CHAPTER 2

Elliptic Differential Equations

2.1 OCCURRENCE OF THE LAPLACE AND POISSON EQUATIONS

In Chapter 1, we have seen the classification of second order partial differential equation into elliptic, parabolic and hyperbolic types. In this chapter we shall consider various properties and techniques for solving Laplace and Poisson equations which are elliptic in nature.

Various physical phenomena are governed by the well known Laplace and Poisson equations. A few of them, frequently encountered in applications are: steady heat conduction, seepage through porous media, irrotational flow of an ideal fluid, distribution of electrical and magnetic potential, torsion of prismatic shaft, bending of prismatic beams, distribution of gravitational potential, etc. In the following two sub-sections, we shall give the derivation of Laplace and Poisson equations in relation to the most frequently occurring physical situation, namely, the gravitational potential.

2.1.1 Derivation of Laplace Equation

Consider two particles of masses m and m_1 situated at Q and P separated by a distance r as shown in Fig. 2.1. According to Newton's universal law of gravitation, the magnitude of the force, proportional to the product of their masses and inversely proportional to the square of the distance, between them is given by

$$F = G\frac{mm_1}{r^2} \tag{2.1}$$

where G is the gravitational constant. It $\mathbf{r}$ represents the vector $\vec{PQ}$, assuming unit mass at Q and $G = 1$, the force at Q due to the mass at P is given by

$$\mathbf{F} = -\frac{m_1\mathbf{r}}{r^3} = \nabla\left(\frac{m_1}{r}\right) \tag{2.2}$$

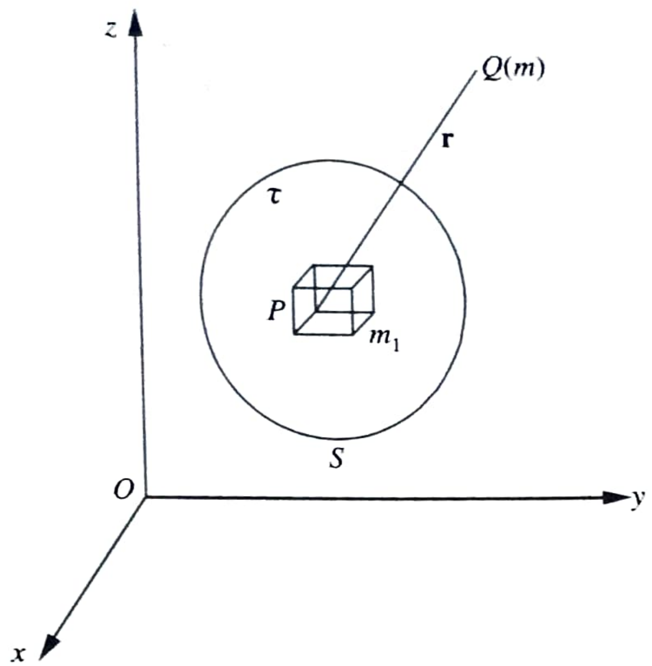

Fig. 2.1 Illustration of Newton's universal law of gravitation.

which is called the intensity of the gravitational force. Suppose a particle of unit mass moves under the attraction of a particle of mass m_1 at P from infinity up to Q; then the work done by the force $\mathbf{F}$ is

$$\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \int_{\infty}^{r} \nabla\left(\frac{m_1}{r}\right) \cdot d\mathbf{r} = \frac{m_1}{r} \tag{2.3}$$

This is defined as the potential V at Q due to a particle at P and is denoted by

$$V = -\frac{m_1}{r} \tag{2.4}$$

From Eq. (2.2), the intensity of the force at P is

$$\mathbf{F} = -\nabla V \tag{2.5}$$

Now, if we consider a system of particles of masses $m_1, m_2, \ldots, m_n$ which are at distances $r_1, r_2, \ldots, r_n$ respectively, then the force of attraction per unit mass at Q due to the system is

$$\mathbf{F} = \sum_{i=1}^{n} \nabla \frac{m_i}{r_i} = \nabla \sum_{i=1}^{n} \frac{m_i}{r_i} \tag{2.6}$$

The work done by the force acting on the particle is

$$\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} = -V \tag{2.7}$$

Therefore,

$$\nabla^2 V = -\nabla^2 \sum_{i=1}^{n} \frac{m_i}{r_i} = \sum_{i=1}^{n} \nabla^2 \frac{m_i}{r_i} = 0, \qquad r_i \neq 0 \tag{2.8}$$

where

$$\nabla^2 = \text{div}\,\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator.

