9-02-19
(Tue)

Non Linear Programming Problem (NLPP) — Unit: IV

i) DEFINITION:-

Non linear programming is a mathematical technique for determining many optimum solution to many business problems.

In a NLPP either objective function or one ormore constraints have non linear relationship or both.

ii) FORMULATING A NLPP:-

1. A company faces a responsive price-volume relationship for its products. The lower a product's price the greater is the sales quantity. Even in face of resultant price decrease by competitors. If ~~with~~ the sales revenue does not vary propotionately with price. Reflect this phenomenon in a non linear objective function of the price.

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'p' - price 'c' - cost $x(p)$ - Demand function

$\boxed{x(p) = \alpha - \beta p}$ α and β - constants

Revenue $= px(p)$

Cost $= c\, x(p)$

$$P = \text{Revenue} - \text{cost}$$
$$= px(p) - cx(p)$$
$$= p(\alpha - \beta p) - c(\alpha - \beta p)$$
$$= p\alpha - \beta p^2 - c\alpha + c\beta p$$
$$= (\alpha + c\beta)p - \beta p^2 - c\alpha$$

$\boxed{\text{Max } P = \alpha + (c\beta)p - \beta p^2 - c\alpha}$ (Non linear objective function)

2. Product Allocation Method
3. Port folio selection problem

iii) GENERAL NLPP

Max or Min $Z = f(x_1, x_2, \dots x_n)$

Subject to

$$\left.\begin{aligned} g^1(x_1, x_2, \dots x_n) &\le / \ge / = b_1 \\ g^2(x_1, x_2, \dots x_n) &\le / \ge / = b_2 \\ g^m(x_1, x_2, \dots x_n) &\le / \ge / = b_m \end{aligned}\right\}$$

$x_i \ge 0$ $i = 1, 2, \dots n$

IV 1

Let z be a real valued function of 'n' variables defined by

a) $z = f(x_1, x_2, \dots x_n)$

subject to

b) $g^1(x_1, x_2, \dots x_n) \leq b_1$ or $\geq b_1$ or $= b_1$

$g^2(x_1, x_2, \dots x_n) \leq b_2$ or $\geq b_2$ or $= b_2$

...

$g^m(x_1, x_2, \dots x_n) \leq b_m$ or $\geq b_m$ or $= b_m$

where $b_1, b_2, \dots b_m$ are constants and g^i's are real valued functions of 'n' variables

c) $x_i \geq 0 \quad i = 1$ to n

If either $f(x_1, x_2, \dots x_n)$ or some $g^i(x_1, x_2, \dots x_n)$ or both are non-linear then the problem of determining n-tuple $(x_1, x_2, \dots x_n)$ which makes z a maximum or minimum and satisfies (b) and (c) is called a general NLPP.

CONSTRAINED OPTIMIZATION WITH EQUALITY CONSTRAINTS :

Consider the problem of maximizing or minimizing $z = f(x_1, x_2)$ subject to the constraints $g(x_1, x_2) = c$ c - constant

$x_1, x_2 \geq 0$

We assume that $f(x_1, x_2)$, $g(x_1, x_2)$ are differentiable with respect to x_1, x_2. Let us introduce a differentiable function $h(x_1, x_2)$ differentiable with respect to x_1, x_2 defined by $h(x_1, x_2) = g(x_1, x_2) - c$

Then the problem can be restated as

Maximize $z = f(x_1, x_2)$

subject to

$h(x_1, x_2) = 0$

$x_1, x_2 \geq 0$

To find :- The Necessary condition for a maximum (minimum) value of z

A new function is formed by introducing a LAGRANGE MULTIPLIER λ as

$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2)$

The number λ is unknown constant and the function $L(x_1, x_2, \lambda)$ is Lagrange function.

The necessary condition for maximum (minimum) of $f(x_1, x_2)$ subject to the constraints $h(x_1, x_2) = 0$ is given by

$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0 \qquad \frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} \qquad L_1 = f_1 - \lambda h_1$

$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \qquad \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} \qquad L_2 = f_2 - \lambda h_2$

$\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0 \qquad \frac{\partial L}{\partial \lambda} = -h \qquad L_\lambda = -h$

The necessary condition for maximum (minimum)

$f_1 = \lambda h_1 \quad f_2 = \lambda h_2 \quad h(x_1, x_2) = 0$

PROBLEM

1. Obtain necessary condition for NLPP

$$\min Z = kx^{-1}y^{-2}$$

subject to

$$x^2 + y^2 - a^2 = 0$$
$$x, y \geq 0$$

Hence find the minimum value of Z.

