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Unit - V

Convergence and Completeness :-

Convergence :-

A sequence $\{f_n\}$ in a normal linear space is said to be converged to an element 'f' in the space if given $\epsilon > 0$ $\exists$ an N $\ni \forall n \geq N$ we have $\|f - f_n\| < \epsilon$

If f_n converges to f we denote it by ~~$\lim f_n \to f$~~ $f = \lim f_n$ or $f_n \to f$

Pointwise convergence :-

A $\{f_n\}$ convergence pointwise to 'f' if for each x we have $f(x) = \lim f_n(x)$.

This pointwise convergence is almost everywhere if $\lim f_n(x) = f_n$ $\forall x \in \bar{E}$ with $m(\bar{E}) = 0$.

(s)

Cauchy sequence :-

A $\{f_n\}$ is said to be cauchy sequence is if given $\epsilon > 0$ $\exists$ an N of

$$\|f_m(x) - f_n(x)\| < \epsilon$$

Complete :-

A normed linear space is said to be complete if every cauchy sequence in the space converges in the same space.

Summable :-

A series Σf_n is a normed linear space is said to be summable to sum 's' if 's' is in the space and the sequence of partial sum of the series converges to s.

$$\sum_{j=1}^{\infty} f_j \to s$$

ie) $$\left\| s - \sum_{j=1}^{\infty} f_i \right\| \to 0 \text{ as } n \to \infty$$

Absolutely summable :-

$$\sum_{n=1}^{\infty} \|f_n\| < \infty$$

Theorem :-

A normed linear space X is (complete or banach) iff every absolutely summable series is summable.

Proof :-

Suppose X is complete.

Let $\{f_n\}$ be a summable series of elements of X

$$\sum_{n=1}^{\infty} \|f_n\| < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \|f_n\| < M$$

$$\forall \varepsilon > 0 \ \exists N \text{ of } \sum_{n=N}^{\infty} \|f_n\| < \varepsilon$$

Let S_n be the partial sum of the series $\{f_n\}$

$$S_n = \sum_{i=1}^{n} f_i$$

Consider

$$\|S_n - S_m\| = \left\| \sum_{i=1}^{n} f_i - \sum_{i=1}^{m} f_i \right\|$$

Suppose,

$$n \geq m \geq N,$$

Then $$\|S_n - S_m\| = \left\| \sum_{i=m+1}^{n} f_i \right\| \leq \sum_{i=m+1}^{n} \|f_i\|$$

$$\leq \sum_{i=N}^{\infty} \|f_i\| < \varepsilon$$

$$\therefore \|S_n - S_m\| < \varepsilon$$

$\therefore \{S_n\}$ is a Cauchy sequence in X and also X is complete.

$\{S_n\}$ converges to S in X

$\therefore \{f_n\}$ converges to S.

$\therefore$ Absolutely summable is summable.

Converse part :-

Assume summable series is summable.

To prove :- X is complete

Let $\{f_n\}$ be a Cauchy sequence in X

$\forall\ \epsilon = 2^{-k}\ \exists\ n_k \ni$:

$$\|f_n - f_m\| < 2^{-k} \quad \forall\ n, m \geq n_k \quad \text{——} (1)$$

Choose $n_k \ni n_{k+1} > n_k$,

Then $\{f_{n_k}\}_{k=1}^{\infty}$ is a sub-sequence of $\{f_n\}$

$$\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k} \quad \text{——} (2)$$

Let

$g_1 = f_{n_1}$

$g_2 = f_{n_2} - f_{n_1}$

$g_3 = f_{n_3} - f_{n_2}$

$\vdots$

$g_k = f_{n_k} - f_{n_{k-1}}$

Now consider the partial sum of the series $\{g_k\}$

ie) $\{g_1 + g_2 + \dots + g_k\} = f_{n_k}$

$\therefore$ The partial sum of the series g_k is f_{n_k}

Also, $\|g_k\| = \|f_{n_k} - f_{n_{k-1}}\| \leq 2^{-(k-1)}$

Now consider,

$$\sum_{k=1}^{\infty} \|g_k\| = \|g_1\| + \sum_{k=2}^{\infty} \|g_k\|$$

$$\leq \|g_1\| + 2^{-1} + 2^{-2} + \dots + \infty$$

$$\leq \|g_1\| + \left(\frac{1}{2} + \frac{1}{4} + \dots + \infty\right)$$

$$\leq \|g_1\| + 1$$

$$< \infty$$

The series $\{g_k\}$ is absolutely summable it converges.

