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I M.Sc, Semester: 02

Operations Research

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3. JoShop uses lathes and drill presses to produce four types of machine parts, $PP1$, $PP2$, $PP3$, and $PP4$. The table below summarizes the pertinent data.

	Machining time in minutes per unit of				
Machine	*PP1*	*PP2*	*PP3*	*PP4*	*Capacity (minutes)*
Lathes	2	5	3	4	5300
Drill presses	3	4	6	4	5300
Unit profit ($)	3	6	5	4	

For the parts that are not produced by the present optimum solution, determine the rate of deterioration in the optimum profit per unit increase of each of these products.

4. Consider the optimal solution of JoShop in Problem 3. The company estimates that for each part that is not produced (per the optimum solution), an across-the-board 20% reduction in machining time can be realized through process improvements. Would these improvements make these parts profitable? If not, what is the minimum percentage reduction needed to realize profitability?

4.4 ADDITIONAL SIMPLEX ALGORITHMS FOR LP

In the simplex algorithm presented in Chapter 3 the problem starts at a basic feasible solution. Successive iterations remain basic and feasible but move toward optimality until the optimal is reached at the last iteration. The algorithm is sometimes referred to as the **primal simplex** method.

This section presents two additional algorithms: The **dual simplex** and the **generalized simplex**. In the dual simplex, the LP starts at a basic solution that is (better than) optimal but infeasible, and successive iterations remain basic and (better than) optimal as they move toward feasibility. At the last iteration, the feasible (optimal) solution is found. The generalized simplex combines both the primal and dual simplex methods in one algorithm. It deals with problems that start both nonoptimal and infeasible. In this algorithm, successive iterations are associated with basic (feasible or infeasible) solutions. At the final iteration, the solution is both optimal and infeasible (assuming, of course, that one exists).

All three algorithms—the primal, the dual, and the generalized—are used effectively in the course of sensitivity analysis calculations, as will be shown in Section 4.5.

4.4.1 Dual Simplex Method

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As in the (primal) simplex method, the crux of the dual simplex method is that each iteration is always associated with a basic solution. The optimality and feasibility conditions are designed to preserve the optimality of the basic solutions while, simultaneously, moving the solution iterations toward feasibility.

Dual Feasibility Condition. The leaving variable, x_r, is the basic variable having the most negative value (break ties arbitrarily). If all the basic variables are nonnegative, the algorithm ends.

Dual Optimality Condition. The entering variable is determined from among the nonbasic variables as the one corresponding to

$$\min_{\text{Nonbasic } x_j} \left\{ \left| \frac{z_j - c_j}{\alpha_{rj}} \right|, \alpha_{rj} < 0 \right\}$$

where $z_j - c_j$ is the objective coefficient of the z-row in the tableau and α_{rj} is the negative constraint coefficient of the tableau associated with the row of the leaving variable, x_r, and the column of the nonbasic variable, x_j. Ties are broken arbitrarily.

Notice that the *dual optimality condition* guarantees that optimality will be maintained throughout all iterations.

To start the LP both optimal and infeasible, two requirements must be satisfied:

1. The objective function must satisfy the optimality condition of the regular simplex method (Chapter 3).
2. All the constraints must be of the type ($\leq$).

The second condition requires converting any ($\geq$) to ($\leq$) simply by multiplying both sides of the inequality ($\geq$) by -1. If the LP includes ($=$) constraints, the equation can be replaced by two inequalities. For example,

$$x_1 + x_2 = 2$$

is equivalent to

$$x_1 + x_2 \leq 1, x_1 + x_2 \geq 1$$

or

$$x_1 + x_2 \leq 1, -x_1 - x_2 \leq -1$$

After converting all the constraints to ($\leq$), the LP will have an infeasible starting solution if, and only if, at least one of the right-hand sides of the inequalities is strictly negative. Else, if z is optimal and none of the right-hand sides are negative, there will be no need to apply the dual simplex method as the starting solution is already optimal and feasible.

Example 4.4-1

$$\text{Minimize } z = 3x_1 + 2x_2$$

subject to

$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

In the present example, the first two inequalities are multiplied by -1 to convert them to ($\leq$) constraints. The starting tableau is thus given as:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	-3	-2	0	0	0	0
x_3	-3	-1	1	0	0	-3
x_4	-4	-3	0	1	0	-6
x_5	1	1	0	0	1	3

The tableau starts optimal (all $z_j - c_j \leq 0$ in the z-row) and the starting basic solution is infeasible ($x_3 = -3$, $x_4 = -6$, $x_5 = 3$).

According to the dual feasibility condition, x_4 ($= -6$) is the leaving variable. The next table shows how the dual optimality condition is used to determine the entering variable.

Variable	x_1	x_2	x_3	x_4	x_5
z-row ($z_j - c_j$)	-3	-2	0	0	0
x_4-row, α_{4j}	-4	-3	0	1	0
Ratio, $\left\lvert \frac{z_j - c_j}{\alpha_{4j}} \right\rvert$, $\alpha_{4j} < 0$	$\frac{3}{4}$	$\frac{2}{3}$	—	—	—

The ratios show that x_2 is the entering variable. Notice that a variable x_j is a candidate for entering the basic solution only if its α_{ij} is strictly negative. This means that the variables x_3, x_4, and x_5 should not be considered.

The next tableau is obtained by using the familiar row operations.

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	$-\frac{1}{3}$	0	0	$-\frac{2}{3}$	0	4
x_3	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	-1
x_2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	2
x_5	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	1	1
Ratio	$\frac{1}{5}$	—	—	2	—	

The preceding tableau shows that x_3 leaves and x_1 enters, thus yielding the following tableau:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	$-\frac{1}{5}$	$-\frac{3}{5}$	0	$\frac{21}{5}$
x_1	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_5	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	1	$\frac{6}{5}$

The last tableau is feasible (and optimal), thus ending the algorithm. The corresponding solution is $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and $z = \frac{21}{5}$.

To reinforce your understanding of the dual simplex approach, Figure 4.2 shows graphically the path followed by the algorithm in the solution of Example 4.4-1. The algorithm starts at extreme point A (which is infeasible and better than optimum), then moves to B (which still is infeasible and better than optimum), and finally becomes feasible at C. At this point, the process ends with C as the feasible optimal solution.