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Solution :-

$$L = f - \lambda h$$
$$L = kx^{-1}y^{-2} - \lambda(x^2 + y^2 - a^2)$$

$\frac{\partial L}{\partial x} = 0 \qquad ky^{-2}(-1x^{-2}) - 2\lambda x = 0 \qquad -ky^{-2}x^{-2} = 2\lambda x$

$-ky^{-2}x^{-3} = 2\lambda$

$\frac{\partial L}{\partial y} = 0 \qquad kx^{-1}(-2y^{-3}) - 2\lambda y = 0 \qquad -2kx^{-1}y^{-3} = 2\lambda y$

$-2kx^{-1}y^{-4} = 2\lambda$

$\frac{\partial L}{\partial \lambda} = 0 \qquad -(x^2 + y^2 - a^2) = 0$

$$+ky^{-2}x^{-3} = +2kx^{-1}y^{-4}$$
$$y^2 x^{-2} = 2$$
$$x^{-2} = 2y^{-2}$$
$$\frac{1}{x^2} = \frac{2}{y^2}$$
$$x^2 = \frac{y^2}{2} \qquad \boxed{x = \frac{y}{\sqrt{2}}}$$

$$x^2 + y^2 = a^2$$

$$\left(\frac{y}{\sqrt{2}}\right)^2 + y^2 = a^2$$

$$\frac{y^2}{2} + y^2 = a^2$$

$$\frac{3y^2}{2} = a^2$$

$$3y^2 = 2a^2$$

$$y^2 = \frac{2a^2}{3}$$

$$\boxed{y = \sqrt{\frac{2}{3}}\, a}$$

$$\min z = k\left(\frac{\sqrt{2}}{y}\right)\left(\frac{3}{2a^2}\right)$$

$$= \frac{3k}{\sqrt{2}a^2 y}$$

$$= \frac{3k}{\sqrt{2}a^2 \sqrt{2/3}\, a}$$

$$\boxed{\min z = \frac{3\sqrt{3}k}{2a^3}}$$

Necessary Condition for a general LPP :-

Consider the general NLPP

Maximize (or minimize) $z = f(x_1, x_2, \dots x_n)$

subject to the constraints:

$$g^i(x_1, x_2, \dots x_n) = c_i \text{ and } x_i \geq 0$$

$i = 1, 2, 3, \dots m \; (< n)$

The constraints can be reduced to

$$h^i(x_1, \dots x_n) = 0 \text{ for } i = 1, 2, \dots m$$

by the transformation $h^i(x_1, x_2, \dots x_n) = g^i(x_1, \dots x_n) - c_i$ for all $i = 1, 2, \dots m \; (< n)$

The problem can then be written in matrix form as

Maximize (or minimize) $z = f(\bar{x}) \quad (\bar{x} \in \mathbb{R}^n)$

subject to the constraints

$$h^i(\bar{x}) = 0 \quad \bar{x} \geq 0$$

To find the necessary conditions for a maximum or minimum of $f(\bar{x})$, the Lagrangian function $L(\bar{x},\lambda)$ is formed by introducing Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m)$. This function is

$$L(\bar{x}, \lambda) = f(\bar{x}) - \sum_{i=1}^{m} \lambda_i h^i(\bar{x})$$

Assuming that L, f and h^i are all differentiable partially with respect to $x_1, x_2, \ldots x_n$ and $\lambda_1, \lambda_2, \ldots \lambda_m$ then necessary conditions for a maximum (minimum) of $f(\bar{x})$ are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(\bar{x})}{\partial x_j} = 0 \qquad j = 1, 2, \ldots n$$

$$\frac{\partial L}{\partial \lambda_i} = - h^i(\bar{x}) = 0 \qquad i = 1, 2, \ldots, m$$

first derivative zero

These $m+n$ necessary conditions can be represented in the following abbreviated form:

$$L_j = f_j - \sum_{i=1}^{m} \lambda_i h^i_j = 0 \text{ or } f_j = \sum_{i=1}^{m} \lambda_i h^i_j \qquad j = 1, 2, \ldots n$$

and $L_{\lambda_i} = -h^i = 0$ or $h^i = 0$ $\qquad i = 1, 2, \ldots m$

where $f_j = \dfrac{\partial f(\bar{x})}{\partial x_j}$, $h^i = h^i(\bar{x})$ and $h^i_j = \dfrac{\partial h^i(\bar{x})}{\partial x_j}$

-02 **Problem :-**
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e) 1- Obtain necessary condition for NLPP

maximize $z = x_1^2 + 3x_2^2 + 5x_3^2$

subject to the constraints

$$x_1 + x_2 + 3x_3 = 2$$
$$5x_1 + 2x_2 + x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution :-

