18MMA23C : UNIT – V

Non Linear Programming Methods

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5-03-19

# Wolfe's Modified simplex Method (17) | V

1. Maximize $z = 2x_1 + 3x_2 - 2x_1^2$

Subject to $x_1 + 4x_2 \le 4$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

Solution :-

$$x_1 + 4x_2 + S_1^2 = 4 \qquad x_1 \ge 0$$
$$x_1 + x_2 + S_2^2 = 2 \qquad -x_1 \le 0$$
$$-x_1 + S_3^2 = 0 \qquad x_2 \ge 0$$
$$-x_2 + S_4^2 = 0 \qquad -x_2 \le 0$$

$$L(S_1, S_2, S_3, S_4, x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) =$$
$$2x_1 + 3x_2 - 2x_1^2 - \lambda_1(x_1 + 4x_2 + S_1^2 - 4)$$
$$-\lambda_2(x_1 + x_2 + S_2^2 - 2) - \lambda_3(-x_1 + S_3^2) - \lambda_4(-x_2 + S_4^2)$$

$$\frac{\partial L}{\partial x_1} = 2 - 4x_1 - \lambda_1 - \lambda_2 + \lambda_3 = 0$$

$$\frac{\partial L}{\partial x_2} = 3 - 4\lambda_1 - \lambda_2 + \lambda_4 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + 4x_2 + S_1^2 - 4 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + S_2^2 - 2 = 0$$

$$\frac{\partial L}{\partial \lambda_3} = -x_1 + S_3^2 = 0$$

$$\frac{\partial L}{\partial \lambda_4} = -x_2 + S_4^2 = 0$$

$$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0$$

$$\frac{\partial L}{\partial S_2} = -2\lambda_2 S_2 = 0$$

$$\frac{\partial L}{\partial S_3} = -2\lambda_3 S_3 = 0$$

$$\frac{\partial L}{\partial S_4} = -2\lambda_4 S_4 = 0.$$

As $-x_1^2$ represents a negative semi definite quadratic form $z = 2x_1 + 3x_2 - 2x_1^2$ is concave in $x_1, x_2$.

$$\sum d_{jk} x_k - \sum \lambda_i a_{ij} + u_j = c_j \quad i = 1 \text{ to } n$$

$$\sum a_{ij} x_j + S_i^2 = b_i \quad i = 1 \text{ to } m$$

$$\sum \lambda_i S_i + \sum x_j u_j = 0 \quad j = 1 \text{ to } n$$

i) $4x_1 + \lambda_1 + \lambda_2 - \lambda_3 = 2$

$4\lambda_1 + \lambda_2 - \lambda_4 = 3$

$x_1 + 4x_2 + S_1^2 = 4$

$x_1 + x_2 + S_2^2 = 2$

(ii) $\lambda_1 s_1^2 + \lambda_2 s_2^2 + x_1\lambda_3 + x_2\lambda_4 = 0$ $\quad x_1, x_2, s_1^2, s_2^2, \lambda_i$ $\quad i = 1,2,3,4$

$\text{Max } z = -A_1 - A_2$

Subject to

$\lambda_1 x_3 + \lambda_2 x_4 + x_1\lambda_3$

$x_2\lambda_4$

$$4x_1 + \lambda_1 + \lambda_2 - \lambda_3 + A_1 = 2$$
$$4\lambda_1 + \lambda_2 - \lambda_4 + A_2 = 3$$
$$x_1 + 4x_2 + x_3 = 4$$
$$x_1 + x_2 + x_4 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0 \quad A_1, A_2, \lambda_i \geq 0 \quad i = 1 \text{ to } 4$$

↓E.V

| $C_j$ | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $A_1$ | $A_2$ | |
| 1 | $A_1$ | 2 | 4 | 0 | 0 | 0 | 1 | 1 | -1 | 0 | 1 | 0 | ← L.V |
| 1 | $A_2$ | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | -1 | 0 | 1 | |
| 0 | $x_3$ | 4 | 1 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | $x_4$ | 2 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
| $Z_j =$ | | 5 | 4 | 0 | 0 | 0 | 5 | 2 | -1 | -1 | 1 | 1 | |
| $Z_j - C_j =$ | | 5 | 4 | 0 | 0 | 0 | 5 | 2 | -1 | -1 | 0 | 0 | |

$\frac{2}{4} = \frac{1}{2}$ [min]

$\cancel{\frac{3}{0}}$

$\frac{4}{1} = 4$

$\frac{2}{1} = 2$

($\lambda_1, \lambda_2$ corresponding basic variable $x_3, x_4$ lies in $Y_B$ to avoid)

$\lambda_1 x_3 = 0$

$\lambda_2 x_4 = 0$

$x_1\lambda_3 = 0$

$x_2\lambda_4 = 0$

$x_3$

$x_1$ enters the basis. $A_1$ leaves the basis.

| $C_j$ | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $A_1$ $A_2$ |
| 0 | $x_1$ | $1/2$ | 1 | 0 | 0 | 0 | $1/4$ | $1/4$ | $-1/4$ | 0 | 0 |
| 1 | $A_2$ | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | -1 | 1 |
| 0 | $x_3$ | $7/2$ | 0 | 4 | 1 | 0 | $-1/4$ | $-1/4$ | $1/4$ | 0 | 0 |
| 0 | $x_4$ | $3/2$ | 0 | 1 | 0 | 1 | $-1/4$ | $-1/4$ | $1/4$ | 0 | 0 |
| $Z_j$ | | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | -1 | 1 |
| $Z_j - C_j$ | | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | -1 | 0 |

$\lambda_1$ and $\lambda_2$ can enter the basis. But $x_3, x_4$ are still in the basis.

∴ These cannot enter the basis because complementary slackness condition. However since, $\lambda_4$ is not in the basis $x_2$ can enter the basis ($\lambda_4 x_2 \geq 0$)

$x_2$ enter the basis.
$x_3$ leaves the basis.

2nd iteration :-

| | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $A_2$ |
| 0 | $x_1$ | $1/2$ | 1 | 0 | 0 | 0 | $1/4$ | $1/4$ | $-1/4$ | 0 | 0 |
| 1 | $A_2$ | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | $-1$ | 1 |
| 0 | $x_2$ | $7/8$ | 0 | 1 | $1/4$ | 0 | $-1/16$ | $-1/16$ | $1/16$ | 0 | 0 |
| 0 | $x_4$ | $5/8$ | 0 | 0 | $-1/4$ | 1 | $-3/16$ | $-3/16$ | $3/16$ | 0 | 0 |
| $z_j$ | | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | $-1$ | 1 |
| $z_j - c_j$ = | | 3 | 0 | 0 | 0 | 0 | 4 | 1 | 0 | $-1$ | 0 |

Final iteration :- $\lambda_1$ enters the basis

| | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
| 0 | $x_1 \rightarrow$ | $5/16$ | 1 | 0 | 0 | 0 | 0 | $3/16$ | $-1/4$ | $1/16$ |
| 0 | $\lambda_1$ | $3/4$ | 0 | 0 | 0 | 0 | 1 | $1/4$ | 0 | $1/4$ |
| 0 | $x_2 \rightarrow$ | $59/64$ | 0 | 1 | $1/4$ | 0 | 0 | $-3/64$ | $1/16$ | $-1/64$ |
| 0 | $x_4$ | $49/64$ | 0 | 0 | $-1/4$ | 1 | 0 | $-9/64$ | $3/16$ | $-3/64$ |
| $z_j$ | | — | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $z_j - c_j$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The optimum solution is $x_1 = 5/16$ $x_2 = 59/64$

$$\max z = 2x_1 + 3x_2 - 2x_1^2$$
$$= \cancel{2}(5/\cancel{16}) + 3(59/64) - 2(5/16)^2$$
$$= \frac{5}{8} + \frac{177}{64} - \frac{50}{256}$$
$$= \frac{160 + 708 - 50}{256} = \frac{818}{256} = 3.19536$$
$$= 3.19$$

$59 \times 3 = 177$

$32 \times 5 = 160$

$256 / 8 = 32$

$256 / 4 = 64$

$177 \times 4 = 708$

Necessary and Sufficient condition for NLPP by one constraint :-

2716) Determine the optimum solution for the following NLPP check for whether it max or min

$$z = x_1^2 - 10x_1 + x_2^2 - 6x_2 + x_3^2 - 4x_3$$

Unit: V : Beale's Method (18)

1. $\min z = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$ Quadratic form $X^TQX$

subject to $C^TX$

$x_1 + x_2 \le 2$

$x_1 \ge 0 \quad x_2 \ge 0$

Solution :-

$\max z = -6 + 6x_1 - 2x_1^2 + 2x_1x_2 - 2x_2^2$

Introducing slack variable $x_3$

(constraints) → m
$m = 1$
$n = 2$
$n - m = 2 - 1 = 1$
one Non Basic varia

$x_1 + x_2 + x_3 = 2 \qquad x_1, x_2, x_3 \ge 0$

$\boxed{x_B = x_3 \qquad x_{NB} = (x_1, x_2)}$ — ①

$x_3 = 2 - x_1 - x_2$

$f(x) = -6 + 6x_1 - 2x_1^2 + 2x_1x_2 - 2x_2^2$

$\left(\frac{\partial f}{\partial x_1}\right)_{x_{NB}=0} = (6 - 4x_1 + 2x_2)_{x_1=0, x_2=0} = 6$ most positive (enters)

$\left(\frac{\partial f}{\partial x_2}\right)_{x_{NB}=0} = (2x_1 - 4x_2)_{x_1=0, x_2=0} = 0$

∴ $x_1$ enters the basis

$x_3 = 2 - x_1 - x_2$ → (constraint)

$\frac{2}{|-1|}$

To find :- Leaving function

$\min\left\{\frac{2}{|-1|}, \frac{6}{|-4|}\right\} = \frac{6}{4}$

$u_1 = \frac{1}{2}\frac{\partial f}{\partial x_1} = 3 - 2x_1 + x_2$ — ⊛

$3 - 2x_1 + x_2 = u_1$
$-2x_1 = u_1 - x_2$
$x_1 = -\frac{1}{2}[u_1 \ldots$

$\boxed{x_B = (x_3 \; x_1) \qquad x_{NB} = (x_2, u_1)}$

From ⊛ $x_1 = \frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2$

$x_3 = -x_1 - x_2 + 2 \quad (x_3 = 2 - x_1 - x_2)$

$x_3 = 2 - x_2 - \left(\frac{3}{2}\right) + \frac{1}{2}u_1 - \frac{1}{2}x_2)$

$= \frac{1}{2} + \frac{1}{2}u_1 - \frac{3}{2}x_2$

put $x_1 = \frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2$ in $f(x)$

$$f(x) = -6 + 4\left(\frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2\right) - 2\left(\frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2\right)^2$$
$$+ 2\left(\frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2\right)x_2 - 2x_2^2$$
$$= -6 + \frac{18}{2} - \frac{6}{2}u_1 + \frac{6}{2}x_2 - 2\Big[\left(\frac{9}{4}\right) + \frac{1}{4}u_1^2 + \frac{1}{4}x_2^2$$
$$+ 2\left(\frac{3}{2}\right)\left(-\frac{1}{2}u_1\right) + 2\left(-\frac{1}{2}u_1\right)\left(\frac{1}{2}x_2\right) + 2\left(\frac{3}{2}\right)\left(\frac{1}{2}x_2\right)\Big]$$
$$+ \left(2\left(\frac{3}{2}\right) - 2\left(\frac{1}{2}\right)u_1 + 2\left(\frac{1}{2}\right)x_2\right)x_2 - 2x_2^2$$
$$= -6 + 9 - 3u_1 + 3x_2 - \frac{9}{2} - \frac{1}{2}u_1^2 - \frac{1}{2}x_2^2 + 3u_1 + u_1x_2$$
$$- 3x_2 + 3x_2 - u_1x_2 + x_2^2 - 2x_2^2$$
$$= -\frac{3}{2} + 3x_2 - \frac{1}{2}u_1^2 - \frac{3}{2}x_2^2$$