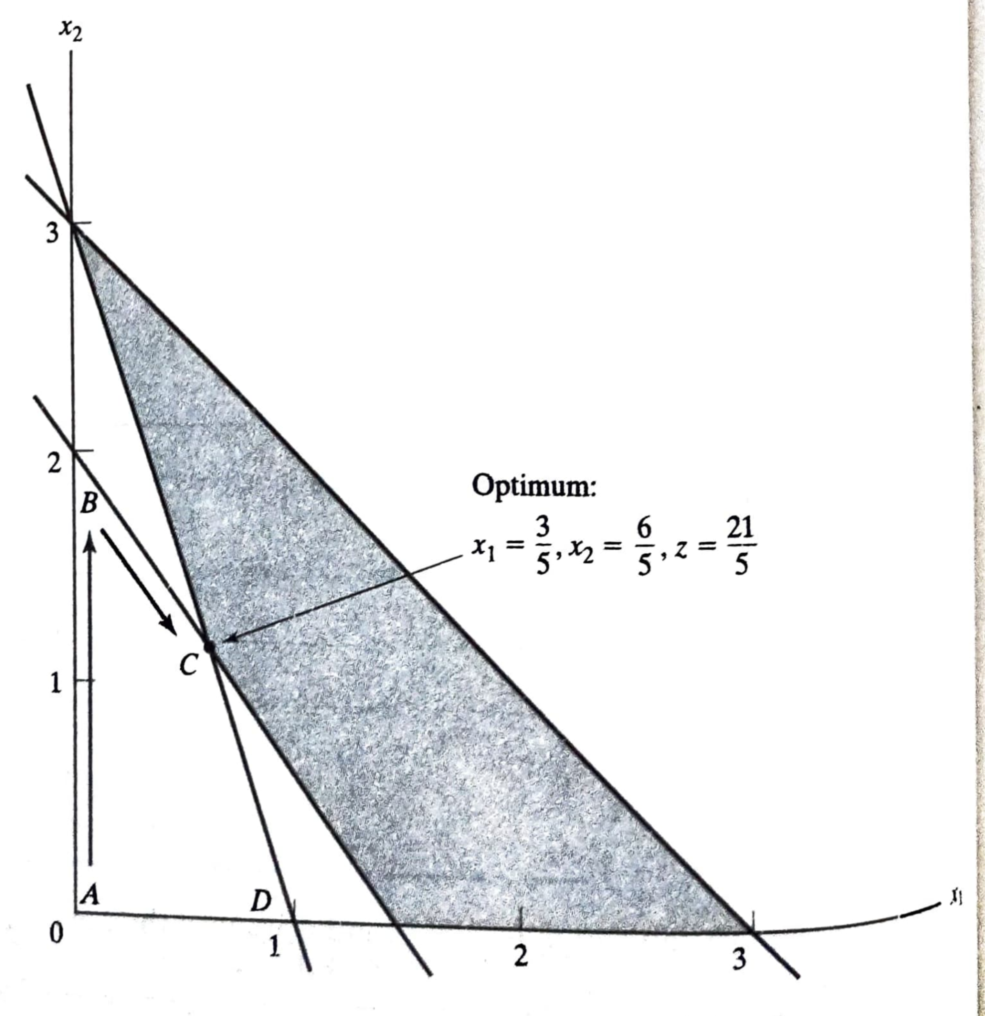
The TORA software is equipped with a tutorial module for the dual simplex method. From the `SOLVE/MODIFY` menu select `Solve` ⇒ `Algebraic` ⇒ `Iterations` ⇒ `Dual Simplex`. Remember that you need to convert (=) constraints to inequalities. You do not need to convert (≥) constraints, however, because TORA will convert the LP to the proper dual simplex starting tableau automatically. If the LP does not satisfy the initial requirements of the dual simplex, a message will appear on the screen.

As in the regular simplex method, the tutorial module allows you to select the entering and the leaving variables beforehand. An appropriate feedback then tells you whether or not your selection is correct.

You are encouraged to use TORA's tutorial mode where possible with the problems in Set 4.4a to avoid the tedious task of carrying out the Gauss-Jordan row operations. In this manner, you can concentrate on understanding the main ideas of the method.

FIGURE 4.2

Dual simplex iterative process for Example 4.4-1



PROBLEM SET 4.4A

1. Consider the solution space in Figure 4.3, where it is desired to find the optimum extreme point that uses the *dual* simplex method to minimize $z = 2x_1 + x_2$. The optimal solution occurs at point $F = (0.5, 1.5)$ on the graph.
   (a) If the starting basic (infeasible but better than optimum) solution is given by point G, would it be possible for the iterations of the dual simplex method to follow the path $G \rightarrow E \rightarrow F$? Explain.
   (b) If the starting basic (infeasible) solution starts at point L, identify a possible path of the dual simplex method that leads to the optimum feasible point at point F.
2. Generate the dual simplex iterations for the following problems using TORA, and trace the path of the algorithm on the graphical solution space.
   (a) Minimize $z = 2x_1 + 3x_2$
   subject to

$$2x_1 + 2x_2 \leq 30$$
$$x_1 + 2x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

   (b) Minimize $z = 5x_1 + 6x_2$
   subject to

$$x_1 + x_2 \geq 2$$
$$4x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

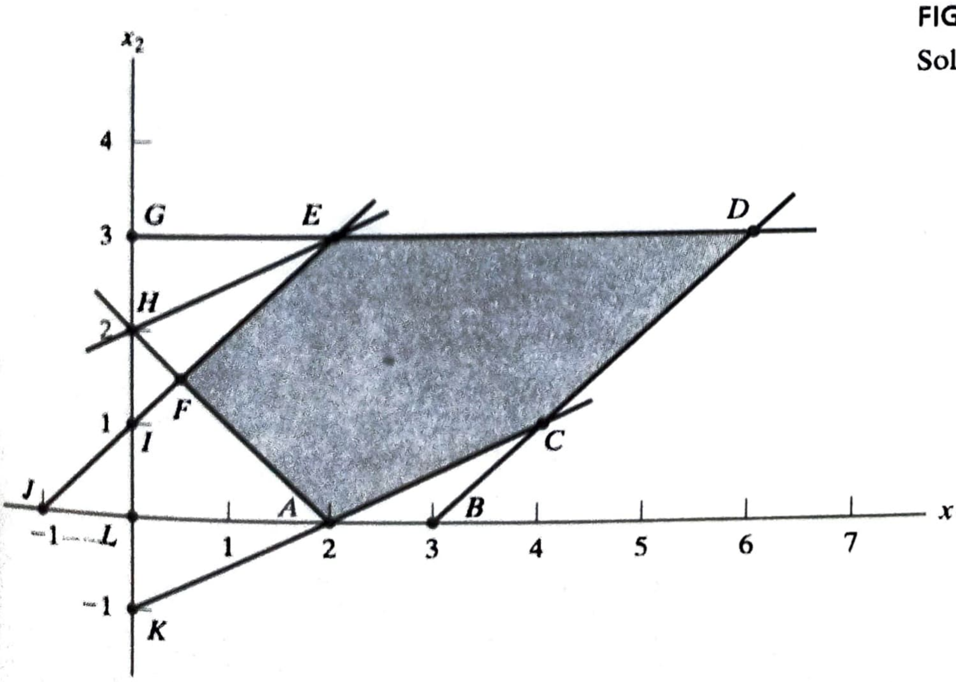


FIGURE 4.3
Solution space for Problem 1, Set 4.4a

determine the number of units of each toy that will maximize revenue. The model and its dual are repeated here for convenience.

TOYCO primal	TOYCO dual
Maximize $z = 3x_1 + 2x_2 + 5x_3$	Minimize $z = 430y_1 + 460y_2 + 420y_3$
subject to	subject to
$x_1 + 2x_2 + x_3 \leq 430$ (Operation 1)	$y_1 + 3y_2 + y_3 \geq 3$
$3x_1 + 2x_3 \leq 460$ (Operation 2)	$2y_1 + 4y_3 \geq 2$
$x_1 + 4x_2 \leq 420$ (Operation 3)	$y_1 + 2y_2 \geq 5$
$x_1, x_2, x_3 \geq 0$	$y_1, y_2, y_3 \geq 0$
Optimal solution:	Optimal solution:
$x_1 = 0, x_2 = 100, x_3 = 230, z = \$1350$	$y_1 = 1, y_2 = 2, y_3 = 0, w = \$1350$

The associated optimum tableau for the primal is given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

4.5.1 Changes Affecting Feasibility

The feasibility of the current optimum solution may be affected only if (1) the right-hand side of the constraints is changed, or (2) a new constraint is added to the model. In both cases, infeasibility occurs when at least one element of the right-hand side of the optimal tableau becomes negative—that is, one or more of the current basic variables become negative.

Changes in the right-hand side. This change requires recomputing the right-hand side of the tableau using Formula 1 in Section 4.2.4:

$$\begin{pmatrix}\text{New right-hand side of} \\ \text{tableau in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in} \\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{New right-hand} \\ \text{side of constraints}\end{pmatrix}$$

Recall that the right-hand side of the tableau gives the values of the basic variables.

Example 4.5-1

Situation 1. Suppose that TOYCO wants to expand its assembly lines by increasing the daily capacity of operations 1, 2, and 3 by 40% to 602, 644, and 588 minutes, respectively. How would this change affect the total revenue?

With these increases, the only change that will take place in the optimum tableau is the right-hand side of the constraints (and the optimum objective value). Thus, the new basic solution is computed as follows:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 602 \\ 644 \\ 588 \end{pmatrix} = \begin{pmatrix} 140 \\ 322 \\ 28 \end{pmatrix}$$

Thus, the current basic variables, x_2, x_3, and x_6, remain feasible at the new values 140, 322, and 28, respectively. The associated optimum revenue is \$1890, which is \$540 more than the current revenue of \$1350.

