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CHAPTER 1

Fundamental Concepts

1.1 INTRODUCTION

Many practical problems in science and engineering, when formulated mathematically, give rise to partial differential equations (often referred to as PDE). In order to understand the physical behaviour of the mathematical model, it is necessary to have some knowledge about the mathematical character, properties, and the solution of the governing PDE. An equation which involves several independent variables (usually denoted by x, y, z, t, ...), a dependent function u of these variables, and the partial derivatives of the dependent function u with respect to the independent variables such as

$$F(x, y, z, t, \ldots, u_x, u_y, u_z, u_t, \ldots, u_{xx}, u_{yy}, \ldots, u_{xy}, \ldots) = 0 \tag{1.1}$$

is called a partial differential equation. A few well-known examples are:

(i) $u_t = k(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional heat equation]

(ii) $u_{xx} + u_{yy} + u_{zz} = 0$ [Laplace equation in three dimensions]

(iii) $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional wave equation]

(iv) $u_t + uu_x = \mu u_{xx}$ [nonlinear one-dimensional Burger equation].

In all these examples, u is the dependent function and the subscripts denote partial differentiation with respect to these variables.

Definition 1.1 The order of the partial differential equation is the order of the highest derivative occurring in the equation. Thus the above examples are partial differential equations of second order, whereas

$$u_t = uu_{xxx} + \sin x$$

is an example for third order partial differential equation.

1.2 CLASSIFICATION OF SECOND ORDER PDE

The most general linear second order PDE, with one dependent function u on a domain Ω of points $X = (x_1, x_2, \ldots, x_n)$, $n > 1$, is

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + F(u) = G \tag{1.2}$$

The classification of a PDE depends only on the highest order derivatives present.

The classification of PDE is motivated by the classification of the quadratic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1.3}$$

which is elliptic, parabolic, or hyperbolic according as the discriminant $B^2 - 4AC$ is negative, zero or positive. Thus, we have the following second order linear PDE in two variables x and y:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.4}$$

where the coefficients A, B, C, ... may be functions of x and y, however, for the sake of simplicity we assume them to be constants. Equation (1.4) is elliptic, parabolic or hyperbolic at a point (x_0, y_0) according as the discriminant

$$B^2(x_0, y_0) - 4A(x_0, y_0)\, C(x_0, y_0)$$

is negative, zero or positive. If this is true at all points in a domain Ω, then Eq. (1.4) is said to be elliptic, parabolic or hyperbolic in that domain. If the number of independent variables is two or three, a transformation can always be found to reduce the given PDE to a canonical form (also called normal form). In general, when the number of independent variables is greater than 3, it is not always possible to find such a transformation except in certain special cases. The idea of reducing the given PDE to a canonical form is that the transformed equation assumes a simple form so that the subsequent analysis of solving the equation is made easy.

1.3 CANONICAL FORMS

Consider the most general transformation of the independent variables x and y of Eq. (1.4) to new variables ξ, η, where

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{1.5}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = (\xi_x \eta_y - \xi_y \eta_x) \neq 0 \tag{1.6}$$

in the domain Ω where Eq. (1.4) holds. Using the chain rule of partial differentiation, partial derivatives become

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

Substituting these expressions into the original differential equation (1.4), we get

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G} \quad (1.8)$$

where

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$\bar{F} = F, \qquad \bar{G} = G \quad (1.9)$$

It may be noted that the transformed equation (1.8) has the same form as that of the original equation (1.4) under the general transformation (1.5).

Since the classification of Eq. (1.4) depends on the coefficients A, B and C, we can also rewrite the equation in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \quad (1.10)$$

It can be shown easily that under the transformation (1.5), Eq. (1.10) takes one of the following three canonical forms:

(i) $u_{\xi\xi} - u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ (1.11a)

or

$u_{\xi\eta} = \phi, (\xi, \eta, u, u_\xi, u_\eta)$ in the hyperbolic case (1.11b)

(ii) $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the elliptic case (1.11c)

(iii) $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

or

$$u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \text{ in the parabolic case}$$

We shall discuss in detail each of these cases separately.

Using Eq. (1.9) it can also be verified that

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x \eta_y - \xi_y \eta_x)^2 (B^2 - 4AC)$$

and therefore we conclude that the transformation of the independent variables does not modify the type of PDE.

