1-02-19
(Fri)

Unit - III
Differentiation and Integration

Definition :- Vitali cover

Let $\mathcal{I}$ be a collection of intervals. Then we say that $\mathcal{I}$ covers a set E in the sense of Vitali, if for each $\epsilon > 0$ and x in E there is an interval $I \in \mathcal{I}$ such that $x \in I$ and $l(I) < \epsilon$.

The intervals may be open, closed or half open. But we do not allow degenerate intervals consisting of only one point.

Example :-

Let $\{r_n\}$ be an enumeration of rationals in $[a, b]$, then $\{I_{n,i}\}_{n,i}$ is a vitali cover of $[a,b]$ where $I_{n,i} = [r_n - \frac{1}{i}, r_n + \frac{1}{i}]$

2M(S) **Lemma :-** (Vitali)

Let E be a set of finite outer measure and $\mathcal{I}$ a collection of intervals that cover E in the sense of Vitali. Then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, I_2, \dots I_N\}$ of intervals in $\mathcal{I}$ such that

$$m^* \left[E \sim \bigcup_{n=1}^{N} I_n \right] < \epsilon$$

Proof :-

It is enough to prove the lemma in the case that each interval in $\mathcal{I}$ is closed. For otherwise we replace each interval by its closure and note that the set of end points of $I_1, I_2, \dots I_N$ has measure zero.

Since $m^* E < \infty$, given $\epsilon > 0$ there exists an open set O containing E with $m^* O < m^* E + \epsilon < \infty$. Since $\mathcal{I}$ is a Vitali cover for E and $E \subset O$ we have each interval I in $\mathcal{I}$ is contained in O.

Suppose that there are intervals which extends beyond O, remove those intervals and remaining intervals, still form a Vitali cover for E. For if $x \in E$ and since O is open there is a neighbourhood with center at x and radius 2ϵ contained in O. Hence there is an interval I in $\mathcal{I}$ such that $x \in I$ and $I \subset O$.

Now we construct a sequence $\{I_n\}$ of intervals as follows. Let I_1 be any interval in $\mathcal{I}$. Let k_1 be the least
supremum
upper bound of the lengths of the intervals in $\mathcal{I}$ which have no points in common with I_1

Clearly $k_1 < \infty$. Let I_2 be an interval from $\mathcal{I}$ which has no point in common with I_1 and let $l(I_2) > \frac{k_1}{2}$. Let k_2 be the least upper bound of the intervals in $\mathcal{I}$ which has no common point in I_1 and I_2.

Continuing in this way, let I_N be an interval from $\mathcal{I}$ which is disjoint from $I_1, I_2, \dots I_N$ with $l(I_N) > \frac{k_{N-1}}{2}$

Let k_N be the least upper bound of the length of the intervals in $\mathcal{I}$ which has no point in common with $I_1, I_2, \dots I_{N-1}$

Clearly $\{k_N\}$ is a decreasing sequence of non-negative real numbers.

Suppose $\bigcup_{i=1}^{N} I_i$ contains, atmost every point of E, then the theorem is proved.

otherwise we get an infinite sequence of disjoint intervals $\{I_n\}$ in $\mathcal{I}$ with $l(I_n) > \frac{k_{n-1}}{2}$, $n = 1, 2, \dots$ and $\bigcup_{i=1}^{\infty} I_i \subset O$

Since $m^* O < \infty$, we have $l(\bigcup_{i=1}^{\infty} I_i) < \infty)$

i.e) $\sum_{i=1}^{\infty} l(I_i) < \infty$

ie) given $\varepsilon_1 > 0$ $\exists$ a +ve integer N such that $\sum_{n=N+1}^{\infty} l(I_n) < \frac{\varepsilon_1}{5}$

Take $R = E \sim \bigcup_{n=1}^{N} I_n$

<u>To prove :-</u> $m^* R < \varepsilon_1$

let $x \in R$. Then $x \notin \bigcup_{n=1}^{N} I_n$. Since each I_n is closed we have $\bigcup_{n=1}^{N} I_n$ is closed. So x is not a limit point of $\bigcup_{n=1}^{N} I_n$. Then $\exists$ an interval I in $\mathcal{I}$, which contains x such that the length of I is so small that it has no point in common with $I_1, I_2, \ldots I_N$

$\therefore l(I) \leq K_N < 2l(I_{N+1})$

If $I \cap I_i = \phi$ for $i < n$ then $l(I) \leq K_n < 2l(I_{n+1})$

But $l(I_n) \to 0$ as $n \to \infty$

$\Rightarrow l(I) = 0$ which is not possible. Hence if $x \in I$, then there is atleast one interval in the sequence $\{I_n\}$ which has a point in common with I.

