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Unit - IV

Differentiation and Integration

15)

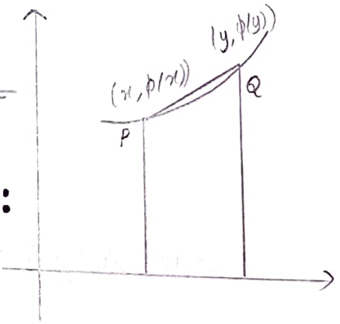
Convex functions :-

A function ϕ defined on an open interval (a,b) is said to be convex, if for each $x, y \in (a,b)$ and each λ, $0 \le \lambda \le 1$, we have

$$\phi(\lambda x + (1-\lambda)y) \le \lambda \phi(x) + (1-\lambda)\phi(y)$$

Geometrical meaning :-

Any point between x and y is $\lambda x + (1-\lambda)y$. If a function is convex then ϕ value at an inbetween point is less than or equal to $\lambda\phi(x) + (1-\lambda)\phi(y)$. The chord (y) between $(x, \phi(x))$ and $(y, \phi(y))$ will be above the graph of ϕ



Lemma :- Property of the chords of a convex function :

If ϕ is convex on (a,b). If x, y, x', y' are points of (a,b) with $x \le x' \le y'$ and $x < y \le y'$. Then the chord over x' and y' has larger slope than the chord over (x,y)

$$\frac{\phi(y) - \phi(x)}{y - x} \le \frac{\phi(y') - \phi(x')}{y' - x'}$$

Proof :-

Suppose if $x_1 \le x \le x_2$ then

$x_2 - x \le x_2 - x_1$

$$\therefore \frac{x_2 - x}{x_2 - x_1} \le 1$$

Let $\lambda = \dfrac{x_2 - x}{x_2 - x_1} \le 1$

Take the points x_1 and x_2, since ϕ is convex.

$$\phi(\lambda x_1 + (1-\lambda)x_2) \le \lambda\phi(x_1) + (1-\lambda)\phi(x_2)$$

$$\therefore \phi\left[\left(\frac{x_2-x}{x_2-x_1}\right)x_1 + \left(1-\left(\frac{x_2-x}{x_2-x_1}\right)\right)x_2\right] \le$$

$$\cancel{x_2} - x_1 - \cancel{x_2} + x$$

$$\left(\frac{x_2-x}{x_2-x_1}\right)\phi(x_1) + \left(1-\left(\frac{x_2-x}{x_2-x_1}\right)\right)\phi(x_2)$$

$$\cancel{x_2 x_1} - x x_1 + x x_2 - \cancel{x_1 x_2} = x(x_2 - x_1)$$

$$\phi\left[\left(\frac{x_2-x}{x_2-x_1}\right)x_1 + \left(\frac{x-x_1}{x_2-x_1}\right)x_2\right] \le \left(\frac{x_2-x}{x_2-x_1}\right)\phi(x_1) + \left(\frac{x-x_1}{x_2-x_1}\right)\phi(x_2)$$

$$\phi(x) \le \left(\frac{x_2-x}{x_2-x_1}\right)\phi(x_1) + \left(\frac{x-x_1}{x_2-x_1}\right)\phi(x_2) \quad \text{— (1)}$$

Here x is a point which lies between x_1 and x_2. So in (1) put $x_1 = x$ and $x = x'$, $x_2 = y'$

Since $x_1 \le x \le x_2$ $[x \le x' < y']$ $\quad x \le x' \le y'$, $x_1 \le x \le x_2$

$$\phi(x') \le \frac{y'-x'}{y'-x}\phi(x) + \frac{x'-x}{y'-x}\phi(y') \quad \text{— (2)}$$

$x = x'$, $x_2 = y'$, $x_1 = x$

$$\frac{(2)}{y'-x'} \Rightarrow \frac{\phi(x')}{y'-x'} \le \frac{\phi(x)}{y'-x} + \frac{x'-x}{(y'-x')(y'-x)}\phi(y') \quad \text{— (3)}$$

Since $x < y \le y'$ put $x_1 = x$ and $x = y$, $x_2 = y'$ in (1) $\quad x' = y$

$$\phi(y) \le \frac{y'-y}{y'-x'}\phi(x) + \frac{y-x}{y'-x}\phi(y') \quad \text{— (4)}$$

