f_n\}$ is a increasing sequence.

$\lim_{n\to\infty} f_n = f$ (by monotone convergence Theorem)

i.e) $\lim_{n\to\infty} f_n = \sum_{n=1}^{\infty} u_n$

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

$$= \lim_{n\to\infty} \int_E \sum_{i=1}^{n} u_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i$$

$$= \sum_{i=1}^{\infty} \int_E u_i$$

Hence the proof.

Proposition :-

Let f be a non-negative function and $\{E_i\}$ a disjoint sequence of measurable sets. Let $E = \bigcup_{i=1}^{\infty} E_i$. Then $\int_E f = \sum \int_{E_i} f$

Proof :-

Let $f_i = f \chi_{E_i}$

Since each E_i is measurable f_i is measurable.

Since f is non-negative, each f_i is non-negative

Now $f \chi_E = f \chi_{\bigcup_{i=1}^{\infty} E_i}$

$= \sum_{i=1}^{\infty} f \chi_{E_i}$

$= \sum_{i=1}^{\infty} f_i$

By above corollary

$$\int_E f \chi_E = \sum_{i=1}^{\infty} \int_E f_i$$

$$\int_E f \chi_E = \sum_{i=1}^{\infty} \int_E f \chi_{E_i} = \sum_{i=1}^{\infty} \int_{E_i} f \chi_{E_i}$$

$$\int_E f = \sum_{i=1}^{n} \int_{E_i} f$$

Hence the proof.

Definition :- Integrable

A non-negative measurable function f is called integrable over the measurable set E if $\int_E f < \infty$

Proposition :-

Let f and g be two non-negative measurable functions. If f is integrable over E and $g(x) < f(x)$ on E, then g is also integrable on E and

$\int_E (f - g) = \int_E f - \int_E g$

Proof :-

Let $f = (f-g) + g$

$f-g$ and g are non-negative

$$\int_E f = \int_E ((f-g)+g)$$

$$= \int_E (f-g) + \int_E g$$

Since $\int_E f < \infty$, $\int_E (f-g) + \int_E g < \infty$

$$\Rightarrow \int_E g < \infty$$

$\therefore$ g is integrable.

Hence proved.

5M(S) ·X·

Proposition :-

Let f be a non-negative function which is integrable over a set E. Then given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \epsilon$

Proof :-

Case (i) :- 'f' is bounded

As f is bounded, let $|f| \le M$

Then given $\epsilon > 0$ if $\delta = \frac{\epsilon}{M}$, then

$$\int_A f\,dx \le M \int_A dx = M \cdot mA < M\delta$$

$$< \cancel{M} \cdot \frac{\epsilon}{\cancel{M}}$$

$$< \epsilon$$

$$\int_A f\,dx < \epsilon$$

Hence proved.

Case (ii) :- If 'f' is unbounded

Let $f_n(x) = \min(f(x), n)$

ie) $f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{if } f(x) > n \end{cases}$

Then each f_n is measurable

If $f(x) < n$

$f_n(x) = f(x)$

$\le f_{n+1}(x)$ $[\text{As } f(x) < n < n+1]$

If $f(x) > n$

$f_n(x) = n$

And $f_{n+1}(x) = n+1$ (if $f(x) > n+1$)

$f_n(x) = n < n+1 = f_{n+1}(x)$

Thus $f_n(x) \le f_{n+1}(x)\ \forall n$

$f_n(x) \le f(x)\ \forall n$ $(\because f_n(x) = n$ and $f(x) > n)$

Thus $\{f_n\}$ is a sequence of non-negative measurable functions converging to f. Also f_n is a increasing sequence.