$f(x) = x_1^2 + 3x_2^2 + 5x_3^2$

$g^1(x) = x_1 + x_2 + 3x_3 \qquad c_1 = 2$

$g^2(x) = 5x_1 + 2x_2 + x_3 \qquad c_2 = 5$

$h^1(x) = x_1 + x_2 + 3x_3 - 2 \qquad h^2(x) = 5x_1 + 2x_2 + x_3 - 5$

$L = f - \lambda_1 h^1(x) - \lambda_2 h^2(x)$

$L = x_1^2 + 3x_2^2 + 5x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

$$
\left.
\begin{aligned}
\frac{\partial L}{\partial x_1} &= 2x_1 - \lambda_1 - 5\lambda_2 = 0 \\
\frac{\partial L}{\partial x_2} &= 6x_2 - \lambda_1 - 2\lambda_2 = 0 \\
\frac{\partial L}{\partial x_3} &= 10x_3 - 3\lambda_1 - \lambda_2 = 0 \\
\frac{\partial L}{\partial \lambda_1} &= -(x_1 + x_2 + 3x_3 - 2) = 0 \\
\frac{\partial L}{\partial \lambda_2} &= -(5x_1 + 2x_2 + x_3 - 5) = 0
\end{aligned}
\right\}
\text{ homogenous equation}
$$

By solving the above equations we get the values of $x_1, x_2, x_3, \lambda_1, \lambda_2$

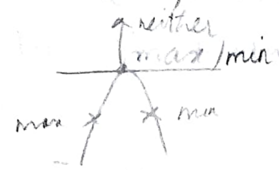
28-02-19 (thurs)

Sufficient condition for general NLPP with one constraint :-

Let the Lagrangian function for general NLPP involving 'n' variables $x_1, x_2, \ldots x_n$ and one constraint be

$$L(\bar{x}, \lambda) = f(\bar{x}) - \lambda h(\bar{x}).$$

The necessary condition for a stationary point (a point which is neither max/min) to be maximum or minimum



$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \qquad j = 1, 2, \ldots n$$

$$\frac{\partial L}{\partial \lambda} = -h(\bar{x}) = 0$$

$$\frac{\partial f}{\partial x_j} = \lambda \frac{\partial h}{\partial x_j} = 0$$

The value of λ is obtained by

$$\lambda = \frac{\frac{\partial f}{\partial x_j}}{\frac{\partial h}{\partial x_j}} \; ; \; j = 1, 2, \ldots n$$

The Sufficient condition for a maximum or minimum require evaluation at each stationery point of $n-1$ principal minor of the determinant given below

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_n} \end{vmatrix}$$

If $\Delta_3 > 0$, $\Delta_4 < 0$, $\Delta_5 > 0$ ... the signs being alternative the stationery point is a local maximum

If $\Delta_3 < 0$, $\Delta_4 < 0$, ... $\Delta_{n+1} < 0$ the sign being always (-ve) the stationery point is a local minimum.

PROBLEM :-

1. Obtain set of necessary and sufficient condition for NLPP

minimize $z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

subject to

$x_1 + x_2 + x_3 = 11$

$x_1, x_2, x_3 \geq 0$

Solution :-

$L = f(x_1, x_2, x_3) - \lambda h(x_1, x_2, x_3)$

$L = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200 - \lambda(x_1 + x_2 + x_3 - 11)$

NECESSARY CONDITION :-

$\frac{\partial L}{\partial x_1} = 4x_1 - 24 - \lambda = 0$

$\frac{\partial L}{\partial x_2} = 4x_2 - 8 - \lambda = 0$

$\frac{\partial L}{\partial x_3} = 4x_3 - 12 - \lambda = 0$

$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 11) = 0$

$x_1 + x_2 + x_3 = 11$

$\begin{array}{r} 44 \\ -24 \\ \hline 20 \end{array}$

$4x_1 + 4x_2 + 4x_3 - 24 - 8 - 12 - 3\lambda = 0$

$4(11) - 24 - 8 - 12 = 3\lambda$

$\boxed{\lambda = 0}$

$4x_1 = 24$ $\quad$ $4x_2 = 8$ $\quad$ $4x_3 = 12$

$\boxed{x_1 = 6}$ $\quad$ $\boxed{x_2 = 2}$ $\quad$ $\boxed{x_3 = 3}$

The solution of the above system of equations is $\lambda = 0$ $(x_1, x_2, x_3) = (6, 2, 3)$ is the stationary point.