Let 'n_0' be the smallest integer such that I meets I_n. Then $n_0 > N$. Also $l(I) \leq K_{n_0 - 1} < 2l(I_{n_0})$

Since $x \in I$ and I has a common point with I_{n_0}, the distance from x to the midpoint of I_{n_0} is atmost $l(I) + \frac{1}{2} l(I_{n_0})$

And $l(I) + \frac{1}{2} l(I_{n_0}) \leq 2l(I_{n_0}) + \frac{1}{2} l(I_{n_0})$

$\leq \frac{5}{2} l(I_{n_0})$

Let J_{n_0} be an interval with same mid point as I_{n_0} with length 5 times $l(I_{n_0})$. Then $x \in J_{n_0}$. Now for each $x \in R$, then there is an interval J_n, $n \in N$ with the same mid point as I_n with length $5l(I_n)$ such that $x \in J_n$

$R \subseteq \bigcup_{n=N+1}^{\infty} J_n$

$$m^*R \leq m^*\left(\bigcup_{n=N+1}^{\infty} J_n\right)$$
$$\leq \sum_{n=N+1}^{\infty} m^* J_n = \sum_{n=N+1}^{\infty} \ell(J_n)$$
$$\leq \sum_{n=N+1}^{\infty} 5\ell(I_n) = 5 \sum_{n=N+1}^{\infty} \ell(I_n) < 5\left(\frac{\epsilon}{5}\right) = \epsilon$$

ie) $m^*R < \epsilon$

Hence the lemma.

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Derivatives of a function f :-

We define a set of four quantities called the derivatives of f at x as follows

$$D^+f(x) = \overline{\lim_{h\to 0^+}} \frac{f(x+h)-f(x)}{h}$$

$$D^- f(x) = \overline{\lim_{h\to 0^+}} \frac{f(x)-f(x-h)}{h}$$

$$D_+ f(x) = \underline{\lim_{h\to 0^+}} \frac{f(x+h)-f(x)}{h}$$

$$D_- f(x) = \underline{\lim_{h\to 0^+}} \frac{f(x)-f(x-h)}{h}$$ 2M(5)

If $D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x) \neq \pm\infty$ then f is differentiable at x and $f'(x)$ is the derivative of $f(x)$ which is the common value of derivatives.

Always $D^+f(x) \geq D_+f(x)$
and $D^-f(x) \geq D_-f(x)$

Lemma :-

If f is continuous on $[a,b]$ and one of its derivative (say D^+) is everywhere non-negative on (a,b) then f is non-decreasing on $[a,b]$
$f(x) \leq f(y)$ for $x \leq y$.

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Theorem :-

$f_1 < f_2 < f_3 < \dots f_n$

Let f be an increasing real valued function on the interval $[a, b]$. Then (i) f is differentiable almost everywhere. The derivative f' is measurable and

(ii) $\rightarrow \int_a^b f'(x)\,dx \leq f(b) - f(a)$

Proof :-

Let us show that the sets where any two derivatives are unequal have measure zero. Consider only the set E where $D^+ f(x) > D_- f(x)$, the sets arising from other combinations of derivatives being similarly handled.

Now the set E is the union of sets $E_{u,v} = \{x : D^+ f(x) > u > v > D_- f(x)\}$ for all rationals u, v. Hence it suffices to prove that outer measure $m^* E_{u,v} = 0$

Let $S = m^* E_{u,v}$ and choosing $\epsilon > 0$, enclosed $E_{u,v}$ in an open set O. ie) $mO < S + \epsilon$. For each point x in $E_{u,v}$ there is an arbitrarily small interval $[x-h, x]$ contained in O such that $f(x) - f(x-h) < vh$

O

$[x-h, x]$

By lemma, we can choose a finite collection $\{I_1, I_2, \dots I_N\}$ of them whose interiors cover a setsub A of $E_{u,v}$ of outer measure greater than $S - \epsilon$. Then summing the intervals, we have

I_N

$$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] < v \sum_{n=1}^{N} h_n$$
$$< v\, mO \qquad \because mO < S + \epsilon$$
$$< v(S + \epsilon)$$

Now each point $y \in A$ is the left end point of an arbitrarily small interval $(y, y+k)$ that is contained in some I_n and for which $f(y+k) - f(y) > uk$. Using lemma again, we can pick out a finite collection $\{J_1, J_2, \dots J_M\}$ of such intervals such that their union contains a subset of A of outer measure greater than $s-2\xi$.