$3 - \frac{9}{2} = \frac{6-9}{2} = -\frac{3}{2}$

$1 - 2 - \frac{1}{2}$

$-1 - \frac{1}{2} =$

$$\left(\frac{\partial f}{\partial x_2}\right)_{x_{NB}=0} = (3 - 3x_2) = 3 \quad x_2 = 0,\ u_1 = 0$$

$$\left(\frac{\partial f}{\partial u_1}\right)_{x_{NB}=0} = (-u_1)_{u_1=0} = 0$$

$(x_3)\ \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

$(x_1)\ \frac{3}{2} \times \frac{2}{1} = 3$

$\therefore$ $x_2$ enters the basis

$$\min\left\{\frac{\frac{3}{2}}{\left|-\frac{1}{2}\right|}, \frac{\frac{1}{2}}{\left|-\frac{3}{2}\right|}, \frac{3}{|-3|}\right\} = \min\{3, \tfrac{1}{3}, 1\} = \frac{1}{3}$$

$(x_1)$ $(x_3)$ $(f(x))$ ; $(x_1)$ $(x_3)$ $f(x)$

$\therefore$ $x_3$ will leave the basis

$\boxed{x_B = (x_2\ x_1) \quad x_{NB} = (u_1\ x_3)}$

$x_1 = \frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2$

$x_3 = 2 - x_1 - x_2$

$x_2 = 2 - x_3 - x_1$

$x_2 = 2 - x_3 - \left(\frac{5}{3} - \frac{1}{3}u_1 - \frac{1}{3}x_3\right)$

$= 2 - x_3 - \frac{5}{3} + \frac{1}{3}u_1 + \frac{1}{3}x_3$

$\boxed{x_2 = \frac{1}{3} - \frac{2}{3}x_3 + \frac{1}{3}u_1}$

$\Rightarrow x_1 = \frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}(2 - x_3 - x_1)$

$= \frac{3}{2} - \frac{1}{2}u_1 + 1 - \frac{1}{2}x_3 - \frac{1}{2}x_1$

$\frac{3}{2}x_1 = \frac{3}{2} - \frac{1}{2}u_1 + 1 - \frac{1}{2}x_3$

$x_1 = 1 - \frac{1}{3}u_1 + \frac{2}{3} - \frac{1}{3}x_3$

$\boxed{x_1 = \frac{5}{3} - \frac{1}{3}u_1 - \frac{1}{3}x_3}$

$-3 + \frac{4}{3} = \frac{-9+4}{3} = \frac{-5}{3}$

$$f(x) = -6 + 6x_1 - 2x_1^2 + 2x_1x_2 - 2x_2^2$$
$$= -6 - 2x_1(-3 + x_1 - x_2) - 2x_2^2$$
$$= -6 - 2\left(\frac{5}{3} - \frac{1}{3}u_1 - \frac{1}{3}x_3\right)\left(-3 + \frac{5}{3} - \frac{1}{3}u_1 - \frac{1}{3}x_3 - \frac{1}{3} + \frac{2}{3}x_3 - \frac{1}{3}u_1\right) - 2\left(\frac{1}{3} - \frac{2}{3}x_3 + \frac{1}{3}u_1\right)^2$$

$$\left(-\frac{5}{3}-\frac{2}{3}u_1+\frac{1}{3}x_3\right)$$

$$= -6 + \left(\frac{10}{3}+\frac{2}{3}u_1+\frac{2}{3}x_3\right)\left(-\frac{5}{3}-\frac{2}{3}u_1+\frac{1}{3}x_3\right)$$

$$-2\left(\frac{1}{9}+\frac{4}{9}x_3^2+\frac{1}{9}u_1^2-\frac{4}{9}x_3-\frac{4}{9}x_3u_1+\frac{2}{9}u_1\right)$$

$$= -6+\frac{50}{9}+\frac{20}{9}u_1-\frac{10}{9}x_3-\frac{10}{9}u_1-\frac{4}{9}u_1^2+\frac{2}{9}u_1x_3$$

$$-\frac{10}{9}x_3-\frac{4}{9}u_1x_3+\frac{2}{9}x_3^2-\frac{2}{9}-\frac{8}{9}x_3^2-\frac{2}{9}u_1^2+\frac{8}{9}x_3$$

$$+\frac{8}{9}x_3u_1-\frac{4}{9}u_1$$

$$= -\frac{2}{3}+\frac{2}{3}u_1-\frac{4}{3}x_3-\frac{2}{3}u_1^2+\frac{2}{3}x_3u_1-\frac{2}{3}x_3^2$$

$-\frac{10}{9}\ \frac{10}{9}+\frac{5}{9}$, $-\frac{12}{9}\ 4$, $6+\frac{48}{9}$, $\frac{-54+48}{9}$, $\frac{10}{9}-\frac{4}{9}$, $=\frac{2}{3}$, $-\frac{4}{9}-\frac{2}{9}$

$$\left(\frac{\partial f}{\partial u_1}\right)_{x_{NB}=0} = \left(\frac{2}{3}-\frac{4}{3}u_1+\frac{2}{3}x_3\right)_{\substack{u_1=0\\x_3=0}} = \frac{2}{3}$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{x_{NB}=0} = \left(-\frac{4}{3}+\frac{2}{3}u_1-\frac{4}{3}x_3\right)_{\substack{u_1=0\\x_3=0}} = -\frac{4}{3}$$

$\therefore$ $u_1$ will enter the basis.

$$\min\left\{\frac{\frac{5}{3}}{|-\frac{1}{3}|},\ \frac{\frac{1}{3}}{|-\frac{2}{3}|},\ \frac{+\frac{2}{3}}{|-\frac{4}{3}|}\right\} = \min\left\{5, \tfrac{1}{2}, \tfrac{1}{2}\right\}$$

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial u_1} = \frac{1}{2}\left(\frac{2}{3}-\frac{4}{3}u_1+\frac{2}{3}x_3\right)$$

$$= \frac{1}{3}-\frac{2}{3}u_1+\frac{1}{3}x_3$$

$$\boxed{u_2 = \frac{1}{3}-\frac{2}{3}u_1+\frac{1}{3}x_3}$$

$\boxed{x_B = (x_1, x_2, u_1)}$ $\quad x_{NB} = (u_2, x_3)$

$$\frac{2}{3}u_1 = \frac{1}{3}-u_2+\frac{1}{3}x_3 \Rightarrow \boxed{u_1 = \frac{1}{2}-\frac{3}{2}u_2+\frac{1}{2}x_3}$$

$$x_1 = \frac{5}{3}-\frac{1}{3}u_1-\frac{1}{3}x_3$$

$$= \frac{5}{3}-\frac{1}{3}\left(\frac{1}{2}-\frac{3}{2}u_2+\frac{1}{2}x_3\right)-\frac{1}{3}x_3$$

$$= \frac{5}{3}-\frac{1}{6}+\frac{1}{2}u_2-\frac{1}{6}x_3-\frac{1}{3}x_3$$

$$= \frac{3}{2}+\frac{1}{2}u_2-\frac{1}{2}x_3$$

$$\boxed{x_1 = \frac{3}{2}+\frac{1}{2}u_2-\frac{1}{2}x_3}$$

$$x_2 = \frac{1}{3}-\frac{2}{3}x_3+\frac{1}{3}u_1$$

$$= \frac{1}{3}-\frac{2}{3}x_3+\frac{1}{3}\left(\frac{1}{2}-\frac{3}{2}u_2+\frac{1}{2}x_3\right)$$

$$= \frac{1}{3} - \frac{2}{3}x_3 + \frac{1}{6} - \frac{1}{2}u_2 + \frac{1}{6}x_3$$

$-\frac{2}{3} + \frac{1}{6}$

$-2 \times 2 = 4$

$\frac{-4+1}{6} = -\frac{\cancel{3}}{\cancel{6}}$ $\frac{1}{2}$

$$= \frac{1}{2} - \frac{1}{2}x_3 - \frac{1}{2}u_2$$

$$\boxed{x_2 = \frac{1}{2} - \frac{1}{2}x_3 - \frac{1}{2}u_2}$$

$$f(x) = -\frac{2}{3} + \frac{2}{3}u_1 - \frac{4}{3}x_3 - \frac{2}{3}u_1^2 + \frac{2}{3}x_3 u_1 - \frac{2}{3}x_3^2$$

$$= -\frac{2}{3} + \frac{2}{3}u_1(1 - u_1 + x_3) - \frac{4}{3}x_3 - \frac{2}{3}x_3^2$$

$$= -\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2} - \frac{3}{2}u_2 + \frac{1}{2}x_3\right)\left(1 - \frac{1}{2} + \frac{3}{2}u_2 - \frac{1}{2}x_3 + x_3\right)$$

$$-\frac{4}{3}x_3 - \frac{2}{3}x_3^2$$

$$= -\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2} - \frac{3}{2}u_2 + \frac{1}{2}x_3\right)\left(\frac{1}{2} + \frac{3}{2}u_2 + \frac{1}{2}x_3\right)$$

$\frac{-2 \times 2}{3 \times 2} + \frac{1}{6}$

$\frac{-4+1}{6} = \frac{-3}{6}$

$$-\frac{4}{3}x_3 - \frac{2}{3}x_3^2$$

$$= -\frac{2}{3} + \frac{2}{3}\left(\frac{1}{4} + \cancel{\frac{3}{4}u_2} + \frac{1}{4}x_3 - \cancel{\frac{3}{4}u_2} - \frac{9}{4}u_2^2 - \cancel{\frac{3}{4}u_2 x_3}\right.$$

$\left(\frac{2}{3}\right)\left(-\frac{9}{4}\right)$

$$\left.+ \frac{1}{4}x_3 + \cancel{\frac{3}{4}u_2 x_3} + \frac{1}{4}x_3^2\right) - \frac{4}{3}x_3 - \frac{2}{3}x_3^2$$

$$= -\frac{2}{3} + \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2}x_3 - \frac{9}{4}u_2^2 + \frac{1}{4}x_3^2\right) - \frac{4}{3}x_3 - \frac{2}{3}x_3^2$$

$$= -\frac{2}{3} + \frac{1}{6} + \frac{1}{3}x_3 - \frac{3}{2}u_2^2 + \frac{1}{6}x_3^2 - \frac{4}{3}x_3 - \frac{2}{3}x_3^2$$

$$= -\frac{1}{2} - x_3 - \frac{3}{2}u_2^2 - \frac{1}{2}x_3^2$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{x_{NB}=0} = (-1 - x_3)_{\substack{x_3=0 \\ u_2=0}} = -1$$

$$\left(\frac{\partial f}{\partial u_2}\right)_{x_{NB}=0} = (-3u_2)_{u_2=0} = 0$$

| $x_1 = \frac{3}{2}$ | $x_2 = \frac{1}{2}$ |
|---|---|

$$f(x) = -6 + \cancel{6}^{3}\left(\frac{3}{\cancel{2}}\right) - \cancel{2}\left(\frac{9}{\cancel{4}_2}\right) + \cancel{2}\left(\frac{3}{\cancel{2}}\right)\left(\frac{1}{2}\right) - \cancel{2}^{1}\left(\frac{1}{\cancel{4}_2}\right)$$

$$= -6 + 9 - \frac{9}{2} + \frac{3}{2} - \frac{1}{2}$$

$$= 3 - \frac{7}{2}$$

$$= -\frac{1}{2}$$

$\max z = -\frac{1}{2}$  $z^* = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

$$\boxed{\min z = \frac{1}{2}}$$

**27**

I M.Sc – OR – Unit IV

# Non-linear Programming

*"Each venture is a new beginning to explore something hidden"*

## 27:1. INTRODUCTION

Like linear programming, *Non-linear Programming* is a mathematical technique for determining the optimal solutions to many business problems. In a *non-linear programming problem*, either the objective function is non-linear, or one or more constraints have non-linear relationship or both.