$L = f - \lambda h$

$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}$

$\frac{\partial^2 L}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2}$

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix}$$

$$= -1(4) + 1(-4)$$

$$\Delta_3 = -8$$

$$\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1[1(16)] + 1[1(0) - 4(4)] - 1[1(0) - 4(-4)]$$

$$= -16 - 16 - 16$$

$$= -48$$

$\Delta_3 < 0 \quad \Delta_4 < 0$

The stationary point is a local minimum

$(6, 2, 3)$ provides the solution to the NLPP.

Sufficient condition for general problem with $m<n$ constraints :-

Introducing m Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Let the Lagrange function for a general NLPP with more than one constraint be

$$L(\bar{x}, \lambda) = f(\bar{x}, \lambda) - \sum_{i=1}^{m} \lambda_i h^i(\bar{x})$$

The necessary condition for stationary point of $f(\bar{x})$ is given by $\frac{\partial L}{\partial x_i} = 0$ $\frac{\partial L}{\partial \lambda_j} = 0$ $i = 1$ to n, $j = 1$ to m

Thus the optimization of $f(\bar{x})$ subject to $h(\bar{x}) = 0$ is equivalent to the optimization of $L(\bar{x}, \lambda)$.

We state the sufficient condition for the Lagrangian multiplier method of stationary point for $f(\bar{x})$ to be maxima or minima. We assume the function $L(\bar{x}, \lambda)$, $f(\bar{x})$ and $h(\bar{x})$ all possess partial derivatives of order 1 and 2 with respect to decision variables.

$$V = \left[\frac{\partial^2 L(\bar{x}, \lambda)}{\partial x_i \partial x_j}\right]_{n \times n}$$

be the matrix of second order partial derivatives $L(\bar{x}, \lambda)$ w.r.to decision variables

$$U = \left[h_j^i(\bar{x})\right]_{m \times n}$$

where $h_j^i = \frac{\partial h_i}{\partial x_j}$ $i = 1$ to m, $j = 1$ to n

Define a square matrix

$$H^B = \left[\begin{array}{c|c} \bar{O} & U \\ \hline U^T & V \end{array}\right]_{(m+n)\times(m+n)}$$

where $\bar{O}$ is an $m \times m$ null matrix

The matrix H^B is called bordered Hessian matrix.

Then the sufficient condition for maximum and minimum stationary points are given below:

Consider $(\bar{x}_0, \bar{\lambda}_0)$ for a function $L(\bar{x}, \bar{\lambda})$ to be its stationary point. Let H_0^B be the corresponding Bordered Hessian matrix computed at this stationary point. Then $\bar{x}_0$ is (a) maximum point, if starting with principal minor of order $2m+1$ and last $n-m$ principal minors of H_0^B form an alternating sign pattern starting with minors $(-1)^{m+n}$ and (b) minimum point, if starting with principal minor of order $2m+1$, the last $n-m$ principal minors of H_0^B have the sign of $(-1)^m$

Problem :-

1. Obtain necessary and sufficient condition and optimum solution for NLPP

(i) minimize $Z = f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$

subject to

$$x_1 + x_2 = 7$$

(ii) optimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

subject to

$$x_1 + x_2 + x_3 = 15$$
$$2x_1 - x_2 + 2x_3 = 20$$
$$x_1, x_2, x_3 \geq 0$$

SOLUTION :-

i) $L = f(x_1, x_2) - \lambda h(x_1, x_2)$

$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$

Necessary condition

$$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0 \qquad \boxed{\lambda = 6e^{2x_1+1}}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0 \qquad \boxed{\lambda = 2e^{x_2+5}}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0$$

$$x_1 + x_2 = 7 \qquad x_2 = 7 - x_1$$

$$\cancel{6}^{3}e^{2x_1+1} = \cancel{2}e^{x_2+5}$$

$$3e^{2x_1+1} = e^{7-x_1+5}$$

$$\log(3e^{2x_1+1}) = \log(e^{7-x_1+5})$$

$$\log 3 + (2x_1 + 1) = (7 - x_1 + 5)$$

$$\log 3 = 7 - x_1 + 5 - 2x_1 - 1$$
$$= 11 - 3x_1$$

$$3x_1 = 11 - \log 3$$

$$\boxed{x_1 = \frac{1}{3}(11 - \log 3)}$$

$$x_2 = 7 - x_1$$
$$= 7 - \frac{1}{3}(11 - \log 3)$$
$$= \frac{21 - 11 + \log 3}{3}$$

$$\boxed{x_2 = \frac{1}{3}(10 + \log 3)}$$

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ii) $L = (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) - \lambda_1(x_1 + x_2 + x_3 - 15)$
$\quad - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad — ①$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad — ②$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad — ③$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0 \quad — ④$$