Then summing those intervals yield

$$\sum_{i=1}^{n} f(y_i+k_i) - f(y_i) > u \sum_{i=1}^{M} k_i$$

$$> u(s-2\xi)$$

$J_1 \subset I_n$

Each interval J_i is contained in some interval I_n and if we sum those i for which $J_i \subset I_n$, we have

$$\sum f(y_i+k_i) - f(y_i) \leq f(x_n) - f(x_n - h_n)$$

Since f is increasing. Thus

$$\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) \geq \sum_{i=1}^{m} f(y_i+k_i) - f(y_i)$$

and so $v(s+\xi) > u(s-2\xi)$ $\quad vs + v\xi \geq us - 2u\xi$

$vs \geq us \quad v > u$

Since this is true for each positive ξ we have $vs \geq us$. But $u > v$ and so s must be zero.

This shows that $g(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ is defined almost everywhere and that f is differentiable where ever g is finite.

Let $g_n(x) = n\left[f(x+\frac{1}{n}) - f(x)\right]$ where we set $f(x) = f(b)$ for $x \geq b$. Then $g_n(x) \to g(x)$ for almost all x and so g is measurable. Since f is increasing, we have $g_n \geq 0$

Hence by Fatou's lemma

$$\int_a^b f(x+\tfrac{1}{n}) \qquad n\int_{a+1/n}^{b+1/n} f(x) \qquad n\int_a^b f(x)\,dx \qquad f$$

$$\int_a^b g \leq \underline{\lim} \int_a^b g_n = \underline{\lim}\; n\int_a^b \left[f(x+\tfrac{1}{n}) - f(x)\right] dx$$

$$= \underline{\lim}\left[n\int_b^{b+\frac{1}{n}} f - n\int_a^{a+\frac{1}{n}} f\right]$$

$$= \underline{\lim}\left[f(b) - n\int_a^{a+\frac{1}{n}} f\right]$$

$$\leq f(b) - f(a)$$

Hence the theorem.

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2M(S)

Functions of Bounded variations :-

Definition :-

Let f be a real valued function defined on the interval $[a,b]$ and let $a = x_0 < x_1 < x_2 < \dots < x_k = b$ be any subdivision of $[a,b]$

Define $p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

where we use, r^+ to denote r if $r \geq 0$ and 0 if $r \leq 0$ the set $r^- = |r| - r^+$

we have

$$p - n = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ - \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^-$$

$$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= f(b) - f(a)$$

The set $P = \sup p$

$N = \sup n$

$T = \sup t$

where we take the suprema over all possible sub-divisions of $[a,b]$

Clearly $P \leq T \leq P + N$

Recall :-

$f^+ = \max(f, 0)$

$f^- = \max(-f, 0)$

$f = f^+ - f^-$ and $|f| = f^+ + f^-$

P, N, T are called positive, negative total variations of f over $[a,b]$

Notation :-

P is same as $P_a^b[f]$

N is $N_a^b[f]$

T is $T_a^b(f)$

If $T_a^b(f) < \infty$ we say f is of bounded variation on $[a,b]$

Lemma :-

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If f is of bounded variation on $[a,b]$ then $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$

Proof :-

we have $p - n = f(b) - f(a)$

$$\therefore p = n + f(b) - f(a)$$

$$P \leq N + f(b) - f(a)$$

Taking supremum over all subdivisions of $[a,b]$ we have

$$\sup P \leq N + f(b) - f(a)$$

$$P \leq N + f(b) - f(a) \quad \text{— (1)}$$

Now, $p - n = f(b) - f(a)$

$$\Rightarrow n = p - (f(b) - f(a))$$

$$n \leq p - (f(b) - f(a))$$

Taking supremum the subdivisions of $[a,b]$ we get

$$N \leq P - (f(b) - f(a)) \quad \text{— (2)}$$

Since $P \leq t \leq T$ we have $P \leq T$

Also $N \leq T$

Since T is of bounded variation $T < \infty$.

Hence $P < \infty$ and $N < \infty$

From (1) and (2)

$$P - N \leq f(b) - f(a)$$

and $N - P \leq -(f(b) - f(a))$

ie) $P - N \geq f(b) - f(a)$

$$\Rightarrow P - N \leq f(b) - f(a) \leq P - N$$

ie) $P-N = f(b)-f(a)$ — 5

we have $t = p+n$

$\leq P+N$

Taking supremum over all subdivisions of $[a,b]$ we have $T \leq P+N$ — 3

Since $p-n = f(b)-f(a)$

$n = p-(f(b)-f(a))$

$t = p+n = 2p-(f(b)-f(a))$

$t = 2p-(P-N)$ [from 5]

since $T \geq t$, we have $T \geq 2P-(P-N)$

Taking supremum over all subdivisions of $[a,b]$

$T \geq 2P-(P-N) = P+N$ — 4

From 3 and 4 $T = P+N$

Hence the lemma.