## 27:2. FORMULATING A NON-LINEAR PROGRAMMING PROBLEM (*NLPP*)

We consider some real-life problems, that we shall formulate as *NLPPs*.

### SAMPLE PROBLEMS

**2701.** *A company faces a responsive price-volume relationship for its products, the lower a product's price — the greater is the sales quantity, even in face of resultant price decreases by competitors. If the sales-revenue does not vary proportionately with price, reflect this phenomenon in a non-linear objective function of the price.*

**Mathematical Formulation of the Problem**

Let $x(p)$ represent the sales quantity as a function of the price $p$, say in the product-mix problem. Clearly, the associated sales revenue is $px(p)$. Now, if the sales quantity function be given by the demand equation $x(p) = \alpha - \beta p$ for $\alpha$, $\beta$ constants, over the range of $p$, then the sales revenue component in the objective function is quadratic, $z = px(p) = \alpha p - \beta p^2$. If each unit costs $c$ to produce (where $p$ and $c$ are in the same units), then total profit $P$ is given by

$$P = z - cx(p) = \alpha p - \beta p^2 - c\alpha + c\beta p = (\alpha + c\beta)p - c\alpha - \beta p^2.$$

**2702.** (**Production Allocation Problem**) *A manufacturing company produces two products : Radios and TV sets. Sales-price relationships for these two products are given below :*

| *Product* | *Quantity demanded* | *Unit price* |
|---|---|---|
| *Radios* | $1{,}500 - 5\,p_1$ | $p_1$ |
| *TV sets* | $3{,}800 - 10\,p_2$ | $p_2$ |

*The total cost functions for these two products are given by* $200x_1 + 0.1x_1^2$ *and* $300x_2 + 0.1x_2^2$ *respectively. The production takes place on two assembly lines. Radio sets are assembled on Assembly line I and TV sets are assembled on Assembly line II. Because of the limitations of the assembly-line capacities, the daily production is limited to no more than 80 radio sets and 60 TV sets. The production of both types of products requires electronic components. The production of each of these sets requires five units and six units of electronic equipment components respectively. The electronic* *components are supplied by another manufacturer, and the supply is limited to 600 units per day. The company has 160 employees, i.e., the labour supply amounts to 160 man-days. The production of one unit of radio set requires 1 man-day of labour, whereas 2 man-days of labour are required for a TV set. How many units of radio and TV sets should the company produce in order to maximize the total profit? Formulate the problem as a non-linear programming problem.*

**Mathematical Formulation of the Problem**

Let us assume that whatever is produced is sold in the market. Let $x_1$ and $x_2$ stand for the quantities of radio sets and TV sets respectively, manufactured by the firm. Then we are given that

$$x_1 = 1,500 - 5p_1 \qquad \text{or} \qquad p_1 = 300 - 0.2x_1$$
$$x_2 = 3,800 - 10p_2 \qquad\qquad p_2 = 380 - 0.1x_2$$

Further, if $C_1$, $C_2$ stand for the total cost of production of these units of radio sets and TV sets respectively, then we are also given that

$$C_1 = 200x_1 + 0.1x_1^2 \quad \text{and} \quad C_2 = 300x_2 + 0.1x_2^2$$

Now, the revenue on radio sets is $p_1 x_1$ and on TV sets is $p_2 x_2$. Thus, the total revenue $R$ is measured by

$$R = p_1x_1 + p_2x_2$$

which can be written as

$$R = (300 - 0.2x_1)x_1 + (380 - 0.1x_2)x_2$$
$$= 300x_1 - 0.2x_1^2 + 380x_2 - 0.1x_2^2.$$

The total profit $z$ is measured by the difference between the total revenue $R$ and the total cost $C = C_1 + C_2$. Thus

$$z = R - C_1 - C_2 = 100x_1 - 0.3x_1^2 + 80x_2 - 0.2x_2^2.$$

The objective function thus obtained is a non-linear function.

In the present case, production is influenced by the available resources. The two assembly lines have limited capacity to produce radio and TV sets. Since no more than 80 radio sets can be assembled on assembly line I and 60 TV sets on assembly line II per day, we have the restrictions : $x_1 \leq 80$ and $x_2 \leq 60$.

There is another side constraint in the daily requirement of the electronic components, so that $5x_1 + 6x_2 \leq 600$. The number of available employees is limited to 160 man-days. Thus $x_1 + 2x_2 \leq 160$. Also obviously, since the manufacturer cannot produce negative number of units, we must have $x_1 \geq 0$ and $x_2 \geq 0$.

Hence, the given problem can be put in the following mathematical format :

> Determine two real numbers, $x_1$ and $x_2$ so as to maximize
> $$z = 100x_1 - 0.3x_1^2 + 80x_2 - 0.2x_2^2$$
> subject to the contraints :
> $$0 \leq x_1 \leq 80, \; 0 \leq x_2 \leq 60, \; 5x_1 + 6x_2 \leq 600,$$
> $$x_1 + 2x_2 \leq 160, \quad x_1 \geq 0, \quad x_2 \geq 0.$$

This problem is a non-linear programming problem, since the objective function is non-linear in $x_1$ and $x_2$.

***Remarks.*** In a non-linear programming problem, the objective function $z$ may be linear in $x_1$ and $x_2$, whereas the constraints are non-linear in $x_1$ and $x_2$, or both $z$ and the constraints may be non-linear in $x_1$ and $x_2$. For example, the decision-making problem

$$\text{Maximize } f(x_1, x_2) = 3x_1 + 5x_2 \text{ subject to the constraints}$$

$$x_1 + x_2 \le 3, \quad x_1^2 + x_2^2 \le 10, \quad \text{and} \quad x_1, x_2 \ge 0$$

is a non-linear programming problem.

**2703.** (***Portfolio Selection Problem***) *An individual investor has an opportunity to invest a fixed amount of money in n different bonds and stocks and wishes to maximize his anticipated returns while considering variance of return as undesirable. Let $x_j$ be the proportion of his assets invested in the jth security. Then, the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called a portfolio and the return R corresponding to a given portfolio $\mathbf{x}$ is a random variable. The investor is risk-averse and is therefore interested in determining a portfolio $\mathbf{x}$ that will minimise the variance of R subject to the restriction that his expected return is not less than some specified amount C (per unit invested). Formulate this portfolio selection problem as an NLPP.*

### Mathematical Formulation of the Problem

Suppose the total funds available to investor is $B$. There are $n$ channels of investment. The expected return per unit of investment from the $i$th channel is $r_i$, the variance of the $i$th investment is $\sigma_i^2$, while the covariance between $i$th and $j$th investment is $\sigma_{ij}$.

Thus, if an amount $x_i$ ($i = 1, 2, \ldots, n$) is invested in the $i$th type, then the expected return is $\sum_{i=1}^{n} r_i x_i$; while the variance of the investment is :

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j.$$

As higher return and lower variance are desirable quantities from the investor's point of view, the objective function may be :

$$z = \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

Also, since the total amount the investor can spend is $B$ and as his expected return must not be less than $C$; we have

$$\sum_{i=1}^{n} x_i = B \quad \text{and} \quad \sum_{i=1}^{n} r_i \ge C.$$

From the point of view of investor, $z$ should be maximized subject to the constraints :

$$\sum_{i=1}^{n} x_i = B \quad \text{and} \quad \sum_{i=1}^{n} r_i \ge C, \quad \text{where } x_i \ge 0 \; (i = 1, 2, \ldots, n).$$

## *PROBLEMS*

**2704.** (***One-Potato, Two-Potato Problem***) A frozen-food company processes potatoes into packages of French fries, hash browns and flakes (for meshed potatoes). At the beginning of the manufacturing process, the raw potatoes are sorted by length and quality, and then allocated to the separate product lines.

The company can purchase its potatoes from two sources, which differ in their yields of various sizes and quality. Each source yields different fractions of the products French fries, hash browns and flakes. Suppose that it is possible, at different costs, to alter these yields somewhat. Let $f_1$, $f_2$ and $f_3$ be the fractional yield per unit of weight of source *1* potatoes made into the three products, similarly, let $g_1$, $g_2$ and $g_3$ be the yields for source *2*. Suppose that each $f_i$ and $g_i$ can vary within $\pm$ 10% of the yields shown below :

| *Product* | *Source 1* | *Source 2* | *Purchase limitations* |
|---|---|---|---|
| French fries | 0.2 | 0.3 | 1.8 |
| Hash browns | 0.2 | 0.1 | 1.2 |
| Flakes | 0.3 | 0.3 | 2.4 |
| Relative Profit | 5 | 6 | |

Let $C_1(f_1, f_2, f_3)$ and $C_2(g_1, g_2, g_3)$ be the expense associated with obtaining these yields.

The problem is to determine how many potatoes should the company purchase from each source?

Formulate the problem as a non-linear programming problem.

**2705.** A manufacturing concern operates its two available machines to polish its metal products. The two machines are equally efficient, although their maintenance costs are different. The daily maintenance and operation cost of the machines is given in rupees as the non-linear function :

$$f(x_1, x_2) = 100 - 1.2x_1 - 1.5x_2 + 0.3x_1^2 + 0.5x_2^2$$

where $x_1$ and $x_2$ are the number of hours of operation of machine *I* and machine *II* respectively.

The past records of the firm indicate that the combined operating hours of two machines should be at least 35 hours a day in order to perform a satisfactory job. However, the production manager wishes to operate machine *I* at least 6 hours more than machine *II* because of the higher repair cost of the latter. Find the optimal hours of operating the two machines and the minimum daily cost.

Formulate the problem as a non-linear programming problem.