$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 \quad — ⑤$$

① $\Rightarrow$ $8x_1 - 4x_2 = \lambda_1 + 2\lambda_2$

② $\Rightarrow$ $-4x_1 + 4x_2 = \lambda_1 - \lambda_2$

$$8x_1 - 4x_2 = \lambda_1 + \cancel{2\lambda_2}$$
$$-8x_1 + 8x_2 = 2\lambda_1 - \cancel{2\lambda_2}$$

$$4x_2 = 3\lambda_1$$
$$x_2 = \frac{3\lambda_1}{4}$$

$$8x_1 - \cancel{4}\left(\frac{3\lambda_1}{\cancel{4}}\right) = \lambda_1 + 2\lambda_2$$

$$8x_1 - 3\lambda_1 = \lambda_1 + 2\lambda_2$$

$$8x_1 = 4\lambda_1 + 2\lambda_2$$

$$x_1 = \frac{\cancel{2}(2\lambda_1 + \lambda_2)}{\cancel{8}\,4}$$

④ $\Rightarrow$ $\dfrac{2\lambda_1 + \lambda_2}{4} + \dfrac{3\lambda_1}{4} + x_3 = 15$

$$5\lambda_1 + \lambda_2 + 4x_3 = 60$$
$$4x_3 = 60 - 5\lambda_1 - \lambda_2$$
$$x_3 = 15 - \frac{5\lambda_1}{4} - \frac{\lambda_2}{4}$$

$(3) \Rightarrow 2\left(15 - \frac{5\lambda_1}{4} - \frac{\lambda_2}{4}\right) - \lambda_1 - 2\lambda_2 = 0$

$-\frac{5}{2} - 1$

$-\frac{1}{2} - 2$

$$30 - \frac{5}{2}\lambda_1 - \frac{\lambda_2}{2} - \lambda_1 - 2\lambda_2 = 0$$
$$30 - \frac{7}{2}\lambda_1 - \frac{5}{2}\lambda_2 = 0$$
$$\frac{7}{2}\lambda_1 + \frac{5}{2}\lambda_2 = 30$$

2
(1

$(3) \Rightarrow \cancel{2}\left(\frac{2\lambda_1 + \lambda_2}{\cancel{4}_{2}}\right) - \frac{3\lambda_1}{4} + 2\left(15 - \frac{5\lambda_1}{4} - \frac{\lambda_2}{4}\right) = 20$

$$\lambda_1 + \cancel{\frac{\lambda_2}{2}} - \frac{3\lambda_1}{4} + 30 - \frac{5\lambda_1}{2} - \cancel{\frac{\lambda_2}{2}} = 20$$

$1 - \frac{3}{4} - \frac{5\times2}{2\times2}$

$\frac{4-3-10}{4}$

$\frac{-9}{4}$

$\frac{1}{2} - \frac{1}{2} =$

$$\frac{+9}{4}\lambda_1 = +10$$
$$\boxed{\lambda_1 = \frac{40}{9}}$$

$$\frac{7}{\cancel{2}}\left(\frac{\cancel{40}^{20}}{9}\right) + \frac{5}{2}\lambda_2 = 30$$
$$\frac{5}{2}\lambda_2 = 30 - \frac{140}{9}$$
$$\frac{5}{2}\lambda_2 = \frac{270 - 140}{9}$$
$$\frac{\cancel{5}}{2}\lambda_2 = \frac{\cancel{130}^{26}}{9}$$
$$\boxed{\lambda_2 = \frac{52}{9}}$$

$$x_1 = \frac{2\left(\frac{40}{9}\right) + \frac{52}{9}}{4} = \frac{80 + 52}{9 \times 4} = \frac{\cancel{132}^{\cancel{66}\,33}}{9 \times \cancel{4}_{\cancel{2}}} = \frac{33}{9}$$

$$\boxed{x_1 = \frac{33}{9}}$$

$$x_2 = \frac{\cancel{3}\left(\frac{40}{\cancel{9}_3}\right)}{4} = \frac{10}{3}$$

$$\boxed{x_2 = \frac{10}{3}}$$

$$x_3 = 15 - \frac{5}{\cancel{4}}\left(\frac{\cancel{40}^{10}}{9}\right) - \frac{1}{\cancel{4}}\left(\frac{\cancel{52}^{13}}{9}\right) = 15 - \frac{50}{9} - \frac{13}{9}$$
$$= 15 - \frac{\cancel{63}^{7}}{\cancel{9}} = 8$$