1.3.1 Canonical Form for Hyperbolic Equation

Since the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} > 0$ for hyperbolic case, we set $\bar{A} = 0$ and $\bar{C} = 0$ in Eq. (1.9), which will give us the coordinates ξ and η that reduce the given PDE to a canonical form in which the coefficients of $u_{\xi\xi}, u_{\eta\eta}$ are zero. Thus we have

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

which, on rewriting, become

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

Solving these equations for (ξ_x/ξ_y) and (η_x/η_y), we get

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{1.12}$$

The condition $B^2 > 4AC$ implies that the slopes of the curves $\xi(x, y) = C_1, \eta(x, y) = C_2$ are real. Thus, if $B^2 > 4AC$, then at any point (x, y), there exists two real directions given by the two roots (1.12) along which the PDE (1.4) reduces to the canonical form. These are called *characteristic equations*. Though there are two solutions for each quadratic, we have considered only one solution for each. Otherwise we will end up with the same two coordinates.

Along the curve $\xi(x, y) = c_1$, we have

$$d\xi = \xi_x\, dx + \xi_y\, dy = 0$$

Hence,

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) \tag{1.13}$$

Similarly, along the curve $\eta(x, y) = c_2$, we have

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) \tag{1.14}$$

Integrating Eqs. (1.13) and (1.14), we obtain the equations of family of characteristics $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$, which are called the characteristics of the PDE (1.4). Now to obtain the canonical form for the given PDE, we substitute the expressions of ξ and η into Eq. (1.8) which reduces to Eq. (1.11a).

To make the ideas clearer, let us consider the following example:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

Comparing with the standard PDE (1.4), we have $A = 3$, $B = 10$, $C = 3$, $B^2 - 4AC = 64 > 0$. Hence the given equation is a hyperbolic PDE. The corresponding characteristics are:

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = \frac{1}{3}$$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = 3$$

To find ξ and η, we first solve for y by integrating the above equations. Thus, we get

$$y = 3x + c_1, \qquad y = \frac{1}{3}x + c_2$$

which give the constants as

$$c_1 = y - 3x, \qquad c_2 = y - x/3$$

Therefore,

$$\xi = y - 3x = c_1, \qquad \eta = y - \frac{1}{3}x = c_2$$

These are the characteristic lines for the given hyperbolic equation. In this example, the characteristics are found to be straight lines in the (x, y)-plane along which the initial data, impulses will propagate.

To find the canonical equation, we substitute the expressions for ξ and η into Eq. (1.9) to get

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 3(-3)^2 + 10(-3)(1) + 3 = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(3)(-3)\left(-\frac{1}{3}\right) + 10\left[(-3)(1) + 1\left(-\frac{1}{3}\right)\right] + 2(3)(1)(1)$$

$$= 6 + 10\left(-\frac{10}{3}\right) + 6 = 12 - \frac{100}{3} = -\frac{64}{3}$$

$$\bar{C} = 0, \quad \bar{D} = 0, \quad \bar{E} = 0, \quad \bar{F} = 0$$

Hence, the required canonical form is

$$\frac{64}{3}u_{\xi\eta} = 0 \quad \text{or} \quad u_{\xi\eta} = 0$$

On integration, we obtain

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

where f and g are arbitrary. Going back to the original variables, the general solution is

$$u(x, y) = f(y - 3x) + g(y - x/3)$$

1.3.2 Canonical Form for Parabolic Equation

For the parabolic equation, the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} = 0$, which can be true if $\bar{B} = 0$ and $\bar{A}$ or $\bar{C}$ is equal to zero. Suppose we set first $\bar{A} = 0$ in Eq. (1.9). Then we obtain

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

or

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

which gives

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Using the condition for parabolic case, we get

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} \tag{1.15}$$

Hence, to find the function $\xi = \xi(x, y)$ which satisfies Eq. (1.15), we set

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A}$$

and get the implicit solution

$$\xi(x, y) = C_1$$

In fact, one can verify that $\bar{A} = 0$ implies $\bar{B} = 0$ as follows:

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

Since $B^2 - 4AC = 0$, the above relation reduces to

$$\bar{B} = 2A\xi_x\eta_x + 2\sqrt{AC}\,(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y)$$

However,

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} = -\frac{2\sqrt{AC}}{2A} = -\sqrt{\frac{C}{A}}$$

Hence,

$$\bar{B} = 2(\sqrt{A}\xi_x - \sqrt{A}\xi_x)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$$

We therefore choose ξ in such a way that both $\bar{A}$ and $\bar{B}$ are zero. Then η can be chosen in any way we like as long as it is not parallel to the ξ-coordinate. In other words, we choose η such that the Jacobian of the transformation is not zero. Thus we can write the canonical equation for parabolic case by simply substituting ξ and η into Eq. (1.8) which reduces to either of the forms (1.11c).