**2706.** A company manufactures two products *A* and *B* It takes 30 minutes to process one unit of product *A* and 15 minutes for each unit of *B* and the maximum machine time available is 35 hours per week. Products *A* and *B* require 2 kgs. and 3 kgs. of raw material per unit respectively. The available quantity of raw material is envisaged to be 180 kgs. per week.

The products *A* and *B* which have unlimited market potential sell for Rs. 200 and Rs. 500 per unit respectively. If the manufacturing costs for products *A* and *B* are $2x^2$ and $3y^2$ respectively, find how much of each product should be produced per week, where

$x$ = Quantity of Product *A* to be produced, and

$y$ = Quantity of Product *B* to be produced.

Formulate the problem as a non-linear programming problem.

## 27:3. GENERAL NON-LINEAR PROGRAMMING PROBLEM

***Definition 1*** **(General Non-linear Programming Problem).** *Let $z$ be a real valued function of n variables defined by*

(a) $$z = f(x_1, x_2, \ldots, x_n).$$

*Let $\{b_1, b_2, \ldots, b_m\}$ be a set of constants such that*

(b) $$\begin{cases} g^1(x_1, x_2, \ldots, x_n) \ \{\le, \ge \text{ or } =\} \ b_1 \\ g^2(x_1, x_2, \ldots, x_n) \ \{\le, \ge \text{ or } =\} \ b_2 \\ \vdots \qquad\qquad \vdots \\ g^m(x_1, x_2, \ldots, x_n) \ \{\le, \ge \text{ or } =\} \ b_m \end{cases}$$

*where $g^i$'s are real valued functions of n variables, $x_1, \ldots, x_n$. Finally, let*

(c) $$x_j \ge 0, \quad j = 1, 2, \ldots, n.$$

*If either $f(x_1, \ldots, x_n)$ or some $g^i(x_1, \ldots, x_n)$, $i = 1, 2, \ldots, m$; or both are non-linear, then the problem of determining the n-tuple $(x_1, x_2, \ldots x_n)$ which makes $z$ a maximum or minimum and satisfies (b) and (c), is called a* ***general non-linear programming problem (GNLPP).***

In matrix notations the *GNLPP* may be written as :

Determine $\mathbf{x}^T \in R^n$ so as to maximize or minimize the objective function $z = f(x)$, subject to the constraints :

$$g^i(\mathbf{x}) \{\le, \ge \text{ or } =\} b_i, \quad \mathbf{x} \ge \mathbf{0} \qquad i = 1, 2, \ldots, m,$$

where either $f(\mathbf{x})$ or some $g^i(\mathbf{x})$ or both are non-linear in $\mathbf{x}$.

Sometimes it is convenient to write the constraints $g^i(x)\{\le, \ge \text{or} =\} b_i$ as $h^i(\mathbf{x})\{\le, \ge \text{or} =\} 0$ for $h^i(x) = g^i(\mathbf{x}) - b_i$.

There is no simplex-like solution procedure for the solution of the general non-linear programming problem. However, numerous solution methods have been developed since the appearance of the fundamental theoretical paper by Kuhn and Tucker. A few primary types of available solution techniques will be discussed in this and the next chapter.

## 27:4. CONSTRAINED OPTIMIZATION WITH EQUALITY CONSTRAINTS

If the non-linear programming problem is composed of some differentiable objective function and equality constraints, the optimization may be achieved by the use of *Lagrange multipliers** as illustrated below :

Consider the problem of maximizing or minimizing $z = f(x_1, x_2)$ subject to the constraints :

$$g(x_1, x_2) = c \quad \text{and} \quad x_1, x_2 \geq 0,$$

where $c$ is a constant.

We assume that $f(x_1, x_2)$ and $g(x_1, x_2)$ are differentiable w.r.t. $x_1$ and $x_2$. Let us introduce a differentiable function $h(x_1, x_2)$, differentiable w.r.t. $x_1$ and $x_2$ and defined by $h(x_1, x_2) \equiv g(x_1, x_2) - c$. Then the problem can be restated as

$$\text{Maximize} \quad z = f(x_1, x_2) \quad \text{subject to the constraints :}$$
$$h(x_1, x_2) = 0 \quad \text{and} \quad x_1, x_2 \geq 0.$$

To find the necessary conditions for a maximum (or minimum) value of $z$, a new function is formed by introducing a Lagrange multiplier $\lambda$, as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2).$$

The number $\lambda$ is an unknown constant, and the function $L(x_1, x_2, \lambda)$ is called the *Lagrangian function with Lagrange multiplier* $\lambda$. The necessary conditions for a maximum or minimum (stationary value) of $f(x_1, x_2)$ subject to $h(x_1, x_2) = 0$ are thus given by

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0, \quad \frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \quad \text{and} \quad \frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0.$$

Now, these partial derivatives are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1},$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2},$$

and

$$\frac{\partial L}{\partial \lambda} = -h,$$

where $L$, $f$ and $h$ stand for the functions $L(x_1, x_2, \lambda)$, $f(x_1, x_2)$, and $h(x_1, x_2)$ respectively, or simply by

$$L_1 = f_1 - \lambda h_1, \quad L_2 = f_2 - \lambda h_2 \quad \text{and} \quad L_\lambda = -h.$$

The *necessary conditions for maximum or minimum of* $f(x_1, x_2)$ are thus given by

$$f_1 = \lambda h_1, \quad f_2 = \lambda h_2 \quad \text{and} \quad -h(x_1, x_2) = 0$$

***Note.*** These *necessary conditions* become sufficient conditions for a maximum (minimum) if the objective function is concave (convex) and the side constraints are in the form of equalities.

**Illustration.** *Obtain the set of necessary conditions for the non-linear programming problem :*

$$\textit{Minimize} \quad z = kx^{-1}y^{-2}, \quad \textit{subject to the constraints :}$$
$$x^2 + y^2 - a^2 = 0 \quad \textit{with} \quad x \geq 0, \; y \geq 0;$$

*and hence find the minimum value of z.*

*The method of *Lagrange multipliers* is a systematic way of generating the necessary conditions for a stationary point.

**Solution.** The Lagrange function is

$$L(x, y, \lambda) = k\,x^{-1}\,y^{-2} + \lambda\,(x^2 + y^2 - a^2),$$

where $f(x, y) = k\,x^{-1}\,y^{-2}$ and $h(x, y) = g(x, y) - C = (x^2 + y^2) - a^2$.

The necessary conditions for the minimum of $f(x, y)$ gives

$$\frac{\partial L}{\partial x} = 0 \Rightarrow -k\,x^{-2}\,y^{-2} + 2x\lambda = 0,$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow -2k\,x^{-1}\,y^{-3} + 2\lambda y = 0,$$

and

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 - a^2 = 0.$$

From the first two equations, we get

$$2\lambda = k\,x^{-3}\,y^{-2} = 2k\,x^{-1}\,y^{-4}. \text{ This yields } x = y/\sqrt{2}.$$

Using this value of $x$ in the third equation,

$$y = a\sqrt{2/3} \text{ and therefore } x = \frac{1}{\sqrt{2}} \times a\sqrt{\frac{2}{3}} = \frac{a}{\sqrt{3}}$$

$\therefore$ $\quad$ Minimum $z = k \times (a/\sqrt{3})^{-1}\,(a\sqrt{2/3})^{-2} = 3\sqrt{3}\,k/2a^3.$

## Necessary Conditions for a General *NLPP*

Consider the general *NLPP* :

Maximize (or Minimize) $z = f(x_1, x_2, \ldots, x_n)$ subject to the constraints :

$$g^i(x_1, \ldots, x_n) = c_i \quad \text{and} \quad x_i \geq 0; \quad i = 1, 2, \ldots, m\ (< n)$$

The constraints can be reduced to

$$h^i(x_1, \ldots, x_n) = 0 \quad \text{for } i = 1, 2, \ldots, m,$$

by the transformation $h^i(x_1, \ldots, x_n) = g^i(x_1, \ldots, x_n) - c_i$ for all $i = 1, 2, \ldots, m\ (< n)$.

The problem can then be written in the matrix form as

Maximize (or Minimize) $z = f(\mathbf{x})$, $(\mathbf{x} \in \mathbf{R}^n)$ subject to the constraints :

$$h^i(\mathbf{x}) = 0,\ \mathbf{x} \geq \mathbf{0}.$$

To find the necessary conditions for a maximum or minimum of $f(\mathbf{x})$, the Lagrangian function $L(\mathbf{x}, \lambda)$, is formed by introducing $m$ Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. This function is defined by

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i\,h^i(\mathbf{x}).$$

Assuming that $L$, $f$ and $h^i$ are all differentiable partially w.r.t. $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$, the necessary conditions for a maximum (minimum) of $f(\mathbf{x})$ are :

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(\mathbf{x})}{\partial x_j} = 0, \qquad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -h^i(\mathbf{x}) = 0, \qquad i = 1, 2, \ldots, m$$

These $m + n$ necessary conditions can be represented in the following abbreviated form :

$$L_j = f_j - \sum_{i=1}^{m} \lambda_i\,h^i_j = 0 \quad \text{or} \quad f_j = \sum_{i=1}^{m} \lambda_i\,h^i_j; \qquad j = 1, 2, \ldots, n$$

and

$$L\lambda_j = -h^i = 0 \quad \text{or} \quad h^i = 0; \qquad i = 1, 2, \ldots, m$$

where $\quad f_j = \dfrac{\partial f(\mathbf{x})}{\partial x_j}$, $\quad h^i = h^i(\mathbf{x})$ and $\quad h^i_j = \dfrac{\partial h^i(\mathbf{x})}{\partial x_j}$.

***Remark.*** These necessary conditions also become sufficient for a maximum (minimum) of the objective function if the objective function is concave (convex) and the constraints are equality ones.

**Illustration.** *Obtain the set of necessary conditions for the non-linear programming problem :*

$$\text{Maximize } z = x_1^2 + 3x_2^2 + 5x_3^2 \text{ subject to the constraints :}$$

$$x_1 + x_2 + 3x_3 = 2, \quad 5x_1 + 2x_2 + x_3 = 5$$

$$x_1, x_2, x_3 \geq 0.$$

**[Delhi B.Sc. (Stat.) 2006]**

**Solution.** Here, we have $\mathbf{x} = (x_1, x_2, x_3)$, $f(\mathbf{x}) = x_1^2 + 3x_2^2 + 5x_3^2$, $g^1(\mathbf{x}) = x_1 + x_2 + 3x_3$, $g^2(\mathbf{x}) = 5x_1 + 2x_2 + x_3$ and $c_1 = 2$, $c_2 = 5$. Defining $h^i(\mathbf{x}) = g^i(\mathbf{x}) - c_i$, $i = 1, 2$, we have the constraints : $h^i(\mathbf{x}) = 0$ for $i = 1, 2$.