Hence the partial sum of the series converges say $\{f_{n_k}\}$ converges to f.

Now have to show that, $f_n \to f$

Since $\{f_n\}$ is a cauchy sequence.

$\forall\ \varepsilon/2 > 0\ \exists\ N \ni \|f_n - f_m\| < \varepsilon/2,\ \forall\ n, m \geq N$ —— (3)

Since $\{f_{n_k}\}$ converges to f

$\forall\ \varepsilon/2, k\ \exists\ K \ni \|f_{n_k} - f\| < \varepsilon/2\ \ \forall\ k > K$ —— (4)

We choose k so large that

$k > K$ and $n_k > K$

Consider,

$$\|f_n - f\| = \|f_n - f_{n_k} + f_{n_k} - f\|$$
$$\leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|$$
$$\leq \varepsilon/2 + \varepsilon/2 \quad [\text{by (3) and (4)}]$$
$$< \varepsilon$$

$$\|f_n - f\| < \varepsilon$$

So every, Cauchy sequence $\{f_n\} \to f$

$\therefore$ X is complete.

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Riesz - Fischer Theorem :-

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Statement :-

The L^p spaces are complete.

Proof :-

[Case (i) :- $p = \infty$

Let $\{f_n\}$ be a Cauchy sequence in L^∞

$$\|f_m - f_n\|_\infty \to 0 \text{ as } (m,n) \to \infty$$

$\Rightarrow$ ess sup $|f_m(t) - f_n(t)| \to 0$ as $(m,n) \to \infty$

i.e) $|f_m(t) - f_n(t)| \to 0$ almost everywhere

$\{f_n(t)\}$ is a Cauchy sequence in R

Sequence $f_n(t)$ converges to some real number, define ~~by~~ it $f(t)$

i.e) $|f_n(t) - f(t)| \to 0$ as $n \to \infty$

$\|f_n - f\| \to 0$ as $n \to \infty$

Hence $f_n \to f$

Every Cauchy sequence in L^∞ ~~is~~ converges.

$\therefore$ L^∞ is complete.

Hence proved.]

Case (II) :-

Suppose $1 \le p < \infty$

Let $\{f_n\}$ be a sequence in L^p with $\sum_{n=1}^{\infty} \|f_n\| = M < \infty$.

Now we have to show that this $\{f_n\}$ is summable.

Define the function g_n's by setting

$$g_n(x) = \sum_{K=1}^{n} |f_K(x)|$$

Let g_n is an increasing sequence of extended real number $\forall x$.

It must converge to extended real number denote its real number as $g(x)$.

Now this 'g' is measurable and $g_n \ge 0$.

By Fatou's lemma,

$$\int g^p \le \underline{\lim} \int g_n^p$$

$$\|g_n\| = \|\sum_{K=1}^{n} f_K\|$$

$$\le \sum_{K=1}^{n} \|f_K\|$$

$$\le M$$

$$\|g_n\| = \left(\int g_n^p\right)^{1/p}$$

Hence $\int g^p \le \underline{\lim} \int g_n^p \le M^p$

$\therefore$ g^p is integrable and $g(x) =$ finite a.e

$\therefore$ $g \in L^p$ and $g(x) =$ finite almost everywhere

$\therefore$ For those 'x' show that $g(x)$ is finite the series $\sum_{K=1}^{\infty} f_K(x)$ is an absolutely summable series of real number.

Hence it must be summable to real number say $s(x)$.

Suppose $g(x) = \infty$ we said $s(x) = 0$

Now we have to prove that for the series

$$\sum_{k=1}^{\infty} f_k(x) = S(x)$$

The partial sum $S_n = \sum_{k=1}^{n} f_k$. Since f_k's are measurable. Partial sum of 'S_n' is measurable. Hence S is measurable.

Consider $|S_n(x)| = \left|\sum_{k=1}^{\infty} f_k(x)\right|$

$$\leq \left|\sum_{k=1}^{n} f_k(x)\right|$$

$$\leq g(x) \quad [\because S_n(x) \to g(x) \text{ a.e}]$$

$$|S(x)| \leq g(x)$$

Also $S \in L^p$ ($\because g \in L^p$)

Moreover, $|S_n(x) - S(x)| \leq |S_n(x)| + |S(x)|$

$$\leq g(x) + g(x)$$

$$\leq 2g(x)$$

$$|S_n(x) - S(x)|^p \leq 2^p |g(x)|^p$$

$\therefore$ $S_n - S$ is a sequence of measurable on L^p function on $L^p \to 0$ and bounded by $2^p |g(x)|^p$

By Lebesgue convergence theorem,

$\int |S_n - S|^p \to 0$ as $n \to \infty$

$$\|S_n - S\|^p \to 0$$

$\therefore$ $S_n - S$ is in L^p

$\therefore$ Every absolutely summable series is summable in L^p.