To illustrate the procedure, we consider the following example:

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$$

The discriminant $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$, and hence the given PDE is parabolic everywhere. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = -\frac{2xy}{2x^2} = -\frac{y}{x}$$

On integration, we have

$$xy = c$$

and hence $\xi = xy$ will satisfy the characteristic equation and we can choose $\eta = y$. To find the canonical equation, we substitute the expressions for ξ and η into Eq. (1.9) to get

$$\bar{A} = Ay^2 + Bxy + cx^2 = x^2y^2 - 2x^2y^2 + y^2x^2 = 0$$

$$\bar{B} = 0, \qquad \bar{C} = y^2, \qquad \bar{D} = -2xy$$

$$\bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = e^x$$

Hence, the transformed equation is

$$y^2 u_{\eta\eta} - 2xyu_\xi = e^x$$

or

$$\eta^2 u_{\eta\eta} = 2\xi u_\xi + e^{\xi/\eta}$$

The canonical form is, therefore,

$$u_{\eta\eta} = \frac{2\xi}{\eta^2} u_\xi + \frac{1}{\eta^2} e^{\xi/\eta}$$

1.3.3 Canonical Form for Elliptic Equation

Since the discriminant $B^2 - 4AC < 0$, for elliptic case, the characteristic equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

give us complex conjugate coordinates, say ξ and η. Now, we make another transformation from (ξ, η) to (α, β) so that

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

which give us the required canonical equation in the form (1.11b).

To illustrate the procedure, we consider the following example:

$$u_{xx} + x^2 u_{yy} = 0$$

The discriminant $B^2 - 4AC = -4x^2 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4x^2}}{2} = -ix$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = ix$$

Integration of these equations yields

$$iy + \frac{x^2}{2} = c_1, \qquad -iy + \frac{x^2}{2} = c_2$$

Hence, we may assume that

$$\xi = \frac{1}{2}x^2 + iy, \qquad \eta = \frac{1}{2}x^2 - iy$$

Now, introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

we obtain

$$\alpha = \frac{x^2}{2}, \qquad \beta = y$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + \beta\alpha_x\alpha_y + c\alpha_y^2 = x^2$$

$$\bar{B} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2c(\alpha_y\beta_y) = 0$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + c\beta_y^2 = x^2$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + c\alpha_{yy} + D\alpha_x + E\alpha_y = 1$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + c\beta_{yy} + D\beta_x + E\beta_y = 0$$

$$\bar{F} = 0, \qquad \bar{G} = 0$$

Thus the required canonical equation is

$$x^2 u_{\alpha\alpha} + x^2 u_{\beta\beta} + u_\alpha = 0$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}$$

EXAMPLE 1.1 Classify and reduce the relation

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

to a canonical form and solve it.

Solution The discriminant of the given PDE is

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

Hence the given equation is of a parabolic type. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = \frac{-2xy}{2y^2} = -\frac{x}{y}$$

Integration gives $x^2 + y^2 = c_1$. Therefore, $\xi = x^2 + y^2$ satisfies the characteristic equation. The η-coordinate can be chosen arbitrarily so that it is not parallel to ξ, i.e. the Jacobian of the transformation is not zero. Thus we choose

$$\xi = x^2 + y^2, \qquad \eta = y^2$$

To find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 4x^2y^2 - 8x^2y^2 + 4x^2y^2 = 0$$

$$\bar{B} = 0, \qquad \bar{C} = 4x^2y^2, \qquad \bar{D} = \bar{E} = \bar{F} = \bar{G} = 0$$