For necessary conditions of maximizing $f(\mathbf{x})$, we construct the Lagrangian function

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda_1 h^1(\mathbf{x}) - \lambda_2 h^2(\mathbf{x}), \quad \lambda = (\lambda_1, \lambda_2).$$

This yields the following necessary conditions :

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0, \qquad \frac{\partial L}{\partial x_2} = 6x_2 - \lambda_1 - 2\lambda_2 = 0,$$

$$\frac{\partial L}{\partial x_3} = 10x_3 - 3\lambda_1 - \lambda_2 = 0, \qquad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0,$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0.$$

***Remark.*** In some cases it is not generally possible to solve the equations resulting from the set of necessary conditions, explicitly for the values of **x** and $\lambda_i$'s. A convenient method in this case is to select successive numerical values of $\lambda_i$'s and then solve the set of necessary conditions for **x**. This is repeated until for some values of $\lambda_i$, the resulting **x** satisfies all the constraints in equation form.

## Sufficient Conditions for a General *NLPP* with one Constraint

Let the Lagrangian function for a general *NLPP* involving $n$ variables and one constraint be :

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda h(\mathbf{x}).$$

The necessary conditions for a stationary point to be a maximum or minimum are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \qquad (j = 1, 2, \ldots, n)$$

and

$$\frac{\partial L}{\partial \lambda} = -h(x) = 0.$$

The value of $\lambda$ is obtained by

$$\lambda = \frac{\partial f/\partial x_j}{\partial h/\partial x_j} \qquad (\text{for } j = 1, 2, \ldots, n).$$

The sufficient conditions for a maximum or minimum require the evaluation at each stationary point, of $n-1$ principal minors of the determinant given below :

$$\Delta_{n+1} = \begin{vmatrix} 0 & \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} & \cdots & \dfrac{\partial h}{\partial x_n} \\ \dfrac{\partial h}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 h}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \dfrac{\partial^2 x}{\partial x_1 \partial x_n} \\ \dfrac{\partial h}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 h}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{\partial h}{\partial x_n} & \dfrac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 h}{\partial x_n^2} - \lambda \dfrac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If $\Delta_3 > 0$, $\Delta_4 < 0$, $\Delta_5 > 0$, ... , the signs pattern being alternate, the stationary point is a local maximum.

If $\Delta_3 < 0$, $\Delta_4 < 0$, ..., $\Delta_{n+1} < 0$, the sign being always negative, the stationary point is a local minimum.

**Illustration.** *Obtain the set of necessary and sufficient conditions for the following NLPP :*

*Minimize* $z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$ *subject to the constraints*

$$x_1 + x_2 + x_3 = 11, \; x_1, x_2, x_3 \geq 0.$$

**Solution.** We formulate the Lagrangian function as

$$L(x_1, x_2, \lambda) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200 - \lambda(x_1 + x_2 + x_3 - 11)$$

The necessary conditions for the stationary point are

$$\frac{\partial L}{\partial x_1} = 4x_2 - 24 - \lambda = 0 \qquad \frac{\partial L}{\partial x_3} = 4x_3 - 8 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_3 - 12 - \lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 11) = 0.$$

The solution of the simultaneous equations yields the stationary point

$$\mathbf{x}_0 = (x_1, x_2, x_3) = (6, 2, 3) ; \; \lambda = 0.$$

The sufficient condition for the stationary point to be a minimum is that both the minors $\Delta_3$ and $\Delta_4$ should be negative. Now, we have

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -8 \quad \text{and} \quad \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -48.$$

Since $\Delta_3$ and $\Delta_4$ both are negative, $\mathbf{x}_0 = (6, 2, 3)$ provides the solution to the *NLLP*. Hence, the stationary point is a local minimum. Thus $x_0 = (6, 2, 3)$ provides the solution to the *NLPP*.

**Exercise.** Examine $z = 6x_1 x_2$ for maxima and minima under the requirement $2x_1 + x_2 = 10$.

[**Hint :** The stationary point is $x_0 = (2, 5, 5)$ which is a local maximum for $z$.]

## Sufficient Conditions for a General Problem with $m$ ($< n$) Constraints

Introducing the $m$ Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$, let the Lagrangian function for a general *NLPP* with more than one constraint be :

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h^i(\mathbf{x}) \qquad (m < n)$$

The reader may verify that the equations

$$\frac{\partial L}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda_i} = 0 \qquad (i = 1, 2, ..., m ; j = 1, 2, ... n)$$

yield the necessary conditions for stationary points of $f(\mathbf{x})$. Thus, the optimization of $f(\mathbf{x})$ subject to $h(\mathbf{x}) = 0$ is equivalent to the optimization of $L(\mathbf{x}, \lambda)$. We state here the sufficiency conditions for the Lagrange multiplier method of stationary point of $f(\mathbf{x})$ to be a maxima or minima without proof. For this we assume that the function $L(\mathbf{x}, \lambda)$, $f(\mathbf{x})$ and $h(\mathbf{x})$ all possess partial derivatives of order one and two w.r.t. the decision variables.

Let
$$V = \left( \frac{\partial^2 L(\mathbf{x}, \lambda)}{\partial x_i \partial x_j} \right)_{n \times n}$$

be the matrix of second order partial derivatives of $L(\mathbf{x}, \lambda)$ w.r.t. decision variables

$$U = [\, h'_j(\mathbf{x}) \,]_{m \times n}$$

where $h'_j(\mathbf{x}) = \dfrac{\partial h^i(\mathbf{x})}{\partial \mathbf{x}_j}$, $i = 1, 2, ..., m ; \; j = 1, 2, ..., n.$

Define the square matrix

$$\mathbf{H}^B = \begin{bmatrix} \mathbf{O} & \mathbf{U} \\ \mathbf{U}^T & \mathbf{V} \end{bmatrix}_{(m+n)\times(m+n)}$$

where $\mathbf{O}$ is an $m \times m$ null matrix. The matrix $\mathbf{H}^B$ is called the *bordered Hessian matrix*. Then, the sufficient conditions for maximum and minimum stationary points are given below.

Consider $(\mathbf{x}_o, \lambda_o)$ for the function $L(\mathbf{x}, \lambda)$ to be its stationary point. Let $\mathbf{H}^B{}_o$ be the corresponding bordered Hessian matrix computed at this stationary point. Then $\mathbf{x}_o$ is a

(*a*) maximum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of $\mathbf{H}^B{}_o$ from an alternating sign pattern starting with $(-1)^{m+n}$; and

(*b*) minimum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of $\mathbf{H}^B{}_o$ have the sign of $(-1)^m$.

***Remark.*** It may be observed that the above conditions are only sufficient for identifying an extreme point, but not necessary. That is, a stationary point may be an extreme point without satisfying the above conditions.

## SAMPLE PROBLEMS

**2707.** (***Input-Allocation Problem***) *A manufacturing concern produces a product consisting of two raw materials, say $A_1$ and $A_2$. The production function is estimated as*

$$z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

*where $z$ represents the quantity (in tons) of the product produced and, $x_1$ and $x_2$ designate the input amounts of raw materials $A_1$ and $A_2$. The company has Rs. 50,000 to spend on these two raw materials. The unit price of $A_1$ is Rs. 10,000 and of $A_2$ is Rs. 5,000. Determine how much input amounts of $A_1$ and $A_2$ be decided so as to maximize the production output.*

**Solution.** Since the company must operate within the available funds, the budgetary constraint is

$$10{,}000x_1 + 5{,}000x_2 \le 50{,}000 \quad \text{or} \quad 2x_1 + x_2 \le 10.$$

We reduce this inequality constraint to an equality by imposing an additional assumption that the company has to spend every available single paisa on these raw materials. Then, the constraint is $2x_1 + x_2 = 10$. Also, obviously $x_1 \ge 0$ and $x_2 \ge 0$. The problem of the company can thus be written as the following *NLPP* :

Maximize $z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$ subject to the constraints :

$$2x_1 + x_2 = 10 \quad \text{and} \quad x_1, x_2 \ge 0$$

or Maximize $z = f(x_1, x_2)$ subject to the constraints :

$$h(x_1, x_2) = 0 \quad \text{and} \quad x_1, x_2 \ge 0$$

where $h(x_1, x_2) = 2x_1 + x_2 - 10$. Observe that $f(x_1, x_2)$ and $h(x_1, x_2)$ are both differentiable w.r.t. $x_1$ and $x_2$.

Also, we observe that the objective function $z = f(x_1, x_2)$ is a concave function and the said constraint is an equality constraint. Therefore, the necessary and sufficient conditions for a maximum are

$$f_1 = \lambda h_1, \quad f_2 = \lambda h_2 \quad \text{and} \quad -h(x_1, x_2) = 0$$

That is,

$$3.6 - 0.8x_1 = 2\lambda, \quad 1.6 - 0.4x_2 = \lambda, \quad \text{and} \quad 2x_1 + x_2 = 10.$$

The first two of these yield

$$\lambda = 1.8 - 0.4x_1 = 1.6 - 0.4x_2$$

and so the elimination of $\lambda$ gives $0.4x_1 - 0.4x_2 - 0.2 = 0$.

Now since $x_2 = 10 - 2x_1$, the last equation gives

$$0.4x_1 - 0.4(10 - 2x_1) - 0.2 = 0$$

or

$$1.2x_1 - 4.2 = 0, \quad \text{or} \quad x_1 = 3.5$$

Thus,

$$x_2 = 10 - 2x_1 = 3.$$

The maximum value of the objective function is thus, given by

$$z = f(3.5, 3) = 3.6(3.5) - 0.4(3.5)^2 + 1.6(3) - 0.2(3)^2$$

$$= 10.7 \text{ (tonnes)}.$$

Thus, in order to have a maximum production of 10.7 tonnes, the company must input 3.5 units of raw material $A$ and 3 units of raw material $B$.

**2708.** *Obtain the necessary and sufficient conditions for the optimum solution of the following NLPP :*

*Minimize* $z = f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$ *subject to the constraints :*

$$x_1 + x_2 = 7, \quad x_1, x_2 \geq 0.$$

**[Kerala M.Sc. (Math.) 2001]**

**Solution.** Let us introduce a new differentiable Lagrangian function $L(x_1, x_2, \lambda)$ defined by

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(x_1 + x_2 - 7)$$

$$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

where $\lambda$ is the Lagrangian multiplier.