Situation 2. Although the new solution is appealing from the standpoint of increased revenue, TOYCO recognizes that its implementation may take time. Another proposal was thus made to shift the slack capacity of operation 3 ($x_6 = 20$ minutes) to the capacity of operation 1. How would this change impact the optimum solution?

The capacity mix of the three operations changes to 450, 460, and 400 minutes, respectively. The resulting solution is

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 460 \\ 400 \end{pmatrix} = \begin{pmatrix} 110 \\ 230 \\ -40 \end{pmatrix}$$

The resulting solution is infeasible because $x_6 = -40$, which requires applying the dual simplex method to recover feasibility. First, we modify the right-hand side of the tableau as shown by the shaded column. Notice that the associated value of $z = 3 \times 0 + 2 \times 110 + 5 \times 230 = \$1370$.

New table.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1370
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	110
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	-40

From the dual simplex, x_6 leaves and x_4 enters, which yields the following optimal feasible tableau (in general, the dual simplex may take more than one iteration to recover feasibility).

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	5	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1350
x_2	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_4	-1	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	20

The optimum solution (in terms of x_1, x_2, and x_3) remains the same as in the original model. This means that the proposed shift in capacity allocation is not advantageous in this case because all it does is shift the surplus capacity in operation 3 to a surplus capacity in operation 1. The conclusion is that operation 2 is the bottleneck and it may be advantageous to shift the surplus to operation 2 instead (see Problem 1, Set 4.5a). The selection of operation 2 over operation 1 is also reinforced by the fact that the dual price for operation 2 ($2/min) is higher than that for operation 2 (= $1/min).

PROBLEM SET 4.5A

1. In the TOYCO model listed at the start of Section 4.5, would it be more advantageous to assign the 20-minute excess capacity of operation 3 to operation 2 instead of operation 1?
2. Suppose that TOYCO wants to change the capacities of the three operations according to the following cases:

   (a) $\begin{pmatrix}460\\500\\400\end{pmatrix}$ **(b)** $\begin{pmatrix}500\\400\\600\end{pmatrix}$ **(c)** $\begin{pmatrix}300\\800\\200\end{pmatrix}$ **(d)** $\begin{pmatrix}450\\700\\350\end{pmatrix}$

   Use post-optimal analysis to determine the optimum solution in each case.
3. Consider the Reddy Mikks model of Example 2.1-1. Its optimal tableau is given in Example 3.3-1. If the daily availabilities of raw materials $M1$ and $M2$ are increased to 28 and 8 tons, respectively, use post-optimal analysis to determine the new optimal solution.

*4. The Ozark Farm has 20,000 broilers that are fed for 8 weeks before being marketed. The weekly feed per broiler varies according to the following schedule:

Week	1	2	3	4	5	6	7	8
lb/broiler	.26	.48	.75	1.00	1.30	1.60	1.90	2.10

For the broiler to reach a desired weight gain in 8 weeks, the feedstuffs must satisfy specific nutritional needs. Although a typical list of feedstuffs is large, for simplicity we will limit the model to three items only: limestone, corn, and soybean meal. The nutritional needs will also be limited to three types: calcium, protein, and fiber. The following table summarizes the nutritive content of the selected ingredients together with the cost data.

	Content (lb) per lb of			
Ingredient	*Calcium*	*Protein*	*Fiber*	$ per lb
Limestone	.380	.00	.00	.12
Corn	.001	.09	.02	.45
Soybean meal	.002	.50	.08	1.60

The feed mix must contain

(a) At least .8% but not more than 1.2% calcium

(b) At least 22% protein

(c) At most 5% crude fiber

Solve the LP for week 1 and then use post-optimal analysis to develop an optimal schedule for the remaining 7 weeks.

suspect are least restrictive in terms of the optimum solution. The model is solved using the remaining (primary) constraints. We may then add the secondary constraints one at a time. A secondary constraint is discarded if it satisfies the available optimum. The process is repeated until all the secondary constraints are accounted for.

Apply the proposed procedure to the following LP:

$$\text{Maximize } z = 5x_1 + 6x_2 + 3x_2$$

subject to

$$5x_1 + 5x_2 + 3x_3 \le 50$$
$$x_1 + x_2 - x_3 \le 20$$
$$7x_1 + 6x_2 - 9x_3 \le 30$$
$$5x_1 + 5x_2 + 5x_3 \le 35$$
$$12x_1 + 6x_2 \le 90$$
$$x_2 - 9x_3 \le 20$$
$$x_1, x_2, x_3 \ge 0$$

4.5.2 Changes Affecting Optimality

This section considers two particular situations that could affect the optimality of the current solution:

1. Changes in the original objective coefficients.
2. Addition of a new economic activity (variable) to the model.

Changes in the Objective Function Coefficients. These changes affect only the optimality of the solution. Such changes thus require recomputing the z-row coefficients (reduced costs) according to the following procedure:

1. Compute the dual values using Method 2 in Section 4.2.3.
2. Use the new dual values in Formula 2, Section 4.2.4, to determine the new reduced costs (z-row coefficients).

Two cases will result:

1. New z-row satisfies the optimality condition. The solution remains unchanged (the optimum objective value may change, however).
2. The optimality condition is not satisfied. Apply the (primal) simplex method to recover optimality.

Example 4.5-4

Situation 1. In the TOYCO model, suppose that the company has a new pricing policy to meet the competition. The unit revenues under the new policy are \$2, \$3, and \$4 for train, truck, and car toys, respectively. How is the optimal solution affected?

The new objective function is

$$\text{Maximize } z = 2x_1 + 3x_2 + 4x_3$$

Thus,

$$(\text{New objective coefficients of basic } x_2, x_3, \text{ and } x_6) = (3, 4, 0)$$

Using Method 2, Section 4.2.3, the dual variables are computed as

$$(y_1, y_2, y_3) = (3, 4, 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = \left(\tfrac{3}{2}, \tfrac{5}{4}, 0\right)$$

The z-row coefficients are determined as the difference between the left- and right-hand sides of the dual constraints (Formula 2, Section 4.2.4). It is not necessary to recompute the objective-row coefficients of the basic variables x_2, x_3, and x_6 because they always equal zero regardless of any changes made in the objective coefficients (verify!).

$$\text{Reduced cost of } x_1 = y_1 + 3y_2 + y_3 - \mathbf{2} = \tfrac{3}{2} + 3\left(\tfrac{5}{4}\right) + 0 - \mathbf{2} = \tfrac{13}{4}$$

$$\text{Reduced cost of } x_4 = y_1 - 0 = \tfrac{3}{2}$$

$$\text{Reduced cost of } x_5 = y_2 - 0 = \tfrac{5}{4}$$

Note that the right-hand side of the first dual constraint is 2, the *new* coefficient in the modified objective function.

The computations show that the current solution, $x_1 = 0$ train, $x_2 = 100$ trucks, and $x_3 = 230$ cars, remains optimal. The corresponding new revenue is computed as $2 \times 0 + 3 \times 100 + 4 \times 230 = \$1220$. The new pricing policy is not advantageous because it leads to lower revenue.

Situation 2. Suppose now that the TOYCO objective function is changed to

$$\text{Maximize } z = 6x_1 + 3x_2 + 4x_3$$

Will the optimum solution change?

We have

$$(y_1, y_2, y_3) = (3, 4, 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = \left(\tfrac{3}{2}, \tfrac{5}{4}, 0\right)$$

$$\text{Reduced cost of } x_1 = y_1 + 3y_2 + y_3 - \mathbf{6} = \tfrac{3}{2} + 3\left(\tfrac{5}{4}\right) + 0 - \mathbf{6} = -\tfrac{3}{4}$$

$$\text{Reduced cost of } x_4 = y_1 - 0 = \tfrac{3}{2}$$

$$\text{Reduced cost of } x_5 = y_2 - 0 = \tfrac{5}{4}$$

The new reduced cost of x_1 shows that the current solution is not optimum.