Hence, the required canonical equation is

$$4x^2y^2u_{\eta\eta} = 0 \quad \text{or} \quad u_{\eta\eta} = 0$$

To solve this equation, we integrate it twice with respect to η to get

$$u_\eta = f(\xi), \qquad u = f(\xi)\eta + g(\xi)$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ. Now, going back to the original independent variables, the required solution is

$$u = y^2 f(x^2 + y^2) + g(x^2 + y^2)$$

EXAMPLE 1.2 Reduce the following equation to a canonical form:

$$(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$$

Solution The discriminant of the given PDE is

$$B^2 - 4AC = -4(1 + x^2)(1 + y^2) < 0$$

Hence the given PDE is an elliptic type. The characteristic equations are

$$\frac{dy}{dx}=\frac{B-\sqrt{B^2-4AC}}{2A}=-\frac{\sqrt{-4(1+x^2)(1+y^2)}}{2(1+x^2)}=-i\sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{dy}{dx}=\frac{B+\sqrt{B^2-4AC}}{2A}=i\sqrt{\frac{1+y^2}{1+x^2}}$$

On integration, we get

$$\xi=\ln(x+\sqrt{x^2+1})-i\ln(y+\sqrt{y^2+1})=c_1$$

$$\eta=\ln(x+\sqrt{x^2+1})+i\ln(y+\sqrt{y^2+1})=c_2$$

Introducing the second transformation

$$\alpha=\frac{\xi+\eta}{2},\quad \beta=\frac{\eta-\xi}{2i}$$

we obtain

$$\alpha=\ln(x+\sqrt{x^2+1})$$

$$\beta=\ln(y+\sqrt{y^2+1})$$

Then the canonical form can be obtained by computing

$$\bar{A}=A\alpha_x^2+B\alpha_x\alpha_y+C\alpha_y^2=1,\quad \bar{B}=0,\quad \bar{C}=1,\quad \bar{D}=\bar{E}=\bar{F}=\bar{G}=0$$

Thus the canonical equation for the given PDE is

$$u_{\alpha\alpha}+u_{\beta\beta}=0$$

EXAMPLE 1.3 Reduce the following equation to a canonical form and hence solve it:

$$u_{xx}-2\sin x u_{xy}-\cos^2 x u_{yy}-\cos x u_y=0$$

Solution Comparing with the general second order PDE (1.4), we have

$$A=1,\quad B=-2\sin x,\quad C=-\cos^2 x,$$
$$D=0,\quad E=-\cos x,\quad F=0,\quad G=0$$

The discriminate $B^2-4AC=4(\sin^2 x+\cos^2 x)=4>0$. Hence the given PDE is hyperbolic. The relevant characteristic equations are

$$\frac{dy}{dx}=\frac{B-\sqrt{B^2-4AC}}{2A}=-\sin x-1$$

$$\frac{dy}{dx}=\frac{B+\sqrt{B^2-4AC}}{2A}$$

On integration, we get

$$y = \cos x - x + c_1, \qquad y = \cos x + x + c_2$$

Thus, we choose the characteristic lines as

$$\xi = x + y - \cos x = c_1, \qquad \eta = -x + y - \cos x = c_2$$

In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(\sin x + 1)(\sin x - 1) - 4\sin^2 x - 2\cos^2 x = -4$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = 0$$

Thus, the required canonical equation is

$$u_{\xi\eta} = 0$$

Integrating with respect to ξ, we obtain

$$u_\eta = f(\eta)$$

where f is arbitrary. Integrating once again with respect to η, we have

$$u = \int f(\eta)\, d\eta + g(\xi)$$

or

$$u = \psi(\eta) + g(\xi)$$

where $g(\xi)$ is another arbitrary function. Returning to the old variables x, y, the solution of the given PDE is

$$u(x, y) = \psi(y - x - \cos x) + g(y + x - \cos x)$$

EXAMPLE 1.4 Reduce the Tricomi equation

$$u_{xx} + xu_{yy} = 0, \qquad x \neq 0$$

for all x, y to canonical form.

Solution The discriminant $B^2 - 4AC = -4x$. Hence the given PDE is of mixed type: hyperbolic for $x < 0$ and elliptic for $x > 0$.

Case I In the half-plane $x < 0$, the characteristic equations are

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{-2\sqrt{-x}}{2} = -\sqrt{-x}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \sqrt{-x}$$

Integration yields

$$y = \frac{2}{3}(-x)^{3/2} + c_1$$

$$y = -\frac{2}{3}(-x)^{3/2} + c_2$$

Therefore, the new coordinates are

$$\xi(x, y) = \frac{3}{2}y - (\sqrt{-x})^3 = c_1$$

$$\eta(x, y) = \frac{3}{2}y + (\sqrt{-x})^3 = c_2$$

which are cubic parabolas.