$x < y \le y'$, $x_1 \le x \le x_2$

$$\frac{(4)}{y-x} \Rightarrow \frac{\phi(y)}{y-x} \le \frac{y'-y}{(y'-x')(y-x)}\phi(x) + \frac{1}{y'-x}\phi(y') \quad \text{— (5)}$$

(3) + (5)

$$\frac{\phi(x')}{y'-x'} + \frac{\phi(y)}{y-x} \le \phi(x)\left[\frac{1}{y'-x} + \frac{y'-y}{(y'-x)(y-x)}\right] + \phi(y')\left[\frac{x'-x}{(y'-x)(y'-x')} + \frac{1}{y'-x}\right]$$

$$\le \phi(x)\left[\frac{\cancel{y}-x+y'-\cancel{y}}{(y'-x)(y-x)}\right] + \phi(y')\left[\frac{\cancel{x'}-x+y'-\cancel{x'}}{(y'-x)(y'-x')}\right]$$

$$\le \frac{1}{y-x}\phi(x) + \frac{1}{y'-x'}\phi(y')$$

$$\frac{\phi(y)}{y-x} - \frac{\phi(x)}{y-x} \leq \frac{1}{y'-x'}\phi(y') - \frac{1}{y'-x'}\phi(x')$$

$$\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(y')-\phi(x')}{y'-x'}$$

Hence proved.

Note :-

$D^- f$ and $D_- f$ are finite and equal then f is differentiable on the left at x and the common value is called left hand derivative of x.

If $D^+ f$ and $D_+ f$ are finite and equal then f is differentiable on the right ~~of~~ at x and the common value is called right hand derivative of x.

5M(S) **Proposition :-**

If ϕ is convex on (a,b) then

(i) ϕ is absolutely continuous on each closed subinterval of (a,b)

(ii) If right and left hand derivative of ϕ exists at each point of open interval (a,b) and are equal to each other except on a countable set.

(iii) The left and right hand derivative and monotone increasing function.

(iv) And at each point the left hand derivative is less than or equal to right hand derivative.

Proof :-

i) Given ϕ is convex

we have to show that if $U(x_i, y_i)$ ~~of~~ is the non-overlapping interval in $[c,d]$ such that $\Sigma |y_i - x_i| < \delta$

we have $\Sigma|\phi(y_i)-\phi(x_i)|<\varepsilon_1$

Let $[c,d] \subset (a,b)$

Now let $x,y \in [c,d] \subset (a,b)$ such that $x<y$

$\therefore a<c\le x<y\le d<b$

Since ϕ is convex, by previous property

$$\frac{\phi(c)-\phi(a)}{c-a} \le \frac{\phi(y)-\phi(x)}{y-x}$$

$$\le \frac{\phi(b)-\phi(d)}{b-d}$$

$$\frac{\phi(c)-\phi(a)}{c-a} \le \frac{\phi(b)-\phi(d)}{b-d} = M$$

$$\therefore \frac{\phi(y)-\phi(x)}{y-x} \le M$$

$$\phi(y)-\phi(x) \le M(y-x)$$

put $\boxed{\delta = \frac{\varepsilon_1}{M}}$

$$\Sigma|y_i-x_i|<\delta \Rightarrow \Sigma|\phi(y_i)-\phi(x_i)|<M\cdot\delta$$

$$<\cancel{M}\cdot\frac{\varepsilon_1}{\cancel{M}}$$

$$<\varepsilon_1$$

$\therefore \phi$ is absolutely continuous.

~~iii)~~ iii) If x_0 is a fixed point on (a,b) and x is any point in (a,b) $x>x_0 \Rightarrow x_0<x$

Suppose $x,y\in(a,b)$ such that $x<y$. Since ϕ is convex $x_0<x<y$

$$\frac{\phi(x)-\phi(x_0)}{x-x_0} \le \frac{\phi(y)-\phi(x_0)}{y-x_0}.$$

Hence the function ϕ is an increasing function of x.

Hence statement (iii) is proved.

iv) when $x\to x_0$ in the right and left, the limit exists, finitely at each point x_0 belongs to (a,b) $x_0\in(a,b)$

$\therefore \phi$ is differentiable on the right and ~~left~~

~~than~~ the left at each point $x_0 \in (a,b)$

$\therefore$ Always left hand derivative $\leq$ right hand derivative.