Since the objective function $z = f(x_1, x_2)$ is convex and the constraint an equality, the necessary and sufficient conditions for the minimum of $f(x_1, x_2)$ are given by

$$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0 \quad \text{or} \quad \lambda = 6e^{2x_1+1}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0 \quad \text{or} \quad \lambda = 2e^{x_2+5}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0 \quad \text{or} \quad x_1 + x_2 = 7$$

Using these three, we get

$$6e^{2x_1+1} = 2e^{x_2+5} \quad \text{or} \quad 3e^{2x_1+1} = e^{x_2+5} = e^{7-x_1+5}$$

or

$$\log 3 + (2x_1 + 1) = 7 - x_1 + 5 \quad \text{or} \quad x_1 = \tfrac{1}{3}(11 - \log 3).$$

Thus

$$x_2 = 7 - \tfrac{1}{3}(11 - \log 3) = (10 + \log 3)/3.$$

**2709.** *Solve the non-linear programming problem :*

*Optimize* $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$ *subject to the constraints :*

$$x_1 + x_2 + x_3 = 15, \quad 2x_1 - x_2 + 2x_3 = 20.$$

**[Delhi B.Sc. (Stat.) 2002]**

**Solution.** Here, we have

$$f(\mathbf{x}) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2, \quad h^1(\mathbf{x}) = x_1 + x_2 + x_3 - 15$$

$$h^2(\mathbf{x}) = 2x_1 - x_2 + 2x_3 - 20.$$

Construct the Lagrangian function

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda_1 h^1(\mathbf{x}) - \lambda_2 h^2(\mathbf{x})$$

$$= (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

The stationary point $(\mathbf{x}_0, \lambda_0)$ has thus given the following necessary conditions :

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial L}{\partial \lambda_1} = -[x_1 + x_2 + x_3 - 15] = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -[2x_1 - x_2 + 2x_3 - 20] = 0.$$

The solution to these simultaneous linear equations yields

$$\mathbf{x}_o = (x_1, x_2, x_3) = (33/9,\ 10/3,\ 8) \quad \text{and} \quad \lambda_o = (\lambda_1, \lambda_2) = (40/9,\ 52/9).$$

The bordered Hessian matrix at this solution $(\mathbf{x}_o, \lambda_o)$ is given by

$$\mathbf{H}^B{}_o = \left[\begin{array}{cc:ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hdashline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

Here since $n = 3$ and $m = 2$, therefore $n - m = 1$, $(2m + 1 = 5)$. This means that one needs to check the determinant of $\mathbf{H}^B{}_o$ only and it must have the sign of $(-1)^2$.

Now, since det $\mathbf{H}^B{}_o = 90 > 0$, $\mathbf{x}_o$ is a minimum point. 19

## *PROBLEMS*

*Solve the following non-linear programming problems, using the method of Lagrangian multipliers.*

**2710.** Minimize $z = 6x_1^2 + 5x_2^2$ subject to the constraints :

$$x_1 + 5x_2 = 3, \quad x_1, x_2 \geq 0.$$

*[Kerala M.Sc. (Math.) 2001]*

**2711.** Minimize $f(x_1, x_2) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$ subject to the constraints :

$$2x_1 - x_2 = 4, \quad x_1, x_2 \geq 0.$$

*[Nagarjuna M.Sc. (Stat.) 1989]*

**2712.** Maximize $z = 5x_1 + x_2 - (x_1 - x_2)^2$ subject to the constraints :

$$x_1 + x_2 = 4, \quad x_1, x_2 \geq 0.$$

**2713.** Maximize $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$ subject to the constraints :

$$x_1 + 2x_2 = 2, \text{ and } x_1, x_2 \geq 0.$$

**2714.** Minimize $z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$ subject to the constraints :

$$x_1 + x_2 + x_3 = 20, \quad x_1, x_2, x_3 \geq 0.$$

**2415.** Minimize $z = x_1^2 + x_2^2 + x_3^2$ subject to the constraints :

$$4x_1 + x_2^2 + 2x_3 = 14, \quad x_1, x_2, x_3 \geq 0.$$

**2716.** Determine the optimum solution for the following *NLPP* and check whether it maximizes or minimizes the objective function :

$$z = x_1^2 - 10x_1 + x_2^2 - 6x_2 + x_3^2 - 4x_3 \text{ subject to the constraint :}$$

$$x_1 + x_2 + x_3 = 7, \text{ and } x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

**2717.** Solve the following *NLPP* :

Optimize $z = 4x_1 + 9x_2 - x_1^2 - x_2^2$ subject to the constraints :

$$4x_1 + 3x_2 = 15, \quad 3x_1 + 5x_2 = 14; \quad x_1 \geq 0 \quad x_2 \geq 0.$$

**2718.** Minimize $z = x_1^2 + x_2^2 + x_3^2$ subject to the constraints :

$$x_1 + x_2 + 3x_3 = 2, \quad 5x_1 + 2x_2 + x_3 = 5, \quad x_1, x_2, x_3 \geq 0.$$

*[Annamalai M.E. (Nov.) 2002; Delhi B.Sc. (Stat.) 2004]*

**2719.** Minimize $z = 6x_1 + 8x_3 - x_1^2 - x_2^2$ subject to the constraints :

$$4x_1 + 3x_2 = 16, \quad 3x_1 + 5x_2 = 15, \quad x_1, x_2 \geq 0.$$

**2720.** A positive quantity $b$ is to be divided into $n$ parts in such a way that the product of $n$ parts is to be a maximum. Use Lagrange's multiplier method to obtain the optimal sub-division.

[**Hint** : Let $n$ parts be $x_1, x_2, \ldots, x_n$. Then, the problem is :

Maximize $z = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ subject to : $x_1 + x_2 + \ldots + x_n = b$, and $x_j \geq 0$ ($j = 1, 2, \ldots, n$).]

## 27:5. CONSTRAINED OPTIMIZATION WITH INEQUALITY CONSTRAINTS

We shall now derive the *Kuhn-Tucker Conditions* (necessary and sufficient) for the optimal solution of general *NLPP*. Consider the general *NLPP* :

$$\text{Optimize} \quad z = f(x_1, x_2, \ldots, x_n) \quad \text{subject to the constraints :}$$

$$g(x_1, \ldots, x_n) \leq C \quad \text{and} \quad x_1, x_2, \ldots, x_n \geq 0$$

where $C$ is a constant.

Introducing the function $h(x_1, \ldots, x_n) = g - C$, the constraint reduces to $h(x_1, \ldots, x_n) \leq 0$. The problem, thus, can be written as

$$\text{Optimize} \quad z = f(\mathbf{x}) \text{ subject to } h(\mathbf{x}) \leq 0 \text{ and } \mathbf{x} \geq \mathbf{0}, \text{ where } \mathbf{x} \in \mathbf{R}^n.$$

We now slightly modify the problem by introducing new variable $S$, defined by $S^2 = -h(\mathbf{x})$, or $h(\mathbf{x}) + S^2 = 0$.

The new variable $S$ is called a *slack variable* and appears as its square in the constraint equation so as to ensure its being non-negative. This avoids an additional constraint $S \geq 0$. Now the problem can be restated as

$$\text{Optimize} \quad z = f(\mathbf{x}), \ \mathbf{x} \in \mathbf{R}^n \quad \text{subject to the constraints :}$$

$$h(\mathbf{x}) + S^2 = 0 \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}.$$

This is a problem of constrained optimization in $n + 1$ variables and a single equality constraint and can thus be solved by the Lagrangian multiplier method.

To determine the stationary points, we consider the Lagrangian function defined by

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \lambda [h(\mathbf{x}) + S^2],$$

where $\lambda$ is the Lagrange multiplier. The necessary conditions for stationary points are

$$\frac{\partial L}{\partial x_j} \equiv \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \quad \text{for } j = 1, 2, \ldots, n. \qquad \ldots(1)$$

$$\frac{\partial L}{\partial \lambda} \equiv -[h(\mathbf{x}) + S^2] = 0. \qquad \ldots(2)$$

$$\frac{\partial L}{\partial S} \equiv -2S\lambda = 0. \qquad \ldots(3)$$

Equation (3) states that $\dfrac{\partial L}{\partial S} = 0$, which requires either $\lambda = 0$ or $S = 0$. If $S = 0$, (2) implies that $h(\mathbf{x}) = 0$. Thus (2) and (3) together imply $\lambda\, h(\mathbf{x}) = 0$.

The variable $S$ was introduced merely to convert the inequality constraint into an equality one, and therefore may be discarded. Moreover, since $S^2 \geq 0$, (2) gives $h(\mathbf{x}) \leq 0$. Whenever $h(\mathbf{x}) < 0$, we get $\lambda = 0$ and whenever $\lambda > 0$, $h(\mathbf{x}) = 0$. However, $\lambda$ is unrestricted in sign whenever $h(\mathbf{x}) = 0$.

The necessary conditions for the point $\mathbf{x}$ to be a point of maximum are thus restated as (in the abbreviated form) :

| | |
|---|---|
| $f_j - \lambda h_j = 0$ | $(j = 1, 2, \ldots, n)$ |
| $\lambda h = 0$ | *Maximize* $f$ |
| $h \leq 0$ | *subject to* : |
| $\lambda \geq 0$* | $h \leq 0$. |

The set of such necessary conditions is called *Kuhn-Tucker Conditions*.

A similar argument holds for the minimization of non-linear programming problem :

Minimize $z = f(\mathbf{x})$ subject to the constraints :

$$g(\mathbf{x}) \geq C \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}.$$

Introduction of $h(\mathbf{x}) = g(\mathbf{x}) - C$ reduces the first constraint to $h(\mathbf{x}) \geq 0$. The new surplus variable $S_o$ can be introduced in $h(\mathbf{x}) \geq 0$ so that we may have the equality constraint $h(\mathbf{x}) - S_o^2 = 0$. The appropriate Lagrangian function is

$$L(\mathbf{x}, S_o, \lambda) = f(\mathbf{x}) - \lambda\, [h(\mathbf{x}) - S_o^2].$$

The following set of Kuhn-Tucker conditions is obtained :

| | |
|---|---|
| $f_j - \lambda h_j = 0$ | $(j = 1, 2, \ldots, n)$ |
| $\lambda h = 0$ | *Minimize* $f$ |
| $h \geq 0$ | *subject to* : |
| $\lambda \geq 0$ | $h \geq 0$. |

***Theorem 27-1*** **(Sufficiency of Kuhn-Tucker Conditions).** *The Kuhn-Tucker conditions for a maximization NLPP of Maximizing $f(\mathbf{x})$ subject to the constraints $h(\mathbf{x}) \leq 0$ and $\mathbf{x} \geq \mathbf{0}$, are sufficient conditions for a maximum of $f(\mathbf{x})$, if $f(\mathbf{x})$ is concave and $h(\mathbf{x})$ is convex.*

**Proof.** The result follows, if we are able to show that the Lagrangian function

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - [h(\mathbf{x}) + S^2],$$

where $S$ is defined by $h(\mathbf{x}) + S^2 = 0$, is concave in $\mathbf{x}$ under the given conditions.

In that case the stationary point obtained from the Kuhn-Tucker conditions must be the global maximum point.

Now, since $h(\mathbf{x}) + S^2 = 0$, it follows from the necessary conditions that $\lambda S^2 = 0$. Since $h(\mathbf{x})$ is convex and $\lambda \geq 0$, it follows that $\lambda h(\mathbf{x})$ is also convex and $-\lambda h(\mathbf{x})$ is concave. Thus, we conclude that $f(\mathbf{x}) - \lambda h(\mathbf{x})$ and hence $f(\mathbf{x}) - \lambda\,[h(\mathbf{x}) + S^2] = L(\mathbf{x}, S, \lambda)$ is concave in $\mathbf{x}$.

***Remark.*** By a similar argument it can be shown that for the minimization *NLPP*, Kuhn-Tucker conditions are also the sufficient conditions for the minimum of the objective function, if the objective function $f(\mathbf{x})$ is convex and the function $h(\mathbf{x})$ is concave. 20

### *SAMPLE PROBLEM*

**2721.** *Maximize* $z = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$ *subject to the constraints :*

$$2x_1 + x_2 \leq 10 \quad \textit{and} \quad x_1, x_2 \geq 0.$$

**Solution.** Here

$$f(x) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$
$$g(x) = 2x_1 + x_2, \quad c = 10$$
$$h(x) = g(x) - c = 2x_1 + x_2 - 10.$$

* More precisely since $\lambda = \dfrac{f_j}{h_j}\left(= \dfrac{\partial f / \partial x_j}{\partial h / \partial x_j} = \dfrac{\partial f}{\partial h}\right)$ measures the rate of variation of $f$ w.r.t. $h$, then as the right-hand side of $h(\mathbf{x}) \leq 0$ increases about zero, the solution space becomes less constrained and hence $f(\mathbf{x})$ cannot decrease. This means that $\lambda \geq 0$.