In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = -\frac{9}{4}x + 0 + \frac{9}{4}x = 0$$

$$\bar{B} = 9x, \qquad \bar{C} = 0, \qquad \bar{D} = -\frac{3}{4}(-x)^{-1/2} = -\bar{E}, \qquad \bar{F} = \bar{G} = 0$$

Thus, the required canonical equation is

$$9xu_{\xi\eta} - \frac{3}{4}(-x)^{-1/2}u_\xi + \frac{3}{4}(-x)^{-1/2}u_\eta = 0$$

or

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta)$$

Case II In the half-plane $x > 0$, the characteristic equations are given by

$$\frac{dy}{dx} = i\sqrt{x}, \qquad \frac{dy}{dx} = -i\sqrt{x}$$

On integration, we have

$$\xi(x, y) = \frac{3}{2}y - i(\sqrt{x})^3, \qquad \eta(x, y) = \frac{3}{2}y + i(\sqrt{x})^3$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

we obtain

$$\alpha = \frac{3}{2}y, \qquad \beta = -(\sqrt{x})^3$$

The corresponding normal or canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\beta} u_\beta = 0$$

EXAMPLE 1.5 Find the characteristics of the equation

$$u_{xx} + 2u_{xy} + \sin^2(x) u_{yy} + u_y = 0$$

when it is of hyperbolic type.

Solution The discriminant $B^2 - 4AC = 4 - 4\sin^2 x = 4\cos^2 x$. Hence for all $x \neq (2n - 1)\pi/2$, the given PDE is of hyperbolic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = 1 \mp \cos x$$

On integration, we get

$$y = x - \sin x + c_1, \qquad y = x + \sin x + c_2$$

Thus, the characteristic equations are

$$\xi = y - x + \sin x, \qquad \eta = y - x - \sin x$$

EXAMPLE 1.6 Reduce the following equation to a canonical form and hence solve it:

$$y u_{xx} + (x + y) u_{xy} + x u_{yy} = 0$$

Solution The discriminant

$$B^2 - 4AC = (x + y)^2 - 4xy = (x - y)^2 > 0$$

Hence the given PDE is hyperbolic everywhere except along the line $y = x$; whereas on the line $y = x$, it is parabolic. When $y \neq x$, the characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{(x + y) \mp (x - y)}{2y}$$

Therefore,

$$\frac{dy}{dx} = 1, \qquad \frac{dy}{dx} = \frac{x}{y}$$

On integration, we obtain

$$y = x + c_1, \qquad y^2 = x^2 + c_2$$

Hence, the characteristic equations are

$$\xi = y - x, \qquad \eta = y^2 - x^2$$

These are straight lines and rectangular hyperbolas. The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = y - x - y + x = 0, \qquad \bar{B} = -2(x-y)^2,$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 2(x-y), \qquad \bar{F} = \bar{G} = 0$$

Thus, the canonical equation for the given PDE is

$$-2(x-y)^2 u_{\xi\eta} + 2(x-y)u_\eta = 0$$

or

$$-2\xi^2 u_{\xi\eta} + 2(-\xi)\,u_\eta = 0$$

or

$$\xi u_{\xi\eta} + u_\eta = \frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\eta}\right) = 0$$

Integration yields

$$\xi\frac{\partial u}{\partial\eta} = f(\eta)$$

Again integrating with respect to η, we obtain

$$u = \frac{1}{\xi}\int f(\eta)\,d\eta + g(\xi)$$

Hence,

$$u = \frac{1}{y-x}\int f(y^2 - x^2)\,d(y^2 - x^2) + g(y-x)$$

is the general solution.

EXAMPLE 1.7 Classify and transform the following equation to a canonical form:

$$\sin^2(x)u_{xx} + \sin(2x)u_{xy} + \cos^2(x)u_{yy} = x$$

Solution The discriminant of the given PDE is

$$B^2 - 4AC = \sin^2 2x - 4\sin^2 x\cos^2 x = 0$$

Hence, the given equation is of parabolic type. The characteristic equation is

$$\frac{dy}{dx} = \frac{B}{2A} = \cot x$$

Integration gives