Hence proved.

ii) Moreover if $x_0 < y_0$ and $x < y_0$ and $x_0 < y$ then $x_0 < x$ $y_0 < y$

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \frac{\phi(y) - \phi(y_0)}{y - y_0}$$

ie) each derivative at $x_0 \leq$ each derivative at y_0

Since the derivative is monotonic increasing function also if one of them is continuous at a point, then they are equal at the point.

Since monotone increasing function can have only a countable number of discontinuities

$\therefore$ They are equal except on a countable sets.

5M(S)

A partial converse of above Theorem :-

If ϕ is a continuous function on (a,b) and if one derivative (say D^+) of ϕ is non decreasing then ϕ is convex.

Proof :- $\phi(\lambda x + (1-\lambda) y) \leq \lambda \phi(x) + (1-\lambda)\phi(y)$

$t = \lambda$ $y = x$ $x = y$

Let x, y be any point in (a,b) such that $a < x < y < b$

$\phi(ty + (1-t)x) \leq t\phi(y) + (1-t)\phi(x)$

Now define a function ψ on $[0,1]$ as follows:

$\psi(t) = \phi[t(y) + (1-t)x] - t\phi(y) - (1-t)\phi(x)$

We have to show that ψ is not positive on $[0,1]$

$\psi(1) = \phi(y) - \phi(y) = 0$

Since ϕ is continuous

ψ is continuous.

$\psi(0) = \phi(x) - \phi(x) = 0$

And $\psi(1) = \phi(y) - \phi(y) = 0$ $\Rightarrow \psi(0) = \psi(1)$.

Now consider

$$D^+\psi = \overline{\lim_{h\to 0}} \frac{\psi(u+h)-\psi(u)}{h}$$

$$\psi(u+h) = \phi[(u+h)y + 1-(u-h)x] - (u+h)\phi(y) - (1-u+h)\phi(x)$$

$$= \phi(uy+(1-u)x - \overset{+hy}{hx}) - u\phi(y) - h\phi(y) - \phi(x) + u\phi(x) + h\phi(x).$$

~~If $h\to 0$~~

$$\psi(u) = \phi(uy+(1-u)x) - u\phi(y) - (1-u)\phi(x)$$

$$\psi(u+h)-\psi(u) = \phi[uy+(1-u)x+h(y-x)] - \phi[uy+(1-u)x] + h[-\phi(y)+\phi(x)]$$

$$D^+\psi = \overline{\lim_{h\to 0}} \frac{\psi(u+h)-\psi(u)}{h}$$

$$= \overline{\lim_{h\to 0}} \left[\frac{\phi[uy+(1-u)x+h(y-x)] - \phi[uy+(1-u)x] + h[\phi(x)-\phi(y)]}{h}\right]$$

$$= \phi(x)-\phi(y) + \lim_{h(y-x)\to 0} (y-x)\frac{\left[\phi(uy+(1-u)x+h(y-x)) - \phi[uy+(1-u)x]\right]}{h(y-x)}$$

$$= \phi(x)-\phi(y) + (y-x)D^+\phi$$

Since $D^+\phi$ is non-decreasing

$D^+\psi$ is also non-decreasing on $[0,1]$

Let γ be any point where ψ assumes a maximum on $[0,1]$

If $\gamma = 1$, $\psi(t) \le \psi(1) = 0$

$\therefore \psi(t) \le 0 \quad \forall\, t\in[0,1]$

Take it $\gamma \ne 1$, then $\gamma\in[0,1]$

Then ψ has a local maximum at γ

$$\therefore D^+\psi(\gamma) = \overline{\lim_{h\to 0}} \frac{\psi(\gamma+h)-\psi(\gamma)}{h} \le 0$$

Since $D^+\psi$ is non decreasing for all $t\in[0,1]$

$t \le \gamma$

$\therefore D^+\psi(t) \le D^+\psi(\gamma) \le 0$

$\psi(t) \le 0$

Moreover $\psi(t) \le \psi(0) = 0$

$\therefore \psi(t) \le 0$

$\therefore \psi(t) = \phi[ty + (1-t)x] - t\phi(y) - (1-t)\phi(x) \le 0$

ie) $\phi[ty + (1-t)x] \le t\phi(y) + (1-t)\phi(x)$

$\therefore \phi$ is convex.