The Kuhn-Tucker conditions are :

$$\frac{\partial f(\mathbf{x})}{\partial x_1} - \lambda \frac{\partial h(\mathbf{x})}{\partial x_1} = 0, \quad \frac{\partial f(\mathbf{x})}{\partial x_2} - \lambda \frac{\partial h(\mathbf{x})}{\partial x_2} = 0 \quad \lambda h(\mathbf{x}) = 0, \quad h(\mathbf{x}) \leq 0, \quad \lambda \geq 0,$$

where $\lambda$ is the Lagrangian multiplier.

That is,

$$3.6 - 0.8x_1 = 2\lambda \qquad ...(1)$$

$$1.6 - 0.4x_2 = \lambda \qquad ...(2)$$

$$\lambda [2x_1 + x_2 - 10] = 0 \qquad ...(3)$$

$$2x_1 + x_2 - 10 = 0 \qquad ...(4)$$

$$\lambda \geq 0 \qquad ...(5)$$

From equation (3) either $\lambda = 0$ or $2x_1 + x_2 - 10 = 0$.

Let $\lambda = 0$, then (2) and (1) yield $x_1 = 4.5$ and $x_2 = 4$. With these values of $x_1$ and $x_2$ however, (4) cannot be satisfied. Thus, optimal solution cannot be obtained here for $\lambda = 0$. Let then $\lambda \neq 0$, which implies [from (3)] that $2x_1 + x_2 - 10 = 0$. This together with (1) and (2) yields the stationary value

$$\mathbf{x}_o = (x_1, x_2) = (3.5, 3)$$

Now, it is easy to observe that $h(\mathbf{x})$ is convex in $\mathbf{x}$, and $f(\mathbf{x})$ is concave in $\mathbf{x}$. Thus, Kuhn-Tucker conditions are the sufficient conditions for the maximum. Hence $\mathbf{x}_o = (3.5, 3)$ is the solution to the given *NLPP*. The maximum value of $z$ (corresponding to $\mathbf{x}_o$) is given by

$$z_o = 10.7.$$

### Kuhn-Tucker Conditions for General *NLPP* with $m\,(< n)$ Constraints

Introducing $\mathbf{S} = (S_1, S_2, \ldots, S_m)$, let the Lagrangian function for a general *NLPP* with $m\,(< n)$ constraints be

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i [h^i(\mathbf{x}) + S_i^2]$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$ are the Lagrangian multipliers.

The necessary conditions for $f(\mathbf{x})$ to be a maximum are :

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} = 0 \qquad \text{for } j = 1, 2, \ldots, n \qquad ...(1)$$

$$\frac{\partial L}{\partial \lambda_i} = h^i + S_i^2 = 0 \qquad \text{for } i = 1, 2, \ldots, m \qquad ...(2)$$

$$\frac{\partial L}{\partial S_i} = -2S_i \lambda_i = 0 \qquad \text{for } i = 1, 2, \ldots, m \qquad ...(3)$$

where $L = L(\mathbf{x}, \mathbf{S}, \lambda)$, $f = f(\mathbf{x})$ and $h^i = h^i(\mathbf{x})$.

Equation (3) states that either $\lambda_i = 0$ or $S_i = 0$. By an argument parallel to that considered in the case of single inequality constraint; the conditions (3) and (2) together are replaced by the conditions (5), (6) and (7) below :

$$\lambda_i h^i = 0 \qquad \text{for } i = 1, 2, \ldots, m \qquad ...(5)$$

$$h^i \leq 0 \qquad \text{for } i = 1, 2, \ldots, m \qquad ...(6)$$

$$h^i \geq 0 \qquad \text{for } i = 1, 2, \ldots, m \qquad ...(7)$$

The Kuhn-Tucker conditions for a maximum may thus be restated as

| | | |
|---|---|---|
| $f_j = \sum_{i=1}^{m} \lambda_i h^i_j$ | $(j = 1, 2, \ldots, n)$ | |
| $\lambda_i h^i = 0$ | $(i = 1, 2, \ldots, m)$ | *Maximize f* |
| $h^i \le 0$ | $(i = 1, 2, \ldots, m)$ | *subject to :* |
| $\lambda_i \ge 0$ | | $h^i \le 0$ |
| where $h^i_j = \dfrac{\partial h^i}{\partial x_j}$ | $(i = 1, 2, \ldots, m)$. | |

***Theorem 27-2*** **(Sufficiency of Kuhn-Tucker Conditions).** *For the NLPP of maximizing $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$, subject to the inequality constraints $h^i(\mathbf{x}) \le 0$ $(i = 1, 2, \ldots, m)$, the Kuhn-Tucker conditions are also the sufficient conditions for a maximum, if $f(\mathbf{x})$ is concave and all $h^i(\mathbf{x})$ are convex functions of $\mathbf{x}$.*

**Proof.** Exercise for the reader.

The Kuhn-Tucker conditions for a minimization non-linear programming problem may be obtained in a similar manner. These conditions in that case come out to be :

| | | |
|---|---|---|
| $f_j = \sum_{i=0}^{m} \lambda_i h_j^i$ | | $(j = 1, 2, \ldots, n)$ |
| $\lambda_i h^i = 0$ | *Minimize f* | |
| $h^i \ge 0$ | *subject to :* | |
| $\lambda_i \ge 0$ | $h^i \ge 0$ | $(i = 1, 2, \ldots, m)$. |

It can be shown that for this minimization problem, Kuhn-Tucker conditions are also sufficient conditions for the minima if $f(\mathbf{x})$ is convex and all $h^i(\mathbf{x})$ are concave in $\mathbf{x}$, that is, $-h^i(\mathbf{x})$ are also all convex.

***Note.*** If $f(\mathbf{x})$ is strictly concave (convex), the Kuhn-Tucker conditions are sufficient conditions for an absolute maximum (minimum).

***Remarks 1.*** We may consider $\mathbf{x} \ge \mathbf{0}$ or $-\mathbf{x} \le \mathbf{0}$, to have been included in the inequality constraint $h^i(\mathbf{x}) \le 0$.

*2.* In both the maximization and minimization *NLPP*, the Lagrange multipliers $\lambda_i$ corresponding to the equality constraints $h^i(\mathbf{x}) = 0$ must be unrestricted in sign.

*3.* A general *NLPP* may contain the constraints of the '≥' or '=' or '≤' type. In the case of maximization *NLPP*, all constraints must be converted into those of '≤' type and in the case of minimization *NLPP*, into those of '≥' type by suitable multiplication by –1.

## SAMPLE PROBLEMS

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**2722.** *Determine $x_1$, $x_2$ and $x_3$ so as to*

*Maximize* $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$ *subject to the constraints :*

$$x_1 + x_2 \le 2, \quad 2x_1 + 3x_2 \le 12, \quad x_1, x_2 \ge 0.$$ *[IAS 1992]*

**Solution.** Here $f(\mathbf{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$ $\quad$ $^*\mathbf{x} \in \mathbf{R}^n$

$$h^1(\mathbf{x}) = x_1 + x_2 - 2, \quad h^2(\mathbf{x}) = 2x_1 + 3x_2 - 12.$$

---

* The objective function is concave if the principal minors of bordered Hessian matrix, alternate in sign, beginning with the negative sign. If the principal minors are positive, the objective function is convex. In the present case

$$\mathbf{H}^B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad n = 3, \; m = 2, \; |\mathbf{H}^B| < 0. \text{ Thus } f(\mathbf{x}) \text{ is concave.}$$

Clearly, $f(\mathbf{x})$ is concave* and $h^1(\mathbf{x})$, $h(\mathbf{x})$ are convex in $\mathbf{x}$. Thus the Kuhn-Tucker conditions will be the necessary and sufficient conditions for a maximum. These conditions are obtained by the partial differentiation of the Lagrangian function

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \lambda_1 [h^1(\mathbf{x}) + S_1^2] - \lambda_2 [h^2(\mathbf{x}) + S_2^2]$$

where $\mathbf{S} = (S_1, S_2)$, $\lambda = (\lambda_1, \lambda_2)$, $S_1$, $S_2$ being slack variables and $\lambda_1$, $\lambda_2$ the Lagrange multipliers.

The Kuhn-Tucker conditions are given by

(1) $$f_j = \sum_{i=1}^{m} \lambda_i h'_j \qquad (j = 1, 2, 3)$$

(2) $$\lambda_i h^i = 0 \qquad (i = 1, 2)$$

(3) $$h^i \le 0 \qquad (i = 1, 2)$$

(4) $$\lambda_i \ge 0 \qquad (i = 1, 2)$$

Thus, in this problem, these are

(1) (*i*) $-2x_1 + 4 = \lambda_1 + 2\lambda_2$ (*ii*) $-2x_1 + 6 = \lambda_1 + 3\lambda_2$ (*iii*) $-2x_3 = 0$.

(2) (*i*) $\lambda_1 (x_1 + x_2 - 2) = 0$ (*ii*) $\lambda_2 (2x_1 + 3x_2 - 12) = 0$

(3) (*i*) $x_1 + x_2 - 2 \le 0$ (*ii*) $2x_1 + 3x_2 - 12 \le 0$

(4) $\lambda_1 \ge 0$, $\lambda_2 \ge 0$.

Now, there arise four cases :

**Case 1.** $\lambda_1 = 0$ and $\lambda_2 = 0$. (*i*), (*ii*) and (*iii*) yield $x_1 = 2$, $x_2 = 3$, $x_3 = 0$.

However, this solution violates (3) [(*i*) and (*ii*) both], and it must therefore be discarded.

**Case 2.** $\lambda_1 = 0$ *and* $\lambda_2 \ne 0$. (2) yield $2x_1 + 3x_2 = 12$ and (1) (*i*) and (*ii*) yield $-2x_1 + 4 = 2\lambda_2$, $-2x_2 + 6 = 3\lambda_2$. The solution of these simultaneous equations yields $x_1 = 2/13$, $x_2 = 3/13$, $\lambda_2 = 24/13 > 0$; also (1) (*iii*) gives $x_3 = 0$. However, this solution violates (3) (*i*). This solution is also thus discarded.

**Case 3.** $\lambda_1 \ne 0$ *and* $\lambda_2 \ne 0$. (2) (*i*) and (*ii*) yield $x_1 + x_2 = 2$ and $2x_1 + 3x_2 = 12$. These together yield $x_1 = -6$ and $x_2 = 8$. Thus (1) (*i*), (*ii*) and (*iii*) give $x_3 = 0$, $\lambda_1 = 68$, $\lambda_2 = -26$. However, this solution is to be discarded since $\lambda_2 = -26$ violates (4).