Theorem :-

A function f is of bounded variation on $[a,b]$ iff f is the difference of two monotone real valued function of $[a,b]$.

Proof :-

Suppose f is of bounded variation on $[a,b]$.

Then $T_a^b(f) < \infty$

If $x \in [a,b]$, then $T_a^x(f) \leq T_a^b(f) < \infty$

Since $P_a^x(f) \leq T_a^x(f) < \infty$ and

$N_a^x(f) \leq T_a^x(f) < \infty$

Let $g(x) = P_a^x(f)$ and $N_a^x(f) = h(x)$

Since $P_a^x(f) < \infty$ and $N_a^x(f) < \infty$ we have g and h are real valued.

If $x_1 < x_2$ then $P_a^{x_1}(f) < P_a^{x_2}(f)$ $g(x_1) < g(x_2)$

Thus $g(x)$ is increasing

$N_a^{x_1}(f) < N_a^{x_2}(f)$

$h(x_1) < h(x_2)$ Thus h is increasing.

Since $P_a^b(f) - N_a^b(f) = f(b) - f(a)$
we have $P_a^x(f) - N_a^x(f) = f(x) - f(a)$

$$g(x) - h(x) = f(x) - f(a)$$
$$f(x) = g(x) - h(x) + f(a)$$
$$= g(x) - (h(x) - f(a))$$

Since h is a increasing real valued function we have $h(x) - f(a)$ is a increasing real valued function on $[a,b]$.

Thus f is the difference of the two increasing real valued functions on $[a,b]$.

Conversely,

Suppose $f(x) = g(x) - h(x)$ where g and h are increasing real valued functions

Now $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |g(x_i) - h(x_i) - g(x_{i-1}) + h(x_{i-1})|$

$\leq \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |h(x_i) - h(x_{i-1})|$

$\because f(x) = g(x) - h(x)$

Since g is increasing and h is increasing $g(x_i) - g(x_{i-1}) \geq 0$ and $h(x_i) - h(x_{i-1}) \geq 0$

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} (g(x_i) - g(x_{i-1})) + \sum_{i=1}^{n} (h(x_i) - h(x_{i-1}))$$

$\leq g(b) - g(a) + h(b) - h(a)$

Taking supremum we have

$T_a^b(f) \leq \sup g(b) - g(a) + h(b) - h(a) < \infty$

Thus f is a bounded variation.

Hence proved.

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Corollary :-

If f is a bounded variation on $[a,b]$, then f' exists almost everywhere on $[a,b]$.

Proof :-

Since f is of bounded variation on $[a,b]$ by the above theorem f is the difference of two increasing real valued functions on $[a,b]$

ie) $f = g - h$

As we know if g is increasing real valued function, then its derivative exists almost everywhere on $[a,b]$

Hence g and h exists almost everywhere on $[a,b]$. Hence f exists almost everywhere on $[a,b]$.

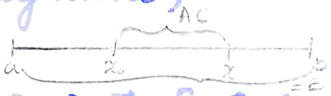
Hence the corollary.

Theorem :-

5M (S) If f is integrable on $[a,b]$ and if $F(x) = \int^{x} f(t)\, dt$, then F is continuous on $[a,b]$ and is of bounded variation on $[a,b]$.

Proof :-

Since f is integrable, we have $|f|$ is integrable.



$x_0 \in [a,b]$

Hence given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $A \subset E$ with $mA < \delta \Rightarrow \int_A |f| < \varepsilon$

Let $x_0 \in [a,b]$ and $A \subset [x_0, x]$ and $E = [a,b]$

$$|F(x) - F(x_0)| = \left| \int_a^x f(t)\, dt - \int_a^{x_0} f(t)\, dt \right|$$

$$= \left| \int_{x_0}^{x} f(t)\, dt \right|$$

$$\leq \int_{x_0}^{x} |f(t)|\, dt$$

$$= \left| \int_a^x f(t)\, dt + \int_{x_0}^{a} f(t)\, dt \right|$$

$$= \left| \int_{x_0}^{x} f(t)\, dt \right|$$

$$\leq \int_{x_0}^{x} f(t)\, dt$$

Thus whenever $|x - x_0| < \delta$

$\Rightarrow |F(x) - F(x_0)| \leq \int_{x_0}^{x} |f(t)|\, dt < \varepsilon$

ie) F is continuous at x_0

Since x_0 is any point in $[a,b]$, we have F is continuous in $[a,b]$

Let $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$ be subdivision of $[a,b]$