By Monotone converging theorem

$$\int_E f = \lim_{n\to\infty} \int_E f_n .$$

22-01-19 (Tue)

There exist a positive integer N such that $\left|\int_E f - \int_E f_n\right| < \varepsilon/2$ for $n \geq N$. In particular

$$\left|\int_E f - \int_E f_n\right| < \varepsilon/2 . \Rightarrow \left|\int_E (f-f_n)\right| < \frac{\varepsilon}{2}$$

Since f is integrable and ~~$f_n < f$~~ on E $\int_E (f-f_n) < \frac{\varepsilon}{2}$

Let $A \subset E$ with $mA < \delta = \frac{\varepsilon}{2N}$. Now

$$\int_A f = \int_A (f-f_N) + f_N = \int_A (f-f_N) + \int_A f_N \leq \varepsilon/2 + N \int_A dx$$

$$\leq \varepsilon/2 + N\, mA < \varepsilon/2 + \cancel{N} \frac{\varepsilon}{2\cancel{N}} < \varepsilon$$

$\therefore \int_A f < \varepsilon$ Hence proved.

21-01-19 (Mon)

The general Lebesgue Integral :-

Definition :-

Let f be any real function. Then the positive part f^+ of the function f is defined by $f^+ = \max(f, 0)$ and negative part f^- of f is defined by $f^- = \max(-f, 0)$

$$= -\min(f, 0)$$

If f is measurable, then f^+ and f^- are also measurable and $f = f^+ - f^-$.

Also $|f| = f^+ + f^-$

Definition :-

A measurable function f is said to be integrable if f^+ and f^- are both integrable over E. In this case, we define

$$\int_E f = \int_E f^+ - \int_E f^-$$

2M(S)

Proposition :-

Let f and g be integrable over E, then

i) the function 'cf' is integrable over E and $\int_E cf = c\int_E f$

ii) the function $(f+g)$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$

iii) If $f \le g$ almost everywhere $\int_E f \le \int_E g$

iv) If A and B are disjoint measurable sets contained in E then $\int_{A\cup B} f = \int_A f + \int_B f$

Proof :-

i) Suppose $c=0$ then $cf=0$ and hence $\int cf = 0$

Also $c\int f = 0 \Rightarrow \int_E cf = c\int_E f$

Suppose $\boxed{c>0}$

$(cf)^+ = \max(cf, 0) = c\max(f,0) = cf^+$

$(cf)^- = \max(-cf, 0) = c\max(-f,0) = cf^-$

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^-$$

$$= c\int_E f^+ - c\int_E f^-$$

$$= c\left[\int_E f^+ - \int_E f^-\right]$$

$$= c\int_E f$$

Suppose $\boxed{c<0}$

$(cf)^+ = \max(cf,0) = -c\max(-f,0) = (-c)(f^-)$

$(cf)^- = \max(-cf,0) = -c\max(f,0) = (-c)(f^+)$

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^-$$

$$= -c\int_E f^- - \int_E (-c)f^+$$

$$= -c\int_E f^- + c\int_E f^+$$

$$\int_E cf = c\left[\int_E f^+ - \int_E f^-\right] = c\int_E f$$

$\therefore \int_E (cf) = c\int_E f \quad \forall c$

ii) If f_1 and f_2 are any two non-negative integrable functions such that

$f = f_1 - f_2$ then $\int f = \int f_1 - \int f_2$

<u>Proof :-</u>

given $f = f_1 - f_2$

But $f = f^+ - f^-$

$\therefore f_1 - f_2 = f^+ - f^-$

$f_1 + f^- = f_2 + f^+$

Taking integral on both sides

$\int f_2 + f^+ = \int f_1 + f^-$

Since f^+, f^-, f_1 and f_2 are non-negative functions

$\int f_2 + \int f^+ = \int f_1 + \int f^-$

$\therefore \int f^+ - \int f^- = \int f_1 - \int f_2$

$\therefore \int f = \int f_1 - \int f_2 = \int f^+ - \int f^-$

Now since f and g are integrable $f = f^+ - f^-$

$g = g^+ - g^-$

$\therefore f+g = (f^+ - f^-) + (g^+ - g^-)$

$= (f^+ + g^+) - (f^- + g^-)$

Here f^+, f^-, g^+ and g^- are all non-negative and integrable.