$\boxed{x_3 = 8}$

$$(x_1, x_2, x_3) = \left(\frac{33}{9}, \frac{10}{3}, 8\right) \qquad (\lambda_1, \lambda_2) = \left(\frac{40}{9}, \frac{52}{9}\right)$$

The Bordered Hessian Matrix is $\qquad m = 2 \qquad n = 3$

$$H_0^B = \left[\begin{array}{cc|ccc} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \hline \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & & & \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & & V = \left[\frac{\partial^2 L(\bar{x}, \lambda)}{\partial x_i \partial x_j}\right] & \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h_2}{\partial x_3} & & & \end{array}\right]$$

$\bar{O} = \overset{2\times 2}{m \times m}$

$U = [h_j^i(\bar{x})]$

$h_j^i(\bar{x}) = \frac{\partial h_i}{\partial x_j}$

$i = 1$ to m
$1, 2$

$j = 1$ to n
$= 1, 2, 3$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]_{5 \times 5}$$

$2m + 1 = 2(2) + 1$
$= 5$

$n - m = 3 - 2$
$= 1$

$(m+n) \times (m \times n)$
5×5

$$|H_0^B| = 1 \begin{vmatrix} 0 & 0 & -1 & 2 \\ 1 & 2 & -4 & 0 \\ 1 & -1 & 4 & 0 \\ 1 & 2 & 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & -1 & -4 & 0 \\ 1 & 2 & 0 & 2 \end{vmatrix}$$

$$+ 1 \begin{vmatrix} 0 & 0 & 2 & -1 \\ 1 & 2 & 8 & -4 \\ 1 & -1 & -4 & 4 \\ 1 & 2 & 0 & 0 \end{vmatrix}$$

$$= 1 \left[-1 \begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} \right]$$

$$- 1 \left[2 \begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix} \right]$$

$$+ 1 \left[2 \begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} \right]$$

$$-3\begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= -3[1(-2)-2(2)] + 3[1(8)-2(4)+8(2+1)].$$

$$= -3(-2-4) + 3(8-8+24)$$

$$= -3(-6) + 3(24)$$

$$= 18 + 72$$

$$H_0^B = 90 > 0 \qquad (-1)^m = (-1)^2$$

$\therefore \left(\frac{33}{9}, \frac{10}{3}, 8\right)$ is a minimum point.

$24 \times 3 = 72$

04-03-19

Work Force Size Model

1. A construction contractor estimate the size of the work force needed for the next 5 week to be 5, 7, 8, 4 and 6 workers respectively. Excess labor kept on the force will cost ₹ 300 per worker per week and new hiring in any week will incur a fixed cost of ₹ 400 plus ₹ 200 per worker per week.

Solution :-

Formula :- Backward approach

$$f_i(x_{i-1}) = \min_{x_i \geq b_i} \{c_i(x_i - b_i) + c_2(x_i - x_{i-1}) + f_{i+1}(x_i)\}$$

$b_1 = 5 \quad b_2 = 7 \quad b_3 = 8 \quad b_4 = 4 \quad b_5 = 6$

$c_1 = 3(x_i - b_i) \quad x_i > b_i$

$c_2 = 4 + 2(x_i - x_{i-1})$

Stage 5 :-

$b_5 = 6$

$x_n = b_n \quad x_5 = b_5 = 6$

x_4	$c_1(x_5 - 6) + c_2(x_5 - x_4)$ $x_5 = 6$	$f_5(x_4)$	Optimum solution x_5^*
4	$3(6-6)+4+2(6-4) = 8$	8	6
5	$3(6-6)+4+2(6-5) = 6$	6	6
6	$3(6-6)+4+2(6-6) = 4$	4	6

19-03-19

OR → NLPP

Problem :-

1. optimize $z = 2x_1 + 3x_2 - (x_1^2 + x_2^2 + x_3^2)$

subject to

$x_1 + x_2 \le 1$

$2x_1 + 3x_2 \le 6$

$x_1, x_2 \ge 0$

Solution :-

$f(x) = 2x_1 + 3x_2 - (x_1^2 + x_2^2 + x_3^2)$

$h^1(x) = x_1 + x_2 - 1$

$h^2(x) = 2x_1 + 3x_2 - 6$

$L(\bar{x}, \bar{S}, \bar{\lambda}) = f(x) - \lambda_1 [h^1(x) + S_1^2] - \lambda_2 [h^2(x) + S_2^2]$

$L = 2x_1 + 3x_2 - (x_1^2 + x_2^2 + x_3^2) - \lambda_1 [x_1 + x_2 - 1 + S_1^2]$
$- \lambda_2 [2x_1 + 3x_2 - 6 + S_2^2]$