$\therefore$ L^p is complete.

Hence proved.

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4 **Approximation in L^p :-**

* **Lemma :-**

(5) Given $f \in L^p$, $1 \leq p < \infty$ and $\varepsilon > 0$ there is bounded measurable f_m with $|f_m| \leq M$ and $\|f - f_m\| < \varepsilon_1$.

Proof :-

Let

$$f_N(x) = \begin{cases} N & N \le f(x) \\ f(x) & -N \le f(x) \le N \\ -N & f(x) \le -N \end{cases}$$

$\therefore |f_N| \le N$

and $\{f_N\} \to f$ almost everywhere

$|f - f_N| \to 0$ almost everywhere

Since $|f - f_N|^p \le |f|^p$

$|f|^p$ is integrable

$\|f - f_N\|^p = \int |f - f_N|^p \to 0$

$\left[\because \|f - f_N\|_p = \int_0^1 |f - f_N|_p\right]$

As $N \to \infty$

$\therefore \|f - f_N\| \to 0$

Thus $\|f - f_N\| \to 0$

Hence there is M with $\|f - f_M\| < \epsilon$

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Proposition :-

Given $f \in L^p$, $1 < p \le \infty$ and $\epsilon > 0$ there is a step function ϕ and a continuous function ψ such that $\|f - \phi\|_p < \epsilon$ and $\|f - \psi\|_p < \epsilon$.

Proof :-

By previous lemma we may choose a bounded function f_M with $\|f - f_M\| < \epsilon/2$

By proposition we can find a step function $|f_M - \phi| < \epsilon/4$ except on a set E of measure less than δ where $\delta = \left(\frac{\epsilon}{4M}\right)^p$

Then

$$\|f_M - \phi\| = \int_0^1 |f_M - \phi|^p = \int_{[0,1]\sim E} |f_M - \phi|^p + \int_E |f_M - \phi|^p$$

$$< \frac{\epsilon^p}{4^p} + M^p \delta$$

$$< \frac{\epsilon^p}{4^p} + \frac{\cancel{M^p}\,\epsilon^p}{4^p \cancel{M^p}} < \frac{\epsilon^p + \epsilon^p}{4^p}$$

$$\le \frac{2\epsilon^p}{4^p} \le \frac{\epsilon^p}{2^p}$$

$$\|f_M - \phi\| < \varepsilon/2$$

$$\|f - \phi\| \le \|f - f_M\| + \|f_M - \phi\|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$< \varepsilon$$

By the Minkowski inequality,

The existance of ψ follow from the fact that any step function ϕ can be approximated in L^p by a continuous function.

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Definition :-

Let $\Delta = \{\xi_0, \xi_1, \dots \xi_n\}$ be a subdivision of the finite interval $[a,b]$ and f an integrable function on $[a,b]$ the function ϕ_Δ on $[a,b]$

$$\phi_\Delta(x) = \frac{1}{\xi_{k+1} - \xi_k} \int_{\xi_k}^{\xi_{k+1}} f(t)\,dt$$

Proposition :-

$T_\Delta(f)$

* Let $f \in L^p$ then the Δ-approximant ϕ_Δ to f converges to f in L^p ie) $\|f - \phi_\Delta\| \to 0$ as the length δ of the longest sub-interval in Δ-appro~~ximant~~aches zero.

Proof :-

For any Δ and any $f \in L^p$ we denote the Δ-approximant to f by $T_\Delta(f)$

Then T_Δ is mapping from L^p to the space of step function whose set of discontinuities are the division points of Δ.

(i) $T_\Delta(f+g) = T_\Delta(f) + T_\Delta(g)$

(ii) $T_\Delta(\alpha f) = \alpha T_\Delta(f)$ and

$$\|T_\Delta(f)\|_p \le \|f\|_p$$

There is a given $\varepsilon > 0$, a set function ϕ such that $\|f - \phi\| < \varepsilon/3$. Now $T_\Delta(\phi)$ differs from ϕ

only on these subinterval of Δ that contain point of discontinuity of ϕ.

Let ϕ have l points discontinuity and let $M = \max |\phi(x)|$.