To determine the new solution, the z-row is changed as highlighted in the following tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-\frac{3}{4}$	0	0	$\frac{3}{2}$	$\frac{5}{4}$	0	1220
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

The elements shown in the shaded cells are the new *reduced cost* for the nonbasic variables x_1, x_4, and x_5. All the remaining elements are the same as in the original optimal tableau. The new optimum solution is then determined by letting x_1 enter and x_6 leave, which yields $x_1 = 10$, $x_2 = 102.5$, $x_3 = 215$, and $z = \$1227.50$ (verify!). Although the new solution recommends the production of all three toys, the optimum revenue is less than when two toys only are manufactured.

PROBLEM SET 4.5C

1. Investigate the optimality of the TOYCO solution for each of the following objective functions. If the solution changes, use post-optimal analysis to determine the new optimum. (The optimum tableau of TOYCO is given at the start of Section 4.5.)
   (a) $z = 2x_1 + x_2 + 4x_3$
   (b) $z = 3x_1 + 6x_2 + x_3$
   (c) $z = 8x_1 + 3x_2 + 9x_3$
2. Investigate the optimality of the Reddy Mikks solution (Example 4.3-1) for each of the following objective functions. If the solution changes, use post-optimal analysis to determine the new optimum. (The optimal tableau of the model is given in Example 3.3-1.)
   ***(a)** $z = 3x_1 + 2x_2$
   (b) $z = 8x_1 + 10x_2$
   ***(c)** $z = 2x_1 + 5x_2$
3. Show that the 100% optimality rule (Problem 8, Set 3.6d, Chapter 3) is derived from (reduced costs) ≥ 0 for maximization problems and (reduced costs) ≤ 0 for minimization problems.

Addition of a New Activity. The addition of a new activity in an LP model is equivalent to adding a new variable. Intuitively, the addition of a new activity is desirable only if it is profitable—that is, if it improves the optimal value of the objective function. This condition can be checked by computing the reduced cost of the new variable using Formula 2, Section 4.2.4. If the new activity satisfies the optimality condition, then the activity is not profitable. Else, it is advantageous to undertake the new activity.

Example 4.5-6

TOYCO recognizes that toy trains are not currently in production because they are not profitable. The company wants to replace toy trains with a new product, a toy fire engine, to be assembled on subject to

$$\begin{aligned} 3x_1 + x_2 - x_3 \quad\quad\quad &= 3 \\ 4x_1 + 3x_2 \quad - x_4 \quad &= 6 \\ x_1 + 2x_2 \quad\quad + x_5 &= 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

*4. The following is an optimal LP tableau:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	0	3	2	?
x_3	0	0	1	1	−1	2
x_2	0	1	0	1	0	6
x_1	1	0	0	−1	1	2

The variables x_3, x_4, and x_5 are slacks in the original problem. Use matrix manipulations to reconstruct the original LP, and then compute the optimum value.

5. In the generalized simplex tableau, suppose that the $\mathbf{X} = (\mathbf{X}_{\text{I}}, \mathbf{X}_{\text{II}})^T$, where $\mathbf{X}_{\text{II}}$ corresponds to a typical *starting* basic solution (consisting of slack and/or artificial variables) with $\mathbf{B} = \mathbf{I}$, and let $\mathbf{C} = (\mathbf{C}_{\text{I}}, \mathbf{C}_{\text{II}})$ and $\mathbf{A} = (\mathbf{D}, \mathbf{I})$ be the corresponding partitions of $\mathbf{C}$ and $\mathbf{A}$, respectively. Show that the matrix form of the simplex tableau reduces to the following form, which is exactly the form used in Chapter 3.

Basic	$\mathbf{X}_{\text{I}}$	$\mathbf{X}_{\text{II}}$	Solution
z	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{D} - \mathbf{C}_{\text{I}}$	$\mathbf{C}_B\mathbf{B}^{-1} - \mathbf{C}_{\text{II}}$	$\mathbf{C}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{X}_B$	$\mathbf{B}^{-1}\mathbf{D}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

7.2 REVISED SIMPLEX METHOD

Section 7.1.1 shows that the optimum solution of a linear program is always associated with a basic (feasible) solution. The simplex method search for the optimum starts by selecting a feasible basis, $\mathbf{B}$, and then moving to another basis, $\mathbf{B}_{\text{next}}$, that yields a better (or, at least, no worse) value of the objective function. Continuing in this manner, the optimum basis is eventually reached.

The iterative steps of the *revised* simplex method are exactly the same as in the *tableau* simplex method presented in Chapter 3. The main difference is that the computations in the revised method are based on matrix manipulations rather than on row operations. The use of matrix algebra reduces the adverse effect of machine roundoff error by controlling the accuracy of computing $\mathbf{B}^{-1}$. This result follows because, as Section 7.1.2 shows, the entire simplex tableau can be computed from the *original* data and the current $\mathbf{B}^{-1}$. In the tableau simplex method of Chapter 3, each tableau is generated from the immediately preceding one, which tends to worsen the problem of rounoff error.

7.2.1 Development of the Optimality and Feasibility Conditions

The general LP problem can be written as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j \text{ subject to } \sum_{j=1}^{n} \mathbf{P}_j x_j = \mathbf{b},\ x_j \geq 0,\ j = 1, 2, \ldots, n$$

For a given basic vector $\mathbf{X}_B$ and its corresponding basis $\mathbf{B}$ and objective vector $\mathbf{C}_B$, the general simplex tableau developed in Section 7.1.2 shows that any simplex iteration can be represented by the following equations:

$$z + \sum_{j=1}^{n} (z_j - c_j)x_j = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b}$$

$$(\mathbf{X}_B)_i + \sum_{j=1}^{n} (\mathbf{B}^{-1}\mathbf{P}_j)_i x_j = (\mathbf{B}^{-1}\mathbf{b})_i$$

$z_j - c_j$, the reduced cost of x_j (see Section 4.3.2), is defined as

$$z_j - c_j = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j - c_j$$

The notation $(\mathbf{V})_i$ is used to represent the ith element of the vector $\mathbf{V}$.

Optimality Condition. From the z-equation given above, an increase in nonbasic x_j above its current zero value will improve the value of z relative to its current value $(= \mathbf{C}_B\mathbf{B}^{-1}\mathbf{b})$ only if its $z_j - c_j$ is strictly negative in the case of maximization and strictly positive in the case of minimization. Otherwise, x_j cannot improve the solution and must remain nonbasic at zero level. Though any nonbasic variable satisfying the given condition can be chosen to improve the solution, the simplex method uses a rule of thumb that calls for selecting the **entering variable** as the one with the *most* negative (*most* positive) $z_j - c_j$ in case of maximization (minimization).

Feasibility Condition. The determination of the **leaving vector** is based on examining the constraint equation associated with the ith *basic* variable. Specifically, we have

$$(\mathbf{X}_B)_i + \sum_{j=1}^{n} (\mathbf{B}^{-1}\mathbf{P}_j)_i x_j = (\mathbf{B}^{-1}\mathbf{b})_i$$

When the vector $\mathbf{P}_j$ is selected by the optimality condition to enter the basis, its associated variable x_j will increase above zero level. At the same time, all the remaining nonbasic variables remain at zero level. Thus, the ith constraint equation reduces to

$$(\mathbf{X}_B)_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{P}_j)_i x_j$$

The equation shows that if $(\mathbf{B}^{-1}\mathbf{P}_j)_i > 0$, an increase in x_j can cause $(\mathbf{X}_B)_i$ to become negative, which violates the nonnegativity condition, $(\mathbf{X}_B)_i \geq \mathbf{0}$ for all i. Thus, we have

$$(\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{P}_j)_i x_j \geq 0, \quad \text{for all } i$$

This condition yields the maximum value of the entering variable x_j as

$$x_j = \min_i \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{P}_j)_i} \middle| (\mathbf{B}^{-1}\mathbf{P}_j)_i > 0 \right\}$$

The basic variable responsible for producing the minimum ratio leaves the basic solution to become nonbasic at zero level.