02-19 (Sat)

Definition :- Supporting line

Let ϕ be convex on (a,b) and let $x_0 \in (a,b)$.
The line $y = m(x - x_0) + \phi(x_0)$ through $(x_0, \phi(x_0))$ is called supporting line at x_0, if it ~~is~~ always lies below the graph of ϕ.

ie) $\phi(x) \ge y$

$$\phi(x) \ge m(x-x_0) + \phi(x_0) \quad \text{— (1)}$$

$$\phi(x) - \phi(x_0) \ge m(x-x_0)$$

$$\frac{\phi(x)-\phi(x_0)}{x-x_0} \ge m$$

From (1) $-\phi(x_0) \le -m(x-x_0) - \phi(x_0)$

$$\phi(x_0) - \phi(x) \le m(x_0 - x)$$

$$\frac{\phi(x_0)-\phi(x)}{x_0-x} \le m$$

$$\therefore \frac{\phi(x)-\phi(x_0)}{x-x_0} \ge m \ge \frac{\phi(x_0)-\phi(x)}{x_0-x}$$

$\therefore m$ lies between left hand and right hand derivative at x_0

ie) A line is a supporting line iff its slope m lies between left hand and right hand derivatives at x_0.

Hence there is a supporting line at each point.

M 15)
15)

Jensen's inequality :-

Let ϕ be a convex function on $(-\infty, \infty)$ and ~~if it is~~ f an integrable function on $[0,1]$ then

$$\int \phi(f(t))\,dt \geq \phi\left(\int f(t)\,dt\right)$$

Proof :-

Let $y = m(x-\alpha) + \phi(\alpha)$ be the equation of the supporting line at the point α, where $\alpha = \int_0^1 f(t)\,dt$

Since $\phi(x) \geq y$

$$\phi(x) \geq m(x-\alpha) + \phi(\alpha)$$

If $x = f(t)$ then

$$\phi(f(t)) \geq m(f(t)-\alpha) + \phi(\alpha)$$

Integrating both sides with respect to t on $[0,1]$

$$\int_0^1 \phi(f(t))\,dt \geq m\int_0^1 f(t)\,dt - m\alpha + \int_0^1 \phi(\alpha)\,dt$$

$$\geq m\alpha - m\alpha + \int_0^1 \phi(\alpha)\,dt$$

$$\geq \int_0^1 \phi(\alpha)\,dt = \phi(\alpha)$$

$$\therefore \int_0^1 \phi(f(t))\,dt \geq \phi\left(\int_0^1 f(t)\,dt\right)$$

Hence proved.

Note :-

i) A function ϕ is said to be strictly convex if $\phi[\lambda x + (1-\lambda)y] < \lambda\phi(x) + (1-\lambda)\phi(y)$ where $x, y \in (a,b)$ and ~~$0 \leq \lambda \leq 1$~~ $\lambda \in (0,1)$

ii) A function ϕ is said to be concave $\phi(-x)$ is conve

iii) A function which is both concave and convex is a linear function.

iv) If ϕ is continuous on any interval I (May be open or closed or half open) and conv in the interior, then ϕ is convex on I also.

02-19
(Mon)

Banach space :-

[complete Normed linear space] Every Cauchy sequence has limits

A normed linear space which is complete in the sense that Cauchy sequence has limits.

i) $\|\alpha f\| = |\alpha| \|f\|$
ii) $\|f+g\| \leq \|f\| + \|g\|$
iii) $\|f\| = 0 \Leftrightarrow f = 0$

Normed linear space

The L^p spaces :-

p = positive real number.

A measurable function defined on $[0,1]$

$L^p = L^p[0,1]$ if $\int_0^1 |f|^p < \infty$ (Lebesgue Integral)

Property :-

$|f+g|^p \leq 2^p(|f|^p + |g|^p)$

$f \in L^p$ we define $\|f\| = \|f\|_p = \left\{\int_0^1 |f|^p\right\}^{1/p}$

i) $\|\alpha f\| = |\alpha| \|f\|$
ii) $\|f+g\| \leq \|f\| + \|g\|$
iii) $\|f\| = 0 \Leftrightarrow f = 0$ almost everywhere

L^∞ - all bounded measurable function on $[0,1]$

$\|f\| = \|f\|_\infty$ = essential sup $|f(t)|$

essential sup $|f(t)| = \inf\{M : m\{t : f(t) > M\} = 0\}$

10M(S) **Minkowski inequality :-**

If 'f' and 'g' are in L^p with $1 \leq p \leq \infty$ then so is $f+g$ and

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$

If $1 < p < \infty$ then equality can hold only if there are non-negative constant 'α' and 'β' such that $\beta f = \alpha g$