$(f^+ + g^+), (f^- + g^-)$ are also non-negative and integrable.

put $f^+ + g^+ = f_1 \quad f^- + g^- = f_2$

$\therefore f+g = f_1 - f_2$

By previous result

$\int (f+g) = \int f_1 - \int f_2$

$= \int (f^+ + g^+) - \int (f^- + g^-)$

$= \int f^+ + \int g^+ - \int f^- - \int g^-$

$= \int f^+ - \int f^- + \int g^+ - \int g^-$

$\int (f+g) = \int f + \int g$

iii) If $f \le g$ almost everywhere

$\Rightarrow f-g \le 0$ almost everywhere

$\Rightarrow g-f \ge 0$ almost everywhere

$\int_E (g-f) \ge 0$ ~~almost everywhere~~

By the above property $\int_E g - \int_E f \ge 0$

$$\int g \ge \int f$$

$$\Rightarrow \int f \le \int g$$

iv) $\int_{A\cup B} f = \int f \chi_{A\cup B} = \int f(\chi_A + \chi_B) = \int f\chi_A + \int f \chi_B$

$$= \int_A f + \int_B f$$

Result :-

f is integrable iff $|f|$ is integrable.

Proof :-

If f is integrable then both f^+ and f^- are integrable and $|f| = f^+ + f^-$

$\therefore |f|$ is integrable.

Similarly converse part is also true.

1-19
e)
M(S)

Dominated Lebesgue convergence Theorem :-

Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$.

Then $\int_E f = \lim \int_E f_n$

Proof :-

Since g is integrable and $|f_n| \le g$ we have $|f_n|$ is integrable.

Hence f_n is integrable.

Now $|f(x)| = \lim_{n\to\infty} |f_n(x)|$ almost everywhere

$\le g(x)$ almost everywhere.

$\therefore |f|$ is integrable and f is integrable.

Now $g \geq |f_n| = \begin{cases} f_n & \text{if } f_n \geq 0 \\ -f_n & \text{if } f_n \leq 0 \end{cases}$

$|f_n| \leq g$

$0 \leq g - |f_n|$

$g - |f_n| \geq 0$

consider $g - f_n$, then $g - f_n \geq 0$

$g - f_n \to g - f$ almost everywhere

By Fatou's lemma

$$\int g - f \leq \underline{\lim} \int g - f_n$$

$$\leq \underline{\lim} \left(\int g - \int f_n\right)$$

$$\int g - \int f \leq \int g - \underline{\lim} \int f_n$$

Since g is integrable

$$\int f \geq \underline{\lim} \int f_n \quad \text{— (1)}$$

Consider $g + f_n$ then $g + f_n \geq 0$ and

$g + f_n \to g + f$ almost everywhere.

Again by Fatou's lemma

$$\int g + f \leq \underline{\lim} \int (g + f_n)$$

$$\int g + \int f \leq \underline{\lim} \left(\int g + \int f_n\right)$$

$$\leq \int g + \underline{\lim} \int f_n \quad \text{(since } g \text{ is integrable)}$$

$$\therefore \int f \leq \underline{\lim} \int f_n \quad \text{— (2)}$$

From (1) and (2)

$$\overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n$$

But $\overline{\lim} \int f_n \geq \underline{\lim} \int f_n$

$$\therefore \overline{\lim} \int f_n = \int f = \underline{\lim} \int f_n$$

$$\therefore \int f = \lim_{n \to \infty} \int f_n$$

Hence proved.] 10M (S)

Generalisation of Lebesgue convergence Theorem :-

Let $\{g_n\}$ be a sequence of integrable functions which converges almost everywhere to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\}$ converges to f almost everywhere. If $\int g = \lim \int g_n$.

Then $\int f = \lim \int f_n$

Proof :-

Clearly 'f_n' is integrable, since g_n are integrable.