PROBLEM SET 7.2A

*1. Consider the following LP:

$$\text{Maximize } z = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to

$$\mathbf{P}_1x_1 + \mathbf{P}_2x_2 + \mathbf{P}_3x_3 + \mathbf{P}_4x_4 = \mathbf{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The vectors $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are shown in Figure 7.4. Assume that the basis **B** of the current iteration is comprised of $\mathbf{P}_1$ and $\mathbf{P}_2$.

(a) If the vector $\mathbf{P}_3$ enters the basis, which of the current two basic vectors must leave in order for the resulting basic solution to be feasible?

(b) Can the vector $\mathbf{P}_4$ be part of a feasible basis?

*2. Prove that, in any simplex iteration, $z_j - c_j = 0$ for all the associated *basic* variables.

3. Prove that if $z_j - c_j > 0\ (< 0)$ for all the nonbasic variables x_j of a maximization (minimization) LP problem, then the optimum is unique. Else, if $z_j - c_j$ equals zero for a nonbasic x_j, then the problem has an alternative optimum solution.

4. In an all-slack starting basic solution, show using the matrix form of the tableau that the mechanical procedure used in Section 3.3 in which the objective equation is set as

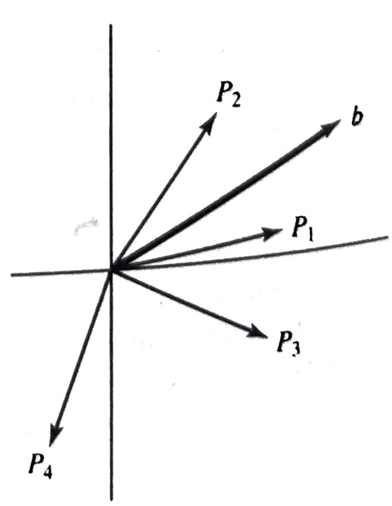
$$z - \sum_{j=1}^{n} c_jx_j = 0$$

automatically computes the proper $z_j - c_j$ for all the variables in the starting tableau.

5. Using the matrix form of the simplex tableau, show that in an all-artificial starting basic solution, the procedure employed in Section 3.4.1 that calls for substituting out

FIGURE 7.4

Vector representation of Problem 1, Set 7.2a



the artificial variables in the objective function (using the constraint equations) actually computes the proper $z_j - c_j$ for all the variables in the starting tableau.

6. Consider an LP in which the variable x_k is unrestricted in sign. Prove that by substituting $x_k = x_k^- - x_k^+$, where x_k^- and x_k^+ are nonnegative, it is impossible that the two variables will replace one another in an alternative optimum solution.

*7. Given the general LP in equation form with m equations and n unknowns, determine the maximum number of *adjacent* extreme points that can be reached from a nondegenerate extreme point (all basic variable are >0) of the solution space.

8. In applying the feasibility condition of the simplex method, suppose that $x_r = 0$ is a basic variable and that x_j is the entering variable with $(\mathbf{B}^{-1}\mathbf{P}_j)_r \neq 0$. Prove that the resulting basic solution remains feasible even if $(\mathbf{B}^{-1}\mathbf{P}_j)_r$ is negative.

9. In the implementation of the feasibility condition of the simplex method, what are the conditions for encountering a degenerate solution (at least one basic variable = 0) for the first time? For continuing to obtain a degenerate solution in the next iteration? For removing degeneracy in the next iteration? Explain the answers mathematically.

*10. What are the relationships between extreme points and basic solutions under degeneracy and nondegeneracy? What is the maximum number of iterations that can be performed at a given extreme point assuming no cycling?

*11. Consider the LP, maximize $z = \mathbf{CX}$ subject to $\mathbf{AX} \leq \mathbf{b}, \mathbf{X} \geq \mathbf{0}$, where $\mathbf{b} \geq \mathbf{0}$. Suppose that the entering vector $\mathbf{P}_j$ is such that at least one element of $\mathbf{B}^{-1}\mathbf{P}_j$ is positive.

   (a) If $\mathbf{P}_j$ is replaced with $\alpha\mathbf{P}_j$, where α is a positive scalar, and provided x_j remains the entering variable, find the relationship between the values of x_j corresponding to $\mathbf{P}_j$ and $\alpha\mathbf{P}_j$.

   (b) Answer Part (a) if, additionally, $\mathbf{b}$ is replaced with $\beta\mathbf{b}$, where β is a positive scalar.

12. Consider the LP

$$\text{Maximize } z = \mathbf{CX} \text{ subject to } \mathbf{AX} \leq \mathbf{b}, \mathbf{X} \geq \mathbf{0}, \text{ where } \mathbf{b} \geq \mathbf{0}$$

After obtaining the optimum solution, it is suggested that a nonbasic variable x_j can be made basic (profitable) by reducing the (resource) requirements per unit of x_j for the different resources to $\frac{1}{\alpha}$ of their original values, $\alpha > 1$. Since the requirements per unit are reduced, it is expected that the profit per unit of x_j will also be reduced to $\frac{1}{\alpha}$ of its original value. Will these changes make x_j a profitable variable? Explain mathematically.

13. Consider the LP

$$\text{Maximize } z = \mathbf{CX} \text{ subject to } (\mathbf{A}, \mathbf{I})\mathbf{X} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$$

Define $\mathbf{X}_B$ as the current basic vector with $\mathbf{B}$ as its associated basis and $\mathbf{C}_B$ as its vector of objective coefficients. Show that if $\mathbf{C}_B$ is replaced with the new coefficients $\mathbf{D}_B$, the values of $z_j - c_j$ for the basic vector $\mathbf{X}_B$ will remain equal to zero. What is the significance of this result?

7.2.2 Revised Simplex Algorithm

Having developed the optimality and feasibility conditions in Section 7.2.1, we now present the computational steps of the revised simplex method.

Step 0. Construct a starting basic feasible solution and let $\mathbf{B}$ and $\mathbf{C}_B$ be its associated basis and objective coefficients vector, respectively.

Step 1. Compute the inverse $\mathbf{B}^{-1}$ by using an appropriate inversion method.[1]

Step 2. For each *nonbasic* variable x_j, compute

$$z_j - c_j = \mathbf{C}_B\mathbf{B}^{-1}\mathbf{P}_j - c_j$$

If $z_j - c_j \geq 0$ in maximization (≤ 0 in minimization) for all nonbasic x_j, stop; the optimal solution is given by

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b},\ z = \mathbf{C}_B\mathbf{X}_B$$

Else, apply the optimality condition and determine the *entering* variable x_j as the nonbasic variable with the most negative (positive) $z_j - c_j$ in case of maximization (minimization).

Step 3. Compute $\mathbf{B}^{-1}\mathbf{P}_j$. If all the elements of $\mathbf{B}^{-1}\mathbf{P}_j$ are negative or zero, stop; the problem has no bounded solution. Else, compute $\mathbf{B}^{-1}\mathbf{b}$. Then for all the *strictly positive* elements of $\mathbf{B}^{-1}\mathbf{P}_j$, determine the ratios defined by the feasibility condition. The basic variable x_i associated with the smallest ratio is the *leaving* variable.

Step 4. From the current basis $\mathbf{B}$, form a new basis by replacing the leaving vector $\mathbf{P}_i$ with the entering vector $\mathbf{P}_j$. Go to step 1 to start a new iteration.

Example 7.2-1

The Reddy Mikks model (Section 2.1) is solved by the revised simplex algorithm. The same model was solved by the tableau method in Section 3.3.2. A comparison between the two methods will show that they are one and the same.

The equation form of the Reddy Mikks model can be expressed in matrix form as

$$\text{maximize } z = (5, 4, 0, 0, 0, 0)(x_1, x_2, x_3, x_4, x_5, x_6)^T$$

subject to

$$\begin{pmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, \ldots, x_6 \geq 0$$

We use the notation $\mathbf{C} = (c_1, c_2, \ldots, c_6)$ to represent the objective-function coefficients and $(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_6)$ to represent the columns vectors of the constraint equations. The right-hand side of the constraints gives the vector $\mathbf{b}$.