Now $|F(x_{i-1}) - F(x_i)| \leq \int_{x_{i-1}}^{x_i} |f(t)|\, dt$

$$\sum_{i=1}^{n} |F(x_{i-1}) - F(x_i)| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)\, dt|$$

$$\leq \int_a^b |f(t)|\, dt$$

Since f is integrable

$\int_a^b |f(t)|\, dt < \infty$

Therefore $\sup \sum_{i=1}^{n} |F(x_{i-1}) - F(x_i)| = T < \infty$

Thus F is a bounded variation in $[a, b]$.

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Theorem :-

If f is integrable on $[a,b]$ and if $\int_a^x f(t)\, dt = 0$ $\forall x \in$ ~~on~~ $[a, b]$ then $f = 0$ almost everywhere on $[a, b]$.

$f(t) = 0$

Proof :- (contrapositive approach)

Suppose $f \neq 0$ almost everywhere on $[a, b]$.

Let $\boxed{f > 0}$ on a set E of positive measure.

Since $mE > 0$ given $\epsilon > 0$ there exists a closed set $F \subset E$ such that

$$m(E - F) < \epsilon$$

Then $\boxed{mF > 0}$

$O \cup F = (a, b)$

$O + F = (a, b)$

Let $O = (a, b) - F$

$= (a, b) \cap F^c$ which is open.

Also $(a, b) = O \cup F$

Then either $\int_a^b f(t)\, dt \neq 0$ or $\int_a^b f(t)\, dt = 0$.

Since $\int_a^b f(t)\, dt = 0 \Rightarrow \int_{O \cup F} f(t)\, dt = 0$

i.e) $\int_O f(t)\, dt + \int_F f(t)\, dt = 0$

i.e) $\int_O f(t)\, dt = -\int_F f(t)\, dt$

Since $mF > 0$ and $f > 0$ on F, we have $\int_F f(t)\, dt > 0$

Therefore $\int_O f(t)\, dt \neq 0$

Since O is open, O can be written as the countable union of disjoint open intervals $\{I_i\}$

(a_n, b_n)

$$\therefore \int_0 f(t)\,dt = \int_{\bigcup_{n=1}^{\infty}(a_n,b_n)} f(t)\,dt \neq 0$$

$$\sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(t)\,dt \neq 0$$

Hence for some n, $\int_{a_n}^{b_n} f(t)\,dt \neq 0$

i.e.) $\int_a^{b_n} f(t)\,dt - \int_a^{a_n} f(t)\,dt \neq 0$

Hence either $\int^{b_n} f(t)\,dt \neq 0$ or $\int_a^{a_n} f(t)\,dt \neq 0$

Hence $x \in [a,b]$ $\ni$ $\int_a^x f(t)\,dt \neq 0$

~~which is a contradiction.~~

$\therefore f = 0$ almost everywhere on $[a,b]$.

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Theorem :-

If f is bounded measurable function on $[a,b]$ and if $F(x) = \int_a^x f(t)\,dt + F(a)$ then $F' = f$ almost everywhere on $[a,b]$

Proof :-

Since f is bounded and measurable it is Lebesgue Integrable.

Hence f is of bounded variation on $[a,b]$

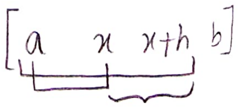
F' exists almost everywhere on $[a,b]$.

Suppose $|f(t)| \leq K$ for $t \in [a,b]$

Define $f_n(x) = \dfrac{F\left(x+\frac{1}{n}\right) - F(x)}{1/n}$

If $\frac{1}{n} = h$

$$f_n(x) = \frac{F(x+h) - F(x)}{h}$$



Now $|f_n(x)| = \left|\dfrac{F(x+h) - F(x)}{h}\right|$

$$= \frac{1}{h}\left|\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right|$$

$$= \frac{1}{h}\left|\int_x^{x+h} f(t)\,dt\right|$$

$$\leq \frac{1}{h}\int_x^{x+h} |f(t)|\,dt$$

$$\leq \frac{1}{h}\int_x^{x+h} K\,dt$$

$$|f_n(x)| \leq \frac{1}{h} K\left[\cancel{x}+h-\cancel{x}\right] = \frac{K\cancel{h}}{\cancel{h}} = K$$

Thus f_n is also bounded by K

Now $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{F\left(x+\frac{1}{n}\right) - F(x)}{\frac{1}{n}}$

$$= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$= F'(x)$ almost everywhere

Thus (f_n) is a sequence of bounded measurable functions converging to $F'(x)$ almost everywhere on $[a,b]$.