$\frac{\partial L}{\partial x_1} = 2 - 2x_1 - \lambda_1 - 2\lambda_2 = 0$

$\frac{\partial L}{\partial x_2} = 3 - 2x_2 - \lambda_1 - 3\lambda_2 = 0$

$\frac{\partial L}{\partial x_3} = -2x_3 = 0$

$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 - 1 + S_1^2) = 0$ $\quad -(x_1 + x_2 - 1) = 0$

$\frac{\partial L}{\partial \lambda_2} = -(2x_1 + 3x_2 - 6 + S_2^2) = 0$ $\quad -(2x_1 + 3x_2 - 6) = 0$

$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0$ $\quad \boxed{S_1 = 0}$

$\frac{\partial L}{\partial S_2} = -2\lambda_2 S_2 = 0$ $\quad \boxed{S_2 = 0}$

$$H_0^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ \hline 1 & 2 & -2 & 0 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$n = 3$

$m = 2$

$$|H_0^B| = 1 \begin{vmatrix} 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 2 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$= 1\left[3\begin{vmatrix}1 & 2 & 0\\ 1 & 3 & 0\\ 0 & 0 & -2\end{vmatrix}\right] - 1\left[2\begin{vmatrix}1 & 2 & 0\\ 1 & 3 & 0\\ 0 & 0 & -2\end{vmatrix}\right]$$

$= 1(-6) - 2(-2)$

$= -6 + 4$ $\qquad (-1)^{m+n} = (-1)^{3+2} = -1$

$= -2 < 0$

$\therefore$ The objective function is to maximize.

The Kuhn Tucker conditions are :

1. i) $2 - 2x_1 - \lambda_1 - 2\lambda_2 = 0$
   ii) $3 - 2x_2 - \lambda_1 - 3\lambda_2 = 0$
   iii) $-2x_3 = 0$

2. i) $\lambda_1(x_1 + x_2 - 1) = 0$
   ii) $\lambda_2(2x_1 + 3x_2 - 6) = 0$

3. i) $x_1 + x_2 - 1 \le 0$
   ii) $2x_1 + 3x_2 - 6 \le 0$

4. $\lambda_1, \lambda_2 \ge 0$

Case i) $\lambda_1 = \lambda_2 = 0$

(1) $\Rightarrow$ $2 - 2x_1 = 0$ $\qquad 2x_1 = 2$ $\quad \boxed{x_1 = 1}$

$3 - 2x_2 = 0$ $\qquad \boxed{x_2 = \frac{3}{2}}$ $\qquad\qquad ax = 0,\ a \ne 0,\ x = 0$

$\boxed{x_3 = 0}$

(3) $\Rightarrow$ i) $\cancel{1} + \frac{3}{2} - \cancel{1} = \frac{3}{2} \nleq 0$

$2(1) + 3\left(\frac{3}{2}\right) - 6 = -4 + \frac{9}{2} = \frac{3}{2} \nleq 0$

$\therefore$ This solution is discarded

Case ii) :- $\lambda_1 = 0$, $\lambda_2 \ne 0$

(1) $\Rightarrow$ $2 - 2x_1 - 2\lambda_2 = 0$ $\qquad 2 - 2x_1 = 2\lambda_2 \quad \times 3$

$3 - 2x_2 - 3\lambda_2 = 0$ $\qquad 3 - 2x_2 = 3\lambda_2 \quad \times 2$

$2x_1 + 3x_2 - 6 = 0 \quad \times 3$

$\cancel{6} - 6x_1 = 6\lambda_2$

$-\cancel{6} - 4x_2 = 6\lambda_2$

$-6x_1 + 4x_2 = 0$

$\cancel{6x_1} + 9x_2 = 18$

$-\cancel{6x_1} + 4x_2 = 0$

$13x_2 = 18$ $\qquad \boxed{x_2 = \frac{18}{13}}$

$\boxed{x_3 = 0}$ $\quad 6x_1 + 9\left(\frac{18}{13}\right) = 18$

$18 \times 9 = 162 \qquad 18 \times 13 = 54 + 18\text{-} = 234$

$6x_1 = 18 - \frac{9(18)}{13}$

$x_1 = \frac{234 - 162}{13 \times 6} = \frac{\cancel{72}^{12}}{13 \times \cancel{6}}$

$234 - 162 = 72$

$\boxed{x_1 = \frac{12}{13}}$

$3 - 2\left(\frac{18}{13}\right) = 3\lambda_2$

$\frac{39 - 36}{13} = 3\lambda_2$

$\boxed{\lambda_2 = \frac{1}{13}}$

(3) $\Rightarrow \frac{12}{13} + \frac{18}{13} - 1 = \frac{30 - 13}{13} = \frac{17}{13} \nleq 0$

$2\left(\frac{12}{13}\right) + 3\left(\frac{18}{13}\right) - 6 = \frac{24 + 54}{13} - 6 = \frac{72}{13} \nleq 0$

$\therefore$ This solution is discarded.