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[1]In most LP presentations, including the first six editions of this book, the *product form* method for inverting a basis (see Section D.2.7) is integrated into the revised simplex algorithm because the *product form* lends itself readily to the revised simplex computations, where successive bases differ in exactly one column. This detail is removed from this presentation because it makes the algorithm appear more complex than it really is. Moreover, the *product form* is rarely used in the development of LP codes because it is not designed for automatic computations, where machine round-off error can be a serious issue. Normally, some advanced numeric analysis method, such as the *LU decomposition* method, is used to obtain the inverse. (Incidentally, TORA matrix inversion is based on LU decomposition.)

In the computations below, we will give the algebraic formula for each step and its final numeric answer without detailing the arithmetic operations. You will find it instructive to fill in the gaps in each step.

Iteration 0

$$\mathbf{X}_{B_0} = (x_3, x_4, x_5, x_6), \mathbf{C}_{B_0} = (0, 0, 0, 0)$$
$$\mathbf{B}_0 = (\mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5, \mathbf{P}_6) = \mathbf{I}, \mathbf{B}_0^{-1} = \mathbf{I}$$

Thus,

$$\mathbf{X}_{B_0} = \mathbf{B}_0^{-1}\mathbf{b} = (24, 6, 1, 2)^T, z = \mathbf{C}_{B_0}\mathbf{X}_{B_0} = 0$$

Optimality computations:

$$\mathbf{C}_{B_0}\mathbf{B}_0^{-1} = (0, 0, 0, 0)$$
$$\{z_j - c_j\}_{j=1,2} = \mathbf{C}_{B_0}\mathbf{B}_0^{-1}(\mathbf{P}_1, \mathbf{P}_2) - (c_1, c_2) = (-5, -4)$$

Thus, $\mathbf{P}_1$ is the entering vector.

Feasibility computations:

$$\mathbf{X}_{B_0} = (x_3, x_4, x_5, x_6)^T = (24, 6, 1, 2)^T$$
$$\mathbf{B}_0^{-1}\mathbf{P}_1 = (6, 1, -1, 0)^T$$

Hence,

$$x_1 = \min\left\{\frac{24}{6}, \frac{6}{1}, -, -\right\} = \min\{4, 6, -, -\} = 4$$

and $\mathbf{P}_3$ becomes the leaving vector.

The results above can be summarized in the familiar simplex tableau format. The presentation should help convince you that the two methods are essentially the same. You will find it instructive to develop similar tableaus in the succeeding iterations.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	-4	0	0	0	0	0
x_3	6						24
x_4	1						6
x_5	-1						1
x_6	0						2

Iteration 1

$$\mathbf{X}_{B_1} = (x_1, x_4, x_5, x_6), \mathbf{C}_{B_1} = (5, 0, 0, 0)$$
$$\mathbf{B}_1 = (\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5, \mathbf{P}_6)$$
$$= \begin{pmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By using an appropriate inversion method (see Section D.2.7, in particular the *product form* method), the inverse is given as

$$\mathbf{B}_1^{-1} = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & 1 & 0 & 0 \\ \frac{1}{6} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{X}_{B_1} = \mathbf{B}_1^{-1}\mathbf{b} = (4, 2, 5, 2)^T, z = \mathbf{C}_{B_1}\mathbf{X}_{B_1} = 20$$

Optimality computations:

$$\mathbf{C}_{B_1}\mathbf{B}_1^{-1} = \left(\tfrac{5}{6}, 0, 0, 0\right)$$

$$\{z_j - c_j\}_{j=2,3} = \mathbf{C}_{B_1}\mathbf{B}_1^{-1}(\mathbf{P}_2, \mathbf{P}_3) - (c_2, c_3) = \left(-\tfrac{2}{3}, \tfrac{5}{6}\right)$$

Thus, $\mathbf{P}_2$ is the entering vector.

Feasibility computations:

$$\mathbf{X}_{B_1} = (x_1, x_4, x_5, x_6)^T = (4, 2, 5, 2)^T$$

$$\mathbf{B}_1^{-1}\mathbf{P}_2 = \left(\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{5}{3}, 1\right)^T$$

Hence,

$$x_2 = \min\left\{\frac{4}{\frac{2}{3}}, \frac{2}{\frac{4}{3}}, \frac{5}{\frac{5}{3}}, \frac{2}{1}\right\} = \min\left\{6, \tfrac{3}{2}, 3, 2\right\} = \tfrac{3}{2}$$

and $\mathbf{P}_4$ becomes the leaving vector. (You will find it helpful to summarize the results above in the simplex tableau format as we did in iteration 0.)

Iteration 2

$$\mathbf{X}_{B_2} = (x_1, x_2, x_5, x_6)^T, \mathbf{C}_{B_2} = (5, 4, 0, 0)$$

$$\mathbf{B}_2 = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_5, \mathbf{P}_6)$$

$$= \begin{pmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Hence,

$$\mathbf{B}_2^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{3}{4} & 0 & 0 \\ \frac{3}{8} & -\frac{5}{4} & 1 & 0 \\ \frac{1}{8} & -\frac{3}{4} & 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{X}_{B_2} = \mathbf{B}_2^{-1}\mathbf{b} = \left(3, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{1}{2}\right)^T, z = \mathbf{C}_{B_2}\mathbf{X}_{B_2} = 21$$

Optimality computations:

$$\mathbf{C}_{B_2}\mathbf{B}_2^{-1} = \left(\tfrac{3}{4}, \tfrac{1}{2}, 0, 0\right)$$

$$\{z_j - c_j\}_{j=3,4} = \mathbf{C}_{B_2}\mathbf{B}_2^{-1}(\mathbf{P}_3, \mathbf{P}_4) - (c_3, c_4) = \left(\tfrac{3}{4}, \tfrac{1}{2}\right)$$

Thus, $\mathbf{X}_{B_2}$ is optimal and the computations end.

Summary of optimal solution:

$$x_1 = 3, x_2 = 1.5, z = 21$$

PROBLEM SET 7.2B

1. In Example 7.2-1, summarize the data of iteration 1 in the tableau format of Section 3.3.
2. Solve the following LPs by the revised simplex method:

   (a) Maximize $z = 6x_1 - 2x_2 + 3x_3$
   subject to

$$2x_1 - x_2 + 2x_3 \le 2$$
$$x_1 \qquad + 4x_3 \le 4$$
$$x_1, x_2, x_3 \ge 0$$

   ***(b)** Maximize $z = 2x_1 + x_2 + 2x_3$
   subject to

$$4x_1 + 3x_2 + 8x_3 \le 12$$
$$4x_1 + x_2 + 12x_3 \le 8$$
$$4x_1 - x_2 + 3x_3 \le 8$$
$$x_1, x_2, x_3 \ge 0$$

   (c) Minimize $z = 2x_1 + x_2$
   subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 4x_2 + \quad + x_4 &= 8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Write the dual problem.
(b) Verify that $\mathbf{B} = (\mathbf{P}_2, \mathbf{P}_3)$ is optimal by computing $z_j - c_j$ for all nonbasic $\mathbf{P}_j$.
(c) Find the associated optimal dual solution.

***5.** An LP model includes two variables x_1 and x_2 and three constraints of the type $\leq$. The associated slacks are x_3, x_4, and x_5. Suppose that the optimal basis is $\mathbf{B} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$, and its inverse is

$$\mathbf{B}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

The optimal primal and dual solutions are

$$\mathbf{X}_B = (x_1, x_2, x_3)^T = (2, 6, 2)^T$$
$$\mathbf{Y} = (y_1, y_2, y_3) = (0, 3, 2)$$

Determine the optimal value of the objective function in two ways using the primal and dual problems.