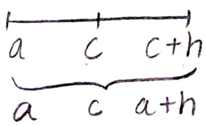
Hence by bounded convergence Theorem for any $c \in [a,b]$

$$\int_a^c F'(x)\,dx = \lim_{n\to\infty}\int_a^c f_n(x)\,dx$$

Now

$$\lim_{n\to\infty}\int_a^c f_n(x)\,dx = \lim_{h\to 0}\int_a^c \frac{F(x+h) - F(x)}{h}\,dx$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left(\int_a^c F(x+h)\,dx - \int_a^c F(x)\,dx\right)\right]$$



$$= \lim_{h\to 0}\frac{1}{h}\left[\int_c^{c+h} F(x)\,dx + \int_a^c \cancel{F(x)}\,dx - \int_a^{a+h} F(x)\,dx - \cancel{\int_a^c F(x)\,dx}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\int_c^{c+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx\right]$$

Now,

$$\lim_{h\to 0}\frac{1}{h}\int_c^{c+h} F(x)\,dx = \lim_{h\to 0}\frac{1}{h} F(c+\theta h) \text{ for } 0<\theta<1$$

$= F(c)$

Hence $\lim_{n\to\infty} \int_a^c f_n(x)\,dx = F(c) - F(a)$

$$\int_a^c F'(x)\,dx = F(c) - F(a) = \int_a^c f(x)\,dx$$

$$\int_a^c (F'(x) - f(x))\,dx = 0$$

By previous theorem $F'(x) - f(x) = 0$ exists almost everywhere on $[a,b]$

$\Rightarrow$ $F'(x) = f(x)$ exists almost everywhere on $[a,b]$

Hence the theorem.

Theorem :-

If f is integrable on $[a,b]$ and if $F(x) = \int_a^x f(t)\,dt + F(a)$ then $F'(x) = f(x)$ almost everywhere in $[a,b]$

Proof :-

Since f is integrable.

f^+ and f^- are also integrable.

Also $f^+ \geq 0$ and $f^- \geq 0$

Assume that f is non-negative

Define $f_n(x)$ as

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n \end{cases}$$

Let $G_n(x) = \int_a^x (f - f_n)$ —(1)

$$G_n(x) = \int_a^x f - \int_a^x f_n$$

$\Rightarrow \int_a^x f = G_n(x) + \int_a^x f_n$ —(3)

Now since

$f - f_n \geq 0$ and $m[a,x] \geq 0$

G_n is an increasing function of x and

$G_n(x) \geq 0$

Therefore G_n has a derivative almost

everywhere and $G_n'(x)$ is non-negative.

Also each f_n is bounded and measurable.

By previous theorem

$$\frac{d}{dx}\int_a^x f_n(t)\,dt = f_n(x) \text{ almost everywhere} \quad \text{—(2)}$$

Now $F(x) = \int_a^x f(t)\,dt + F(a)$

$$= G_n(x) + \int_a^x f_n + F(a) \quad [\text{by (3)}]$$

Differentiating both sides

$$\frac{d}{dx}F(x) = \frac{d}{dx}G_n(x) + \frac{d}{dx}\int_a^x f_n(t)\,dt + \frac{d}{dx}F(a)$$

Thus,

$F'(x) = G_n'(x) + f_n(x)$ almost everywhere by (2)

$F'(x) \geq f_n(x)$ almost everywhere

$F'(x) - f_n(x) \geq 0$ almost everywhere

Since n is arbitrary

$$F'(x) \geq f(x) \text{ almost everywhere} \quad \text{—(4)}$$

Consider

$$\int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx \geq F(b) - F(a) \quad \text{—(5)}$$

Since F is a increasing real valued

we have

$$\int_a^b F'(x)\,dx \leq F(b) - F(a)$$

By theorem

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{—(6)}$$

From (5) and (6)

$$\int_a^b F'(x)\,dx = F(b) - F(a) = \int_a^b f(x)\,dx$$

$$\int_a^b (F'(x) - f(x))\,dx = 0$$

$F'(x) - f(x) \geq 0$ almost everywhere

$\int_a^x (F'(t) - f(t))\,dt = 0$ almost everywhere

$F'(t) = f(t)$ almost everywhere

ie) $F'(x) = f(x)$ almost everywhere in $[a,b]$.

Hence proved.

2M(5) **Absolute continuity :-**

A real valued function f defined on $[a,b]$ is said to be **absolutely continuous** on $[a,b]$ if for given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$ for ~~all~~ every finite collection of disjoint intervals $\{(x_i, x_i')\}$ with $\sum_{i=1}^{n} |x_i - x_i'| < \delta$

Note :-

An absolutely continuous function is continuous.