In the case of continuous distribution of matter of density ρ in a volume τ, we have

$$V(x, y, z) = \iiint_{\tau} \frac{\rho(\xi, \eta, \zeta)}{r}\,d\tau \tag{2.9}$$

where $r = \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{1/2}$ and Q is outside the body. It can be verified that

$$\nabla^2 V = 0 \tag{2.10}$$

which is called the Laplace equation.

2.1.2 Derivation of Poisson Equation

Consider a closed surface S consisting of particles of masses $m_1, m_2, \ldots, m_n$. Let Q be any point on S. Let $\sum_{i=1}^{n} m_i = M$ be the total mass inside S, and let $g_1, g_2, \ldots, g_n$ be the gravity field at Q due to the presence of $m_1, m_2, \ldots, m_n$ respectively within S. Also, let $\sum_{i=1}^{n} g_i = g$, the entire gravity field at Q. Then, according to Gauss law, we have

$$\iint_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G M \tag{2.11}$$

where $M = \iiint_{\tau} \rho\,d\tau, \rho$ is the mass density function and τ is the volume in which the masses are distributed throughout. Since the gravity field is conservative, we have

$$\mathbf{g} = \nabla V \tag{2.12}$$

where V is a scalar potential. But the Gauss divergence theorem states that

$$\iint_S \mathbf{g}\cdot d\mathbf{S} = \iiint_{\tau} \nabla\cdot\mathbf{g}\,d\tau \tag{2.13}$$

Also, Eq. (2.11) gives

$$\iint_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G \iiint_{\tau} \rho\,d\tau \tag{2.14}$$

Combining Eqs. (2.13) and (2.14), we have

$$\iiint_{\tau} (\nabla \cdot \mathbf{g} + 4\pi G\rho)\, d\tau = 0$$

implying

$$\nabla \cdot \mathbf{g} = -4\pi G\rho = \nabla \cdot \nabla V$$

Therefore,

$$\nabla^2 V = -4\pi G\rho \tag{2.15}$$

This equation is known as *Poisson's equation.*

2.2 BOUNDARY VALUE PROBLEMS (BVPs)

The function V, whose analytical form we seek for the problems stated in Section 2.1, in addition to satisfying the Laplace and Poisson equations in a bounded region $\mathbb{R}$ in R^3, should also satisfy certain boundary conditions on the boundary $\partial\mathbb{R}$ of this region. Such problems are referred to as boundary value problems (BVPs) for Laplace and Poisson equations. We shall denote the set of all boundary points of $\mathbb{R}$ by $\partial\mathbb{R}$. By the closure of $\mathbb{R}$, we mean the set of all interior points of $\mathbb{R}$ together with its boundary points and is denoted by $\overline{\mathbb{R}}$. Symbolically, $\overline{\mathbb{R}}. = \mathbb{R} U \partial\mathbb{R}$.

If a function $f \in c^{(n)}$ (f "belongs to" $c^{(n)}$), then all its derivatives of order n are continuous. If it belongs to $c^{(0)}$, then we mean f is continuous.

There are mainly three types of boundary value problems for Laplace equation. If $f \in c^{(0)}$ and is specified on the boundary $\partial\mathbb{R}$ of some finite region $\mathbb{R}$, the problem of determining a function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within $\mathbb{R}$ and satisfying $\psi = f$ on $\partial\mathbb{R}$ is called the boundary value problem of first kind, or the Dirichlet problem. For example, finding the steady state temperature within the region $\mathbb{R}$ when no heat sources or sinks are present and when the temperature is prescribed on the boundary $\partial\mathbb{R}$, is a Dirichlet problem. Another example would be to find the potential inside the region $\mathbb{R}$ when the potential is specified on the boundary $\partial\mathbb{R}$. These two examples correspond to the interior Dirichlet problem.

Similarly, if $f \in c^{(0)}$ and is prescribed on the boundary $\partial\mathbb{R}$ of a finite simply connected region $\mathbb{R}$, determining a function $\psi(x, y, z)$ which satisfies $\nabla^2\psi = 0$ outside $\mathbb{R}$ and is such that $\psi = f$ on $\partial\mathbb{R}$, is called an exterior Dirichlet problem. For example, determination of the distribution of the potential outside a body whose surface potential is prescribed, is an exterior Dirichlet problem. The second type of BVP is associated with von Neumann. The problem is to determine the function $\psi(x, y, z)$ so that $\nabla^2\psi = 0$ within $\mathbb{R}$ while $\partial\psi/\partial n$ is specified at every point of $\partial\mathbb{R}$, where $\partial\psi/\partial n$ denotes the normal derivative of the field variable ψ. This problem is called the Neumann problem. If ψ is the temperature, $\partial\psi/\partial n$ is the heat flux representing the amount of heat crossing per unit volume per unit time along the normal direction, which is zero when insulated. The third type of BVP is concerned with the determination of the function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within $\mathbb{R}$, while a boundary condition of the form $\partial\psi/\partial n + h\psi = f$, where $h \geq 0$ is specified at every point of $\partial\mathbb{R}$. This is called a mixed BVP or Churchill's problem. If we assume Newton's law of cooling, the heat lost is $h\psi$, where ψ is the temperature difference from the surrounding medium and $h > 0$ is a constant depending on the medium. The heat f supplied at a point of the boundary is partly conducted into the medium and partly lost by radiation to the surroundings. Equating these amounts, we get the third boundary condition.