Then $(\|\phi - T(\phi)\|_p)^p = \int |\phi - T(\phi)|^p \leq \delta l (2M)^p$

and so

$\|f - f_N\| = \int_0^1 |f - f_N|$

$$\|\phi - T_\Delta(\phi)\|_p \leq 2M(\delta l)^{1/p}$$

where δ is the length of the longest sub-interval of Δ

$(+) \& (-) \ \phi \qquad (+) \& (-) \ T_\Delta(\phi)$

$$\|f - T_\Delta(f)\| \leq \|f - \phi\| + \|\phi - T_\Delta(\phi)\| + \|T_\Delta(\phi - f)\|$$

$\leq \|\phi - f\|$

$\leq \|f - \phi\|$

$$\leq 2\|f - \phi\| + \|\phi - T_\Delta(\phi)\|$$

$\|f - \phi\| +$

$\|\phi - T_\Delta(\phi)\| +$

$\|T_\Delta \phi - T_\Delta(f)\|$

$$\leq \frac{2}{3}\epsilon + 2M(\delta l)^{1/p}$$

$\|T_\Delta(f)\| \leq \|f\|$

$$< \epsilon \quad \text{whenever } \delta > \frac{1}{(2M)^p l}$$

Bounded linear functionals on the L^p spaces :-

We define a linear functional on a normed linear space X to be a mapping F of the space X into the set of real numbers, such that,

$F : X \to \mathbb{R}$

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

We say that the linear functional is bounded if there is a constant such that

$$|F(f)| \leq M\|f\|$$

$\|f\|$ for all f in X. The smallest constant M for which this inequality is true is called the norm of F.

Thus $$\|F\| = \sup \frac{|F(f)|}{\|f\|}$$

as f ranges over all non-zero element of X. If g is function in L^q. we can define a bounded linear function F on L^p by setting

$$F(f) = \int fg$$

The functional F is clearly linear and the Holder's inequality states that

$$\|F\| \leq \|g\|_q$$

In fact, we actually have

$$\|F\| = \|g\|_q$$

To see this for the case $1 < p < \infty$ we set[2]

$$f = |g|^{q/p} \text{ sign } g$$

Then $|f|^p = |g|^q = fg$

Hence f is in L^p and $\|f\|_p = (\|g\|_q)^{q/p}$

Now

$$F(f) = \int fg = \int |g|^q$$
$$= (\|g\|_q)^q = \|g\|_q \|f\|_p$$

and so $\|F\|$ must be atleast as great as $\|g\|_q$

We state this result as proposition. The cases $p = 1$ and $p = \infty$

[2] For any real number x we define sign $x = 1$ if $x > 0$, sign $0 = 0$ and sign $x = -1$ if $x < 0$.

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Proposition 11 :-

Each function g in L^q define as bounded linear functional F on L^p by

$$F(f) = \int fg$$

we have $\|F\| = \|g\|_q$.

Proof :-

If g is a function in L^q we can define a bounded linear functional F in L^p by setting

$$F(f) = \int fg$$

The functional F is clearly linear, and the Holder inequality states that $\|F\| \leq \|g\|_q$. In fact, we actually have $\|F\| = \|g\|_q$

To see this for the case $1 \leq p < \infty$ we define

$$f = |g|^{q/p} \text{ sign } g$$

Then $|f|^p_{q/p} = |g|^q = fg$ (Holder inequality)

$\|f\|_p = \|g\|_q$

Now $F(f) = \int fg = \int |g|^q = (\|g\|_q)^q$

$= \|g\|_q \|f\|_p$

$\|F\| = \|g\|_q$

<u>Lemma</u> :-

Let 'g' be an integrable function on $[0,1]$ and suppose that there is a constant M s.t. that

$$\left|\int fg\right| \le M \|f\|_p$$

for all bounded measurable function 'f'. Then 'g' is in L^q and $\|g\|_q \le M$ $g \in L^q$ $\|g\|_q \le M$

<u>Proof</u> :-

Let us assume $1 < p < \infty$

We define a sequence bounded measurable function by setting.

$$g_n(x) = \begin{cases} g(x) & \text{if } |g(x)| \le n \\ 0 & \text{if } |g(x)| > n \end{cases} \quad \text{—— (1)}$$

and letting

$f_n = |g_n|^{q/p} \, \text{sign} \, g_n$

$\|f_n\|_p = (\|g_n\|_q)^{q/p}$

$|g_n|^q = f_n \cdot g_n$ $\left|\int fg\right| \le M \|f\|_p$

$|g_n|^q = f_n \cdot g$ [since by (1)]

Hence $(\|g_n\|_q)^q = \int |g_n|^q = \int f_n \cdot g \le M(\|f_n\|_p)$

$\le M(\|g_n\|_q)^{q/p}$

$\|g_n\|_q \le M$

$\int |g_n|^q \le M^q$

By Fatou's lemma,

Since $|g_n|^q$ converges $|g|^q$ almost everywhere we have

$\int |g|^q \le \lim \int |g_n|^q \le M^q$

$\|g\|^q \le M$

$\therefore\ g \in L^q$ and $\|g\|_q \le M$.