Proof :-

Case (i) :- $p = \infty$

$\|f+g\|_\infty$ = ess sup $|(f+g)t|$

$\leq$ ess sup $|f(t)|$ + ess sup $|g(t)|$

$\|f+g\|_\infty \leq \|f(t)\|_\infty + \|g(t)\|_\infty$

Case (ii) :- $\|f\| = 0$ and $1 \le p < \infty$

If $\|f\|_p = 0$ then $f = 0$ almost everywhere

$f + g = g$ almost everywhere

$\|f+g\|_p \le \|g\|_p$ almost everywhere

$\|f+g\|_p \le \|g\|_p + \|f\|_p$

Similarly, $\|g\|_p = 0$ then also the result is true.

$\|g\|_p = 0$ $\quad g = 0$ a.e $\quad f + g = f$

$\|f+g\|_p \le \|f\|_p$

Case (iii) :- Assume $1 \le p < \infty$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$

$\|f\| = \alpha \neq 0$

$\|g\| = \beta \neq 0$

Let $\frac{|f|}{\alpha} = f_0$, $\frac{|g|}{\beta} = g_0$

$\Rightarrow |f| = \alpha f_0 \quad \Rightarrow |g| = \beta g_0$

Consider $\alpha = \|f\| = \left(\int_0^1 |f|^p\right)^{1/p}$

$= \left(\int_0^1 |\alpha f_0|^p\right)^{1/p}$

$= |\alpha| \left(\int_0^1 |f_0|^p\right)^{1/p}$

$= |\alpha| \, \|f_0\|$

$\alpha = \alpha \|f_0\|$

$\|f_0\| = 1$

Similarly $\|g_0\| = 1$

put $\lambda = \frac{\alpha}{\alpha+\beta}$ $\qquad 1-\lambda = \frac{\beta}{\alpha+\beta}$

$\alpha = \lambda(\alpha+\beta)$ $\qquad \beta = (1-\lambda)(\alpha+\beta)$

$|f(x)+g(x)| \le |f(x)| + |g(x)|$

$\le \alpha f_0(x) + \beta g_0(x)$

$\le \lambda(\alpha+\beta) f_0(x) + (1-\lambda)(\alpha+\beta) g_0(x)$

$\le (\alpha+\beta)\left[\lambda f_0(x) + (1-\lambda) g_0(x)\right]$

Now raising both sides to the power p

$|f(x)+g(x)|^p \le (\alpha+\beta)^p \left[\lambda f_0(x) + (1-\lambda) g_0(x)\right]^p$

$\le (\alpha+\beta)^p \left[\lambda f_0(x)^p + (1-\lambda) g_0(x)^p\right]$

Since the function $\phi(t) = t^p$ for $1 \le p < \infty$ is convex, we have

$$(\lambda x+(1-\lambda)y)^p \le [\lambda x^p + (1-\lambda)y^p]$$

$$|f(x)+g(x)|^p \le (\alpha+\beta)^p[\lambda f_0(x)^p + (1-\lambda)g_0(x)^p]$$

Integrating both sides in the interval $[0,1]$

$$\int_0^1 |f(x)+g(x)|^p \le \int_0^1 (\alpha+\beta)^p [\lambda f_0(x)^p + (1-\lambda) g_0(x)^p]$$

$$\le (\alpha+\beta)^p \left[\lambda\left(\int_0^1 f_0(x)\right)^p + (1-\lambda)\left(\int_0^1 g_0(x)\right)^p\right]$$

$$\|f(x)+g(x)\|^p \le (\alpha+\beta)^p [\lambda \underbrace{\|f_0\|^p}_{1} + (1-\lambda)\underbrace{\|g_0\|^p}_{1}]$$

$$\le (\alpha+\beta)^p [\cancel{\lambda} + 1 - \cancel{\lambda}]$$

$$\le (\alpha+\beta)^p$$

$$\le (\|f\|_p + \|g\|_p)^p$$

Take the pth root,

$$\|f+g\| \le \|f\| + \|g\|$$

If $1<p<\infty$ the inequality is strict unless $\boxed{f_0 = g_0}$ almost everywhere

$$f_0 = \frac{|f|}{\alpha}, \quad g_0 = \frac{|g|}{\beta}$$

$$\frac{|f|}{\alpha} = \frac{|g|}{\beta}$$

$$\frac{f}{\alpha} = \frac{g}{\beta}$$

$$\beta f = \alpha g$$

This is true only when $f_0 = g_0$ a.e and sign of f = sign of g i.e

02-19

Minkowski inequality for $0<p<1$:-

Let f and g be two non negative function which belong to the space L^p with $0<p<1$. Then

$$\|f+g\| \ge \|f\| + \|g\|$$

2M(S)