**Case 4.** $\lambda_1 \ne 0$ *and* $\lambda_2 = 0$. (2) (*i*) yield $x_1 + x_2 = 0$. This together with (1) (*i*) and (*ii*) gives $x_1 = 1/2$ and $x_2 = 3/2$, $\lambda_1 = 3 > 0$. Further from (1) (*iii*) $x_3 = 0$. We observe that this solution does not violate any of the Kuhn-Tucker conditions.

Hence, the optimum (maximum) solution to the given *NLPP* is

$$x_1 = 1/2,\ x_2 = 3/2,\ x_3 = 0 \text{ with } \lambda_1 = 3,\ \lambda_2 = 0,$$

the maximum value of the objective function being $z_0 = 17/2$.

**2723.** *Optimize* $z = 2x_1 + 3x_2 - (x_1^2 + x_2^2 + x_3^2)$ *subject to the constraints :*

$$x_1 + x_2 \le 1,\ 2x_1 + 3x_2 \le 6 \text{ and } x_1 \ge 0,\ x_2 \ge 0.$$

**Solution.** Here we have

$$f(x) = 2x_1 + 3x_2 - x_1^2 - x_2^2 - x_3^2, \quad h^1(x) = x_1 + x_2 - 1, \quad \text{and} \quad h^2(x) = 2x_1 + 3x_2 - 6.$$

Before applying the Kuhn-Tucker conditions, it is essential to determine whether the problem is of maximisation or of minimisation type. For that, we construct the bordered Hessian matrix :

$$\mathbf{H}^B = \begin{pmatrix} \mathbf{O} & \vdots & \mathbf{P} \\ \cdots & \vdots & \cdots \\ \mathbf{P}^T & \vdots & \mathbf{Q} \end{pmatrix}_{(m+n)\times(m+n)}$$

or
$$\mathbf{H}^B = \begin{vmatrix} 0 & 0 & \vdots & 1 & 1 & 0 \\ 0 & 0 & \vdots & 2 & 3 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \vdots & -2 & 0 & 0 \\ 1 & 3 & \vdots & 0 & -2 & 0 \\ 0 & 0 & \vdots & 0 & 0 & -2 \end{vmatrix} = -10. \qquad \begin{aligned} n &= 3, \\ m &= 2, \\ n-m &= 1, \\ 2m+1 &= 5. \end{aligned}$$

For maximisation, the sign of $\mathbf{H}^B$ should be $(-1)^{m+n}$, *i.e.*, $-ve$, while for minimisation it should be $(-1)^m$, *i.e.*, $+ve$. Since $\mathbf{H}^B < 0$, the solution point should maximise the objective function.

The Kuhn-Tucker conditions for this case are

$$2 - 2x_1 - \lambda_1 - 2\lambda_2 = 0, \quad \ldots(i) \qquad 3 - 2x_2 - \lambda_1 - 3\lambda_2 = 0, \quad \ldots(ii) \qquad -2x_3 = 0, \quad \ldots(iii)$$

$$\left.\begin{aligned} \lambda_1 (2x_1 + 3x_2 - 6) &= 0 \\ \lambda_2 (x_1 + x_2 - 1) &= 0 \end{aligned}\right\} \quad \ldots(iv) \qquad x_1 + x_2 - 1 \le 0, \quad \ldots(v)$$

$$2x_1 + 3x_2 - 6 \le 0, \quad \ldots(vi) \qquad \lambda_1, \lambda_2 \ge 0. \quad \ldots(vii)$$

Four solutions corresponding to the following values of $\lambda_1$ and $\lambda_2$ can be obtained:

**Case 1.** $\lambda_1 = 0,\ \lambda_2 = 0.$

Equations $(i)$, $(ii)$ and $(iii)$ give $x_1 = 1,\ x_2 = 3/2,\ x_3 = 0$. This solution does not satisfy equation $(vi)$ and is, therefore, discarded.

**Case 2.** $\lambda_1 = 0,\ \lambda_2 \neq 0.$

The solution of equations $(i)$, $(ii)$, $(iii)$ and $(v)$ gives

$$x_1 = 12/13,\ x_2 = 18/13,\ x_3 = 0 \quad \text{and} \quad \lambda_1 = 1/13.$$

This solution again does not satisfy equation $(vi)$ and is therefore discarded.

**Case 3.** $\lambda_1 \neq 0,\ \lambda_2 = 0.$

The solution of equations $(i)$, $(ii)$, $(iii)$ and $(iv)$ yields

$$x_1 = 1/4,\ x_2 = 3/4,\ x_3 = 0 \quad \text{and} \quad \lambda_1 = 3/2$$

This solution satisfies all the conditions, and gives $z = 17/8$.

**Case 4.** $\lambda_1 \neq 0,\ \lambda_2 \neq 0.$

The solution of equations $(i)$, $(ii)$, $(iii)$, $(iv)$ and $(v)$ gives

$$x_1 = -3,\ x_2 = 4,\ x_3 = 0,\ \lambda_1 = -34,\ \lambda_2 = 13.$$

This solution violates condition $(vii)$ and is, thus, unfeasible and is therefore discarded.

Since only one solution satisfies all the conditions, the same is the optimal solution.

Hence, $x_1^o = 1/4,\ x_2^o = 3/4,\ x_3^o = 0$, and maximum $z = 17/8$.

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## PROBLEMS

*Use the Kuhn-Tucker conditions to solve the following non-linear programming problems :*

**2724.** Minimize $z = 2x_1^2 + 12x_1x_2 - 7x_2^2$ subject to the constraints :

$$2x_1 + 5x_2 \le 98,\ x_1, x_2 \ge 0.$$

**2725.** Maximize $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$ subject to the constraints :

$$3x_1 + 2x_2 \le 6,\ x_1 \ge 0,\ x_2 \ge 0$$

*[IAS 1991]*

**2726.** Minimize $z = x_1^2 + x_2^2 + x_3^2$ subject to the constraints :

$$2x_1 + x_2 \le 5,\ x_1 + x_2 \le 2,\ x_1 \ge 1,\ x_2 \ge 2,\ x_3 \ge 0.$$

*[IAS 1993]*

**2727.** Minimize $z = 0.3x_1^2 - 2x_1 + 0.4x_2^2 - 2.4x_2 + 0.6x_1x_2 + 100$ subject to the constraints :

$$2x_1 + x_2 \ge 4,\ x_1, x_2 \ge 0.$$

**2728.** Minimize $z = -\log x_1 - \log x_2$ subject to the constraints :

$$x_1 + x_2 \le 2 \quad \text{and} \quad x_1 \ge 0,\ x_2 \ge 0.$$

**2729.** Minimize $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$ subject to the conditions :

$$x_1 + 3x_2 \le 6,\ 5x_1 + 2x_2 \le 10, \quad \text{and} \quad x_1 \ge 0,\ x_2 \ge 0.$$ *[Madurai B.E. (Electronics) 1990]*

**2730.** Maximize $z = 2x_1 - x_1^2 + x_2$ subject to the constraints : *[Dibrugarh M.Sc. (Stat.) 1994]*

$$2x_1 + 3x_2 \le 6,\ 2x_1 + x_2 \le 4 \quad \text{and} \quad x_1, x_2 \ge 0.$$

**2731.** Maximize $z = 7x_1^2 + 6x_1 + 5x_2$ subject to the constraints : *[Annamalai M.E. (Nov.) 2002]*

$$x_1 + 2x_2 \le 10,\ x_1 - 3x_2 \le 9;\ x_1 \ge 0 \text{ and } x_2 \ge 0.$$

**2732.** Maximize $z = 3x_1 + x_2$ subject to the constraints : *[Madras B.E. (Civil) 1991]*

$$x_1^2 + x_2^2 \le 5,\ x_1 - x_2 \le 1 \quad \text{and} \quad x_1 \ge 0,\ x_2 \ge 0.$$

**2733.** Maximize $z_1 = 8x_1^2 + 2x_2^2$ subject to the constraints :

$$x_1^2 + x_2^2 \le 9,\ x_1 \le 2 \quad \text{and} \quad x_1, x_2 \ge 0.$$

**2734.** Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 5)^2$ subject to the constraints : *[Madurai B.E. (Electronics) 1989]*

$$-x_1^2 + x_2 \le 4 \quad \text{and} \quad -(x_1 - 2)^2 + x_2 \le 3.$$

**2735.** A manufacturing firm produces a product $A$. The firm has the contract to supply 60 units at the end of the first, second and third months. The cost of producing $x$ units of $A$ in any month is given by $x^2$. The firm can produce more units of $A$ in any month and carry them to a subsequent month. However, carrying cost of Rs. 25 per unit is charged for carrying units of $A$ from one month to the next. Assuming that there is no initial inventory, determine the number of units of $A$ to be produced in each month so as to minimize the total cost.

[**Hint :** Minimum (total cost) $z$ = Production cost + Carrying cost

$$= x_1^2 + x_2^2 + x_3^2 + 40x_1 + 25(x_1 - 60) + 25(x_1 + x_2 - 120)$$

subject to the constraints :

$$x_1 + x_2 \ge 120,\ x_1 + x_2 + x_3 \ge 180,\ x_1 \ge 60;\ x_1, x_2, x_3 \ge 0$$

where $x_1$, $x_2$ and $x_3$ = number of units of product $A$ produced in first, second and third months respectively.]

## 27:6. SADDLE POINT PROBLEMS

In Chapter 17, the *saddle point* of a payoff matrix was defined. Let $(a_{ij})$ be the payoff matrix for a two-person zero-sum game. If $\mathbf{a}_{i*j}$ denote the payoff minima at $i^*$ over the rows and $a_{i*j}$ denote the payoff minima at $j^*$ over the columns, then by Theorem 17-1 the saddle point $a_{i*j*}$ is given by

$$a_{i*j} \ge a_{i*j*} \ge a_{ij*}.$$

We now define the saddle points of functions. Let $\phi$ be a real valued function of several variables. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots, u_m)$, then for these variables we shall denote the function $\phi$ by $\phi(\mathbf{x}, \mathbf{u})\ \mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$.

***Definition 1* (Saddle point).** *Let $\phi(\mathbf{x}, \mathbf{u})$ be a function of $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{u} \in \mathbf{R}^m$. The function $\phi(\mathbf{x}, \mathbf{u})$ is said to have a* ***saddle point*** *at $(\mathbf{x}^o, \mathbf{u}^o)$ if and only if*

$$\phi(\mathbf{x}^o, \mathbf{u}) \ge \phi(\mathbf{x}^o, \mathbf{u}^o) \ge \phi(\mathbf{x}, \mathbf{u}^o).$$

***Definition 2* (Saddle value problem).** *Let $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$. The problem of determining saddle point value $\phi(\mathbf{x}^o, \mathbf{u}^o)$ under the constraints $\mathbf{x} \ge \mathbf{0}$, and $\mathbf{u} \ge \mathbf{0}$, is called a* ***saddle value problem.***

We introduce the following notations :

$$\phi^o = \phi(\mathbf{x}^o, \mathbf{u}^o)$$

Assume that $\phi(\mathbf{x}, \mathbf{u})$ is differentiable partially w.r.t. $\mathbf{x}$ and $\mathbf{u}$.