Hence proved.

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Riesz Representation Theorem :-

Statement :-

Let F be a bounded linear functional on L^p, $1 \le p < \infty$. Then there is a function g in L^q such that

$$F(f) = \int fg$$

We also have $\|F\| = \|g\|_q$

Proof :-

Let χ_s be characteristic function of the interval $[0, s]$

For each s the value of $F(\chi_s)$ is a real number $\phi(s)$ and is defines a function ϕ on $[0,1]$

Now I maintain that ϕ is absolutely continuous. For let $\{(s_i, s_i')\}$ be a collection of non overlapping subinterval of $[0,1]$ of total length less than δ. Then $\sum_i |\phi(s_i') - \phi(s_i)| = |F(f)|$

where $f = \sum_i (\chi_{s_i'} - \chi_{s_i})\ \operatorname{sgn}(\phi(s_i') - \phi(s_i))$

since

$$(\|f\|_p)^p < \delta$$

$$\begin{aligned}\sum_i |\phi(s_i') - \phi(s_i)| &= |F(f)| \\ &\le \|F\|\,\|f\|_p \\ &< \|F\|\,\delta^{1/p} \quad \text{——(1)} \\ &< \epsilon\end{aligned}$$

$$\delta = \frac{\epsilon^p}{\|F\|^p}$$

This shows that the total variation ϕ is less than ϵ over any finite collection of disjoint interval of total length δ if we take $\delta = \dfrac{\epsilon^p}{\|F\|^p}$

$\therefore$ (1) becomes

$$\begin{aligned}\sum_i |\phi(s_i') - \phi(s)| &< \|F\| \frac{\epsilon^p}{\|F\|} \\ &< \epsilon\end{aligned}$$

Thus ϕ is absolutely continuous.
By using Theorem 5.14,
"A function F is an infinite integral iff it is absolutely continuous"
$\therefore$ There is an integral function g on $[0,1]$ such that $\phi(s) = \int_0^s g$

Thus $F(\chi_s) = \int_0^1 g\chi_s$

Since every step function on $[0,1]$ is a suitable linear combination $\Sigma c_i \chi_{s_i}$

We must have $F(\psi) = \int_0^1 g\psi$ for each step function ψ. By linearity of F and of integral.

Let f be any bound measurable function on $[0,1]$. By using proposition 3.22,

There is a bounded sequence (ψ_n) of step function which converges a.e to f. Since $(|f-\psi_n|^p)$ is uniformly bounded and tends to zero a.e.

By bounded convergence Theorem,

$\|f-\psi_n\|_p \to 0$ since F is bounded.

$|F(f)-F(\psi_n)| = |F(f-\psi_n)| < \|F\| \cdot \|f-\psi_n\| \to 0$

$F(f) = \lim F(\psi_n)$

Since $g\psi_n$ is always less than $|g|$ times the uniform bound for the sequence (ψ_n), we have

$$\int fg = \lim \int g\psi_n$$
$$= \lim F(\psi_n)$$
$$= F(f) \quad \text{(By Lebesgue convergence Theorem)}$$

By lemma 12,

$$|F(f)| \le \|F\| \, \|f\|_p \quad , f \in L^p$$

We have g in L^q $\|g\|_q \le F$

Thus we have only to show that $F(f) = \int fg$ for each f in L^p.

Let f be arbitrary function in L^p.

By proposition 8,

There is for each $\epsilon > 0$ a step function ψ such that $\|f - \psi\|_p < \epsilon$. Since ψ is bounded we have ② can be written as

$$F(\psi) = \int \psi g$$

Here $\left|F(f) - \int fg\right| = \left|F(f) - F(\psi) + \int \psi g - \int fg\right|$

$$< |F(f-\psi)| + \left|\int (\psi - f) g\right|$$

$$< \|F\| \, \|f-\psi\|_p + \|g\|_q \|f-\psi\|_p$$

$$\leq (\|F\| + \|g\|_q)\,\epsilon$$

Since ϵ is an arbitrary number, we have

$$F(f) = \int fg$$

By proposition 11, the equality $\|F\| = \|g\|_q$

Hence proved.