Proof :-

Since f is absolutely continuous.
There exist $\delta > 0$ and $\varepsilon > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon \text{ whenever } \sum_{i=1}^{n} |x_i - x_i'| < \delta$$

consider

$$|f(x_i') - f(x_i)| \le \sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$$

This is true whenever

$$|x_i' - x_i| \le \sum_{i=1}^{n} |x_i' - x_i| < \delta$$

Thus f is continuous.

02-19
(Fri)

Theorem :-

Every indefinite integral is absolutely continuous.

Proof :-

$F(x) = \int_a^x f(t)\,dt$ be the indefinite integral.
By proposition if f is non-negative integrable function over a set E, then for $\varepsilon > 0$ there exists a $\delta > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta$ we have $\int_A f < \varepsilon$

Now taking f is integrable and $E = [a,b]$ and $A = [x, y] \subseteq [a,b]$

$m(A) < \delta$

$m(A) = |x-y|$

ie) $|x-y| < \delta \Rightarrow \int_x^y f < \epsilon$

$$\int_x^a f + \int_a^y f = \int_x^y f$$

$$\int_a^y f - \int_a^x f = \int_x^y f$$

$\Rightarrow \int_a^y f - \int_a^x f < \epsilon$

$F(y) - F(x) < \epsilon$

$\therefore$ F is continuous

For a finite collection of non-overlapping intervals $A = \{(x_i, x_i')\}$ with $m(A) < \delta$

we have $\int_A f < \epsilon$

$$\sum_{i=1}^{n} \int_{x_i}^{x_i'} f < \epsilon$$

$$\sum_{i=1}^{n} [F(x_i) - F(x_i')] < \epsilon$$

Hence f is absolutely continuous.

Hence proved.

2M(S)

Lemma :-

If f is absolutely continuous on $[a,b]$ then f is of bounded variations on $[a,b]$

Proof :-

Since f is absolutely continuous.

For $\boxed{\epsilon > 1}$ there exists $\delta > 0$ such that

$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$ whenever $\sum_{i=1}^{n} |x_i' - x_i| < \delta$

Here the intervals $\{(x_i, x_i')\}$ are non-overlapping (disjoint) intervals.

Let $a = y_0 < y_1 < y_2 < \dots < y_n = b$ be any subdivision on $[a,b]$.

Let K is the largest integer such that

$K < 1 + \frac{b-a}{\delta}$

Now split the above subdivision into K

set of intervals, ~~where the~~ each of total length ~~of each set of the interval~~ is less than δ.

Now each set of these intervals

$\Sigma |f(x_i') - f(x_i)| \leq 1$

$t = \sum_{i=1}^{n} |f(y_{i+1}) - f(y_i)| \leq K \leq \sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)|$

Taking supremum of all the intervals

$T \leq K \leq \infty$ $\therefore f$ is of bounded variation.

Hence proved.

Corollary :-

If f is absolutely continuous on $[a,b]$ then f has a derivative almost everywhere.

Proof :-

If f is absolutely continuous on $[a,b]$ then by above theorem f is of bounded variation on $[a,b]$

Then by the theorem, there exists $f'(x)$ almost everywhere. Therefore f has a derivative almost everywhere.

Hence proved.

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10 M (S) + 5 M (S)

Theorem :-

If f is absolutely continuous on $[a,b]$ and if $f'(x) = 0$ almost everywhere on $[a,b]$ then f is a constant.

Proof :-

Since f is absolutely continuous for $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$ whenever $\sum_{i=1}^{n} |x_i' - x_i| < \delta$ — (1),

for every finite collection of disjoint intervals $\{(x_i, x_i')\}$. Take any point in $[a,b]$ where $f'(x) = 0$ almost everywhere.

Let $E = \{x \mid a < x < c$ and $f'(x) = 0$ a.e.$\}$ $[\because E \subset (a,c)]$

therefore for all x in E, $f'(x) = 0$ a.e for $\eta > 0$ there exists intervals $[x, x+h] \subseteq [a, c]$ such that $\left|\frac{f(x+h) - f(x)}{h}\right| < \eta$

when $\eta \to 0$, $h \to 0$

$\therefore |f(x+h) - f(x)| < h \times \eta$ — (2)

From Vitali's cover lemma from the covering $[x, x+h]$ of E, for $\delta > 0$ we can find a non overlapping intervals $\{(x_K, y_K)\}$ such that these intervals cover E except ~~of~~ for a set of measure $< \delta$.