Lemma :-

Let $1 \le p < \infty$. Then for a, b, t non-negative we have $(a+tb)^p \geq a^p + ptba^{p-1}$

Proof:-

set $\phi(t) = (a+tb)^p - a^p - ptba^{p-1}$

If $t=0$ then $\phi(0) = 0$

$$\phi'(t) = p(a+tb)^{p-1} \cdot b - pba^{p-1}$$
$$= pb\left[(a+tb)^{p-1} - a^{p-1}\right]$$

Since $p \geq 1$ and $a, b, t \geq 0$

$\phi(t)$ is increasing and non-negative for $t \geq 0$

$$(a+tb)^p - a^p - ptba^{p-1} \geq 0$$
$$(a+tb)^p \geq a^p + ptba^{p-1}$$

Holder's inequality :-

If p and q are non-negative extended real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and if $f \in L^p$, $g \in L^q$ then $fg \in L^1$ and $\int |fg| \leq \|f\|_p \|g\|_q$

Equality holds iff for some constants α and β not both zero, we have

$\alpha |f|^p = \beta |g|^q$ almost everywhere.

Proof:-

If $p=1$, $q=\infty$

$\ni$: $\frac{1}{p}+\frac{1}{q}=1$ $\qquad \frac{1}{1}+\frac{1}{\infty} = 1+0 = 1$

Now consider $\int |fg| = \int |f|\,|g|$

$$\leq \int |f| \text{ ess sup } |g(t)|$$
$$\leq \|f\|_p \|g\|_q$$
$$\leq \|f\|_1 \|g\|_\infty$$

$$\therefore \int |fg| \leq \|f\|_1 \|g\|_\infty$$

Similarly $p=\infty$, $q=1$

$$\int |fg| = \int |f|\,|g|$$
$$\leq \text{ess sup } |f(t)| \int |g|$$
$$\leq \|f\|_p \cdot \|g\|_q$$
$$\leq \|f\|_\infty \cdot \|g\|_1$$

Suppose $1<p<\infty$; $1<q<\infty$

Now consider $f \geq 0$, $g \geq 0$

Consider $h(x) = g(x)^{\frac{q-1}{}}$

We know that $\frac{1}{p} + \frac{1}{q} = 1$

$$\frac{q+p}{qp} = 1$$

$$p+q = pq$$

$$q = pq - p$$

$$q = p(q-1)$$

$$\frac{q}{p} = q-1$$

Similarly, $p-1 = \frac{p}{q}$

$$\therefore h(x) = (g(x))^{q/p}$$

$$g(x) = (h(x))^{p/q}$$

$$g(x) = (h(x))^{p-1}$$

Now consider,

$\Rightarrow pt\, f(x) g(x)$, $t \geq 0$

$\Rightarrow pt\, f(x)(h(x))^{p-1}$

We know that by previous lemma

$$(a+bt)^p \geq a^p + pt\, a^{p-1} \cdot b$$

$$a^p + pt\, a^{p-1} \cdot b \leq (a+bt)^p$$

$$pt\, a^{p-1} \cdot b \leq (a+bt)^p - a^p \quad \text{— (1)}$$

Now consider $a = h(x)$, $b = f(x)$

$$pt\,[h(x)^{p-1}](f(x)) \leq [h(x) + tf(x)]^p - [h(x)]^p$$

$$pt\, g(x) f(x) \leq (h(x) + f(x)t)^p - (h(x))^p$$

$$\therefore |ptfg| \leq |h+tf|^p - |h|^p$$

Integrating on both sides,

$$pt \int |fg| \leq \int |h+tf|^p - |h|^p$$

$$\leq (\|h+tf\|^p) - \|h\|^p$$

Differentiating with respect to 't'

$$p \int |fg| \leq p(\|h+tf\|^{p-1}) \, \|f\|$$

put $t=0$

$$\int |fg| \leq \|h\|^{p-1} \|f\| \leq \|g\| \|f\|$$

$\therefore \|fg\| \le \|f\|_p \cdot \|g\|_q$

$\therefore fg \in L^1$

By Minskowski inequality,

$\alpha |f|^p = \beta |g|^q$ almost everywhere.