6. Prove the following relationship for the optimal primal and dual solutions:

$$\sum_{i=1}^{m} c_i(\mathbf{B}^{-1}\mathbf{P}_k)_i = \sum_{i=1}^{m} y_i a_{ik}$$

where $\mathbf{C}_B = (c_1, c_2, \ldots, c_m)$ and $\mathbf{P}_k = (a_{1k}, a_{2k}, \ldots, a_{mk})^T$, for $k = 1, 2, \ldots, n$, and $(\mathbf{B}^{-1}\mathbf{P}_k)_i$ is the ith element of $\mathbf{B}^{-1}\mathbf{P}_k$.

***7.** Write the dual of

$$\text{Maximize } z = \{\mathbf{CX} | \mathbf{AX} = \mathbf{b}, \ \mathbf{X} \text{ unrestricted}\}$$

8. Show that the dual of

$$\text{Maximize } z = \{\mathbf{CX} | \mathbf{AX} \leq \mathbf{b}, \ \mathbf{0} < \mathbf{L} \leq \mathbf{X} \leq \mathbf{U}\}$$

always possesses a feasible solution.

7.5 PARAMETRIC LINEAR PROGRAMMING

Parametric linear programming is an extension of the post-optimal analysis presented in Section 4.5. It investigates the effect of *predetermined* continuous variations in the objective function coefficients and the right-hand side of the constraints on the optimum solution.

Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and define the LP as

$$\text{Maximize } z = \left\{\mathbf{CX} \middle| \sum_{j=1}^{n} \mathbf{P}_j x_j = \mathbf{b}, \mathbf{X} \geq \mathbf{0}\right\}$$

In parametric analysis, the objective function and right-hand side vectors, **C** and **b**, are replaced with the parameterized functions $\mathbf{C}(t)$ and $\mathbf{b}(t)$, where t is the parameter of variation. Mathematically, t can assume any positive or negative value. In practice, however, t usually represents time, and hence it is nonnegative. In this presentation we will assume $t \geq 0$.

The general idea of parametric analysis is to start with the optimal solution at $t = 0$. Then, using the optimality and feasibility conditions of the simplex method, we determine the range $0 \leq t \leq t_1$ for which the solution at $t = 0$ remains optimal and feasible. In this case, t_1 is referred to as a **critical value**. The process continues by determining successive critical values and their corresponding optimal feasible solutions, and will terminate at $t = t_r$ when there is indication that either the last solution remains unchanged for $t > t_r$ or that no feasible solution exists beyond that critical value.

7.5.1 Parametric Changes in C

Let $\mathbf{X}_{B_i}$, $\mathbf{B}_i$, $\mathbf{C}_{B_i}(t)$ be the elements that define the optimal solution associated with critical t_i (the computations start at $t_0 = 0$ with $\mathbf{B}_0$ as its optimal basis). Next, the critical value t_{i+1} and its optimal basis, if one exists, is determined. Because changes in **C** can affect only the optimality of the problem, the current solution $\mathbf{X}_{B_i} = \mathbf{B}_i^{-1}\mathbf{b}$ will remain optimal for some $t \geq t_i$ so long as the reduced cost, $z_j(t) - c_j(t)$, satisfies the following optimality condition:

$$z_j(t) - c_j(t) = \mathbf{C}_{B_i}(t)\mathbf{B}_i^{-1}\mathbf{P}_j - c_j(t) \geq 0, \text{ for all } j$$

The value of t_{i+1} equals the largest $t > t_i$ that satisfies all the optimality conditions.

Note that *nothing* in the inequalities requires $\mathbf{C}(t)$ to be linear in t. Any function $\mathbf{C}(t)$, linear or nonlinear, is acceptable. However, with nonlinearity the numerical manipulation of the resulting inequalities may be cumbersome. (See Problem 5, Set 7.5a for an illustration of the nonlinear case.)

Example 7.5-1

$$\text{Maximize } z = (3 - 6t)x_1 + (2 - 2t)x_2 + (5 + 5t)x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 40 \\ 3x_1 \qquad + 2x_3 &\leq 60 \\ x_1 + 4x_2 \qquad &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

We have

$$\mathbf{C}(t) = (3 - 6t, 2 - 2t, 5 + 5t), t \geq 0$$

The variables x_4, x_5, and x_6 will be used as the slack variables associated with the three constraints.

Optimal Solution at $t = t_0 = 0$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	160
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	5
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	30
x_6	2	0	0	-2	1	1	10

$$\mathbf{X}_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$\mathbf{C}_{B_0}(t) = (2 - 2t, 5 + 5t, 0)$$

$$\mathbf{B}_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

The optimality conditions for the current nonbasic vectors, $\mathbf{P}_1$, $\mathbf{P}_4$, and $\mathbf{P}_5$, are

$$\{\mathbf{C}_{B_0}(t)\mathbf{B}_0^{-1}\mathbf{P}_j - c_j(t)\}_{j=1,4,5} = (4 + 14t, 1 - t, 2 + 3t) \geq \mathbf{0}$$

Thus, $\mathbf{X}_{B_0}$ remains optimal so long as the following conditions are satisfied:

$$4 + 14t \geq 0$$

$$1 - t \geq 0$$

$$2 + 3t \geq 0$$

Because $t \geq 0$, the second inequality gives $t \leq 1$ and the remaining two inequalities are satisfied for all $t \geq 0$. We thus have $t_1 = 1$, which means that $\mathbf{X}_{B_0}$ remains optimal (and feasible) for $0 \leq t \leq 1$.

The reduced cost $z_4(t) - c_4(t) = 1 - t$ equals zero at $t = 1$ and becomes negative for $t > 1$. Thus, $\mathbf{P}_4$ must enter the basis for $t > 1$. In this case, $\mathbf{P}_2$ must leave the basis (see the optimal tableau at $t = 0$). The new basic solution $\mathbf{X}_{B_1}$ is the alternative solution obtained at $t = 1$ by letting $\mathbf{P}_4$ enter the basis—that is, $\mathbf{X}_{B_1} = (x_4, x_3, x_6)^T$ and $\mathbf{B}_1 = (\mathbf{P}_4, \mathbf{P}_3, \mathbf{P}_6)$.

Alternative Optimal Basis at $t = t_1 = 1$

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}_1^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{X}_{B_1} = (x_4, x_3, x_6)^T = \mathbf{B}_1^{-1}\mathbf{b} = (10, 30, 30)^T$$

$$\mathbf{C}_{B_1}(t) = (0, 5 + 5t, 0)$$

The associated nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_5$, and we have

$$\{\mathbf{C}_{B_1}(t)\mathbf{B}_1^{-1}\mathbf{P}_j - c_j(t)\}_{j=1,2,5} = \left(\tfrac{9+27t}{2}, -2+2t, \tfrac{5+5t}{2}\right) \geq \mathbf{0}$$

According to these conditions, the basic solution $\mathbf{X}_{B_1}$ remains optimal for all $t \geq 1$. Observe that the optimality condition, $-2 + 2t \geq 0$, automatically "remembers" that $\mathbf{X}_{B_1}$ is optimal for a range of t that starts from the last critical value $t_1 = 1$. This will always be the case in parametric programming computations.

The optimal solution for the entire range of t is summarized below. The value of z is computed by direct substitution.

t	x_1	x_2	x_3	z
$0 \leq t \leq 1$	0	5	30	$160 + 140t$
$t \geq 1$	0	0	30	$150 + 150t$

PROBLEM SET 7.5A

*1. In example 7.5-1, suppose that t is unrestricted in sign. Determine the range of t for which $\mathbf{X}_{B_0}$ remains optimal.