i.e) $m^*(E \sim \cup [x_K, y_K]) < \delta$

If $x_K \leq x_{K+1}$ then

$y_0 = a \leq x_1 < y_1 \leq x_2 < y_2 < \dots \leq y_n < c = x_{n+1}$

$\therefore \sum_{K=0}^{n} |x_{K+1} - y_K| < \delta$

$a = y_0 \leq x_1 < y_1 \leq x_2 < \dots < y_n = c \leq x_{n+1}$

consider by (2)

$$\sum_{i=1}^{n} |f(y_K) - f(x_K)| \leq \eta \sum_{K=1}^{n} (y_K - x_K)$$
$$\leq \eta (c-a)$$

We know that since $\sum_{K=0}^{n} |x_{K+1} - y_K| < \delta$ [by the absolute continuity of f]

$$\sum_{K=0}^{n} |f(x_{K+1}) - f(y_K)| < \xi \quad \text{— (3)}$$

$$f(c) - f(a) = \sum_{K=0}^{n} |f(x_{K+1}) - f(y_K)| + \sum_{K=1}^{n} |f(y_K) - f(x_K)|$$
$$< \xi + \eta(c-a)$$

Since ξ and η are very small then $\xi \to 0$ and $\eta \to 0$ we have

$f(c) - f(a) = 0 \Rightarrow f(c) = f(a)$

$\therefore$ For any element c in $[a, b]$

$f(c) = f(a)$

$\Rightarrow f(x) = f(a)$

$\therefore$ f is constant.

09-02-19 (Sat)

Theorem :-

A function F is an indefinite integral iff it is absolutely continuous.

Proof :-

Suppose F is indefinite integral and

$$F(x) = \int_a^x f(t)\,dt$$

By the proposition "If f is a non-negative function which is integrable over a set E. Then for $\epsilon > 0$ there exists $\delta > 0$ such that for all A in E with $m(A) < \delta$

we have $\int_A f < \epsilon$"

Here the set $E = [a,b]$

Let $A = \bigcup_{i=1}^{n} (x_i, x_i')$

Therefore for finite collection of non overlapping intervals with $|x_i - x_i'| < \delta$ with $m(A) < \delta$ then $\int_A f < \epsilon$.

ie) $\displaystyle\sum_{i=1}^{n} \int_{x_i}^{x_i'} |f(t)|\,dt < \epsilon$

$$\sum_{i=1}^{n} |F(x_i) - F(x_i')| < \epsilon \quad \forall \delta > 0$$

whenever

$$\sum_{i=1}^{n} |x_i - x_i'| < \delta$$

Hence F is absolutely continuous on $[a,b]$

converse part:

If F is absolutely continuous, then

F is of bounded variation on $[a,b]$

then f is a difference of two monotonically increasing functions.

$\therefore f(x) = f_1(x) - f_2(x)$

where $f_1(x)$ and $f_2(x)$ are monotonically continuous and also $F'(x)$ exists almost everywhere

$$F'(x) = f_1'(x) - f_2'(x)$$

Consider $|f'(x)| = |f_1'(x) - f_2'(x)|$

$$\leq |f_1'(x)| + |f_2'(x)|$$

$$\leq f_1'(x) + f_2'(x)$$

$$\int_a^b |f'(x)|\,dx \leq \int_a^b f_1'(x)\,dx + \int_a^b f_2'(x)\,dx$$

$\leq f_1(b) - f_1(a) + f_2(b) - f_2(a)$

Therefore f is integrable.

Hence f is differentiable

Let $G(x) = \int_a^x F'(t)\,dt$

Therefore G is absolutely continuous.

Also $G'(x) = F'(x)$ almost everywhere.

Now define $f = F - G$

$$f'(x) = F'(x) - G'(x)$$
$$= 0 \text{ almost everywhere.}$$

Moreover F and G are absolutely continuous

$\therefore$ f is absolutely continuous and $f'(x) = 0$ almost everywhere.

By previous theorem f is constant

$F(x) = G(x)$ is constant. $\quad G(x) = \int_a^x f'(t)\,dt$

$F(a) - G(a) = F(a)$ (If $G(a) = 0$) $\quad G(a) = \int_a^a f'(t)\,dt = 0$

$F(x) - \int_a^x f'(t)\,dt = F(a)$ $\quad F(a) - G(a) = F(a)$

$\therefore F(x) = \int_a^x f'(t)\,dt + F(a)$ $\quad F(x) - G(x) = F(a)$

$\therefore f$ is indefinite integral.

Corollary :-

Every absolutely continuous function is the indefinite integral of its derivative.