Case iii) :- $\lambda_1 \neq 0 \quad \lambda_2 = 0$

(1) $\Rightarrow 2 - 2x_1 = \lambda_1$

$3 - 2x_2 = \lambda_1$

$x_1 + x_2 = 1$

$x_1 + \frac{3}{4} = 1$

$x_1 = 1 - \frac{3}{4}$

$\boxed{x_1 = \frac{1}{4}}$

$2 - 2x_1 = \cancel{\lambda_1}$
$3 - 2x_2 = \cancel{\lambda_2}$

$-1 - 2x_1 + 2x_2 = 0$

$-\cancel{2x_1} + 2x_2 = 1$
$\cancel{2x_1} + 2x_2 = 2$

$4x_2 = 3$

$\boxed{x_2 = \frac{3}{4}}$

$\boxed{x_3 = 0}$

$2 - \cancel{2}^{1}\left(\frac{1}{\cancel{4}_2}\right) = \lambda_1$

$\boxed{\frac{3}{2} = \lambda_1}$

$24 - 11 = 13$

(3) $\Rightarrow \frac{1}{4} + \frac{3}{4} - 1 = 0$

$2\left(\frac{1}{4}\right) + 3\left(\frac{3}{4}\right) - 6 = \frac{11}{4} - 6 = \frac{11 - 24}{4} = \frac{-13}{4} \leq 0$

This satisfies the kuhn Tucker condition.

$\boxed{x_1 = \frac{1}{4}}$ $\quad \boxed{x_2 = \frac{3}{4}}$ $\quad \boxed{x_3 = 0}$

$22 - 5 = 17$

$z = 2\left(\frac{1}{4}\right) + 3\left(\frac{3}{4}\right) - \left(\frac{1}{16} + \frac{9}{16}\right)$

$= \frac{11}{4} - \frac{\cancel{10}^{5}}{\cancel{16}_8} = \frac{17}{8}$

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$$\boxed{\text{Maximum } z = \frac{17}{8}}$$

Case iv) :- $\lambda_1 \neq 0 \quad \lambda_2 \neq 0$

$$x_1 + x_2 = 1 \quad - (1)$$
$$2x_1 + 3x_2 = 6 \quad - (2)$$

$$(1) \times 2 \Rightarrow \cancel{2x_1} + 2x_2 = 2$$
$$(2) \Rightarrow \underset{(-)}{\cancel{2x_1}} \underset{(-)}{+ 3x_2} \underset{(-)}{=} 6$$

$$-x_2 = -4$$
$$\boxed{x_2 = 4}$$

$$x_1 + 4 = 1$$
$$\boxed{x_1 = -3}$$
$$\boxed{x_3 = 0}$$

(3) i) $\Rightarrow$ $-3 + 4 - 1 = 0$

ii) $\Rightarrow$ $2(-3) + 3(4) - 6 = -6 + 12 - 6 = 0$

But $2 - 2x_1 - \lambda_1 - 2\lambda_2 = 0$

$$2 - 2(-3) = \lambda_1 + 2\lambda_2$$
$$8 = \lambda_1 + 2\lambda_2$$

$$3 - 2x_2 = \lambda_1 + 3\lambda_2$$
$$3 - 2(4) = \lambda_1 + 3\lambda_2$$
$$3 - 8 = \lambda_1 + 3\lambda_2$$
$$-5 = \lambda_1 + 3\lambda_2$$

$$8 = \cancel{\lambda_1} + 2\lambda_2$$
$$\underset{(-)}{-5} = \underset{(-)}{\cancel{\lambda_1}} \underset{(-)}{+ 3\lambda_2}$$

$$13 = -\lambda_2$$
$$\boxed{\lambda_2 = -13}$$

This solution is discarded $\because \lambda_2 \leq 0$

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Non Linear Programming – 27 : 27.1 to 27.5.

(i) Introduction – 27.1

(ii) Formulating a Non Linear Programming Problem (NLPP) – 27.2

(iii) General Non Linear Programming Problem – 27.3

(iv) Constrained Optimization with Equality Constraints – 27.4

(v) Constrained Optimization with Inequality Constraints – 27.5.