2. Solve Example 7.5-1, assuming that the objective function is given as

*(a) Maximize $z = (3 + 3t)x_1 + 2x_2 + (5 - 6t)x_3$

(b) Maximize $z = (3 - 2t)x_1 + (2 + t)x_2 + (5 + 2t)x_3$

(c) Maximize $z = (3 + t)x_1 + (2 + 2t)x_2 + (5 - t)x_3$

3. Study the variation in the optimal solution of the following parameterized LP given $t \geq 0$.

$$\text{Minimize } z = (4 - t)x_1 + (1 - 3t)x_2 + (2 - 2t)x_3$$

subject to

$$3x_1 + x_2 + 2x_3 = 3$$
$$4x_1 + 3x_2 + 2x_3 \geq 6$$
$$x_1 + 2x_2 + 5x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

4. The analysis in this section assumes that the optimal solution of the LP at $t = 0$ is obtained by the (primal) simplex method. In some problems, it may be more convenient to obtain the optimal solution by the dual simplex method (Section 4.4.1). Show how the parametric analysis can be carried out in this case, then analyze the LP of Example 4.4-1, assuming that the objective function is given as

$$\text{Minimize } z = (3 + t)x_1 + (2 + 4t)x_2 + x_3, t \geq 0$$

*5. In Example 7.5-1, suppose that the objective function is nonlinear in $t (t \geq 0)$ and is defined as

$$\text{Maximize } z = (3 + 2t^2)x_1 + (2 - 2t^2)x_2 + (5 - t)x_3$$

Determine the first critical value t_1.

7.5.2 Parametric Changes in b

The parameterized right-hand side $\mathbf{b}(t)$ can affect only the feasibility of the problem. The critical values of t are thus determined from the following condition:

$$\mathbf{X}_B(t) = \mathbf{B}^{-1}\mathbf{b}(t) \geq \mathbf{0}$$

Example 7.5-2

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 40 - t \\ 3x_1 \qquad + 2x_3 &\leq 60 + 2t \\ x_1 + 4x_2 \qquad &\leq 30 - 7t \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Assume that $t \geq 0$.

At $t = t_0 = 0$, the problem is identical to that of Example 7.5-1. We thus have

$$\mathbf{X}_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$\mathbf{B}_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

To determine the first critical value t_1, we apply the feasibility conditions $\mathbf{X}_{B_0}(t) = \mathbf{B}_0^{-1}\mathbf{b}(t) \geq 0$, which yields

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5 - t \\ 30 + t \\ 10 - 3t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These inequalities are satisfied for $t \leq \frac{10}{3}$, meaning that $t_1 = \frac{10}{3}$ and that the basis $\mathbf{B}_0$ remains feasible for the range $0 \leq t \leq \frac{10}{3}$. However, the values of the basic variables x_2, x_3, and x_6 will change with t as given above.

The value of the basic variable x_6 $(= 10 - 3t)$ will equal zero at $t = t_1 = \frac{10}{3}$, and will become negative for $t > \frac{10}{3}$. Thus, at $t = \frac{10}{3}$, we can determine the alternative basis $\mathbf{B}_1$ by applying the revised dual simplex method (see Problem 5, Set 7.2b, for details). The leaving variable is x_6.

Alternative Basis at $t = t_1 = \frac{10}{3}$

Given that x_6 is the leaving variable, we determine the entering variable as follows:

$$\mathbf{X}_{B_0} = (x_2, x_3, x_6)^T, \mathbf{C}_{B_0} = (2, 5, 0)$$

Thus,

$$\{z_j - c_j\}_{j=1,4,5} = \{\mathbf{C}_{B_0}\mathbf{B}_0^{-1}\mathbf{P}_j - c_j\}_{j=1,4,5} = (4, 1, 2)$$

Next, for nonbasic x_j, $j = 1, 4, 5$, we compute

$$(\text{Row of } \mathbf{B}_0^{-1} \text{ associated with } x_6)(\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5) = (\text{Third row of } \mathbf{B}_0^{-1})(\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5)$$
$$= (-2, 1, 1)(\mathbf{P}_1, \mathbf{P}_4, \mathbf{P}_5)$$
$$= (2, -2, 1)$$

The entering variable is thus associated with

$$\theta = \min\left\{-, \left|\frac{1}{-2}\right|, -\right\} = \frac{1}{2}$$

Thus, $\mathbf{P}_4$ is the entering vector. The alternative basic solution and its $\mathbf{B}_1$ and $\mathbf{B}_1^{-1}$ are

$$\mathbf{X}_{B_1} = (x_2, x_3, x_4)^T$$

$$\mathbf{B}_1 = (\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}, \mathbf{B}_1^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

The next critical value t_2 is determined from the feasibility conditions, $\mathbf{X}_{B_1}(t) = \mathbf{B}_1^{-1}\mathbf{b}(t) \geq \mathbf{0}$, which yields

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{30 - 7t}{4} \\ 30 + t \\ \frac{-10 + 3t}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These conditions show that $\mathbf{B}_1$ remains feasible for $\frac{10}{3} \leq t \leq \frac{30}{7}$.

At $t = t_2 = \frac{30}{7}$, an alternative basis can be obtained by the revised dual simplex method. The leaving variable is x_2, because it corresponds to the condition yielding the critical value t_2.

Alternative Basis at $t = t_2 = \frac{30}{7}$

Given that x_2 is the leaving variable, we determine the entering variable as follows:

$$\mathbf{X}_{B_1} = (x_2, x_3, x_4)^T, \mathbf{C}_{B_1} = (2, 5, 0)$$

Thus,

$$\{z_j - c_j\}_{j=1,5,6} = \{\mathbf{C}_{B_1}\mathbf{B}_1^{-1}\mathbf{P}_j - c_j\}_{j=1,5,6} = \left(5, \tfrac{5}{2}, \tfrac{1}{2}\right)$$

Next, for nonbasic x_j, $j = 1, 5, 6$, we compute

$$(\text{Row of } \mathbf{B}_1^{-1} \text{ associated with } x_2)(\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6) = (\text{First row of } \mathbf{B}_1^{-1})(\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6)$$
$$= \left(0, 0, \tfrac{1}{4}\right)(\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6)$$
$$= \left(\tfrac{1}{4}, 0, \tfrac{1}{4}\right)$$

Because all the denominator elements, $\left(\frac{1}{4}, 0, \frac{1}{4}\right)$, are ≥ 0, the problem has no feasible solution for $t > \frac{30}{7}$ and the parametric analysis ends at $t = t_2 = \frac{30}{7}$.

The optimal solution is summarized as

t	x_1	x_2	x_3	z
$0 \leq t \leq \frac{10}{3}$	0	$5 - t$	$30 + t$	$160 + 3t$
$\frac{10}{3} \leq t \leq \frac{30}{7}$	0	$\frac{30 - 7t}{4}$	$30 + t$	$165 + \frac{3}{2}t$
$t > \frac{30}{7}$		(No feasible solution exists)		

PROBLEM SET 7.5B

*1. In Example 7.5-2, find the first critical value, t_1, and define the vectors of $\mathbf{B}_1$ in each of the following cases:

*(a) $\mathbf{b}(t) = (40 + 2t, 60 - 3t, 30 + 6t)^T$

(b) $\mathbf{b}(t) = (40 - t, 60 + 2t, 30 - 5t)^T$

*2. Study the variation in the optimal solution of the following parameterized LP, given $t \geq 0$.

$$\text{Minimize } z = 4x_1 + x_2 + 2x_3$$

subject to

$$3x_1 + x_2 + 2x_3 = 3 + 3t$$
$$4x_1 + 3x_2 + 2x_3 \geq 6 + 2t$$
$$x_1 + 2x_2 + 5x_3 \leq 4 - t$$
$$x_1, x_2, x_3 \geq 0$$

3. The analysis in this section assumes that the optimal LP solution at $t = 0$ is obtained by the (primal) simplex method. In some problems, it may be more convenient to obtain the optimal solution by the dual simplex method (Section 4.4.1). Show how the parametric analysis can be carried out in this case, and then analyze the LP of Example 4.4-1, assuming that $t \geq 0$ and the right-hand side vector is

$$\mathbf{b}(t) = (3 + 2t, 6 - t, 3 - 4t)^T$$

4. Solve Problem 2 assuming that the right-hand side is changed to

$$\mathbf{b}(t) = (3 + 3t^2, 6 + 2t^2, 4 - t^2)^T$$

Further assume that t can be positive, zero, or negative.

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