Also $|f| = |\lim f_n|$ almost everywhere

$= \lim |f_n|$

$\le \lim g_n$

$= g$ almost everywhere

$\therefore |f| \le g$ almost everywhere.

$\therefore |f|$ is integrable.

Hence f is integrable.

Now $g_n \ge |f_n| = \begin{cases} f_n & \text{if } f_n \ge 0 \\ -f_n & \text{if } f_n \le 0 \end{cases}$

Consider $g_n - f_n$. If $f_n \ge 0$ then $g_n - f_n \ge 0$

Also $\lim\limits_{n\to\infty} (g_n - f_n) = g - f$ almost everywhere.

By Fatou's lemma

$\int g - f \le \underline{\lim} \int (g_n - f_n)$

$\le \underline{\lim} \int g_n + \underline{\lim}\, (-\int f_n)$

$\le \underline{\lim} \int g_n - \overline{\lim} \int f_n$

$-\int f \le -\overline{\lim} \int f_n$

$\int f \ge \overline{\lim} \int f_n$ — (1)

Consider $g_n + f_n$ for $f_n \le 0$

$g_n + f_n \ge 0$

$g_n + f_n \to g + f$ almost everywhere

By Fatou's lemma

$\int (g + f) \le \underline{\lim} \int (g_n + f_n)$

$\le \underline{\lim} \int g_n + \underline{\lim} \int f_n$

$\therefore \int f \le \underline{\lim} \int f_n$ — (2)

$\overline{\lim} \int f_n \le \int f \le \underline{\lim} \int f_n$

But $\overline{\lim} \int f_n \ge \underline{\lim} \int f_n$

$\overline{\lim} \int f_n = \int f = \underline{\lim} \int f_n$

$\therefore \int f = \lim\limits_{n\to\infty} \int f_n$

Hence proved.

[Convergence in Measure :- 2M(S)

Definition :-

A sequence $\{f_n\}$ of measurable functions defined on a measurable set E is said to converge in measure to a measurable function f, if given $\varepsilon > 0$, there exists a positive integer N such that for $n \geq N$ we have

$$m\{x : |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon \;]$$ 2M(S)

5M (S) [**Proposition :-**

Let $\{f_n\}$ be a sequence of measurable functions defined on E that converges in measure to f, then there is a subsequence $\{f_{n_K}\}$ of f_n which converges to f almost everywhere in E.

Proof :-

For each positive v, find an n such that

$m\{x : |f_n(x) - f(x)| \geq \frac{1}{2^v}\} < \frac{1}{2^v}$ $\qquad \varepsilon = \frac{1}{2^v}$

Let $E_v = \{x : |f_{n_v}(x) - f(x)| \geq \frac{1}{2^v}\}$

Since f_n and f are measurable, we have E_v as measurable set

Let $f_K = \bigcup_{v=K}^{\infty} E_v$

If $x \notin f_K$, $x \notin E_v$ $\forall v \geq K$

If $x \notin f_K$ then $\forall n \geq K$

$\lim_{n \to \infty} f_{n_v} = f$ if $x \notin f_K$ $\qquad m\{x : |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon$

Let $A = \bigcap_{K=1}^{\infty} \bigcup_{v=K}^{\infty} E_v$ $\qquad m\{x : |f_{n_K}(x) - f(x)| \geq \varepsilon\} < \varepsilon$

$A \subset \bigcup_{v=K}^{\infty} E_v$

$mA \leq m\left(\bigcup_{v=K}^{\infty} E_v\right) \leq \sum_{v=K}^{\infty} mE_v$

$< \sum_{v=K}^{\infty} \frac{1}{2^v}$ if $x \in A$

$< \frac{1}{2^{K-1}} \to 0$ as $K \to \infty$

$\therefore mA = 0$ $\quad \lim f_{n_K} = f$ almost everywhere.

Hence proved.] 5M(S)