2.3 SOME IMPORTANT MATHEMATICAL TOOLS

Among the mathematical tools we employ in deriving many important results, the Gauss divergence theorem plays a vital role, which can be stated as follows: Let $\partial\mathbb{R}$ be a closed surface in the xyz-space and $\mathbb{R}$ denote the bounded region enclosed by $\partial\mathbb{R}$ in which $\mathbf{F}$ is a vector belonging to $c^{(1)}$ in $\mathbb{R}$ and continuous on $\mathbb{R}$. Then

$$\iint_{\partial\mathbf{R}} \mathbf{F}\cdot\hat{n}\, dS = \iiint_{\mathbf{R}} \nabla\cdot\mathbf{F}\, dV \tag{2.16}$$

where dV is an element of volume, dS is an element of surface area, and $\hat{n}$ the outward drawn normal.

Green's identities which follow from divergence theorem are also useful and they can be derived as follows: Let $\mathbf{F} = \mathbf{f}\phi$, where $\mathbf{f}$ is a vector function of position and ϕ is a scalar function of position. Then,

$$\iiint_{\mathbf{R}} \nabla\cdot(\mathbf{f}\phi)\, dV = \iint_{\partial\mathbf{R}} \hat{n}\cdot\mathbf{f}\phi\, dS$$

Using the vector identity

$$\nabla\cdot(\mathbf{f}\phi) = \mathbf{f}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{f}$$

we have

$$\iiint_{\mathbf{R}} \mathbf{f}\cdot\nabla\phi\, dV = \iint_{\partial\mathbf{R}} \hat{n}\cdot\mathbf{f}\phi\, dS - \iiint_{\mathbf{R}} \phi\nabla\cdot\mathbf{f}\, dV$$

If we choose $\mathbf{f} = \nabla\psi$, the above equation yields

$$\iint\limits_{\mathbb{R}} \nabla\phi\cdot\nabla\psi \, dV = \iint\limits_{\partial\mathbb{R}} \phi\hat{n}\cdot\nabla\psi \, dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\psi \, dV \tag{2.17}$$

Noting that $\hat{n}\cdot\nabla\psi$ is the derivative of ψ in the direction of $\hat{n}$, we introduce the notation

$$\hat{n}\cdot\nabla\psi = \partial\psi/\partial n$$

into Eq. (2.17) to get

$$\iiint\limits_{\mathbb{R}} \nabla\phi\cdot\nabla\psi \, dV = \iint\limits_{\partial\mathbb{R}} \phi\frac{\partial\psi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\psi \, dV \tag{2.18a}$$

This equation is known as *Green's first identity*. Of course, it is assumed that both ϕ and ψ possess continuous second derivatives.

Interchanging the role of ϕ and ψ, we obtain from relation (2.18a) the equation

$$\iiint\limits_{\mathbb{R}} \nabla\psi\cdot\nabla\phi \, dV = \iint\limits_{\partial\mathbb{R}} \psi\frac{\partial\phi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \psi\nabla^2\phi \, dV \tag{2.18b}$$

Now, subtracting Eq. (2.18b) from Eq. (2.18a), we get

$$\iiint\limits_{\mathbb{R}} (\phi\nabla^2\psi - \psi\nabla^2\phi) dV = \iint\limits_{\partial\mathbb{R}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS \tag{2.19}$$

This is known as *Green's second identity*. If we set $\phi = \psi$ in Eq. (2.18a) we get

$$\iiint\limits_{\mathbb{R}} (\nabla\phi)^2 \, dV = \iint\limits_{\partial\mathbb{R}} \phi\frac{\partial\phi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\phi \, dV \tag{2.20}$$

which is a special case of Green's first identity.

2.4 PROPERTIES OF HARMONIC FUNCTIONS

Solutions of Laplace equation are called harmonic functions which possess a number of interesting properties, and they are presented in the following theorems.

Theorem 2.1 If a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere.

Proof If ϕ is a harmonic function, then $\nabla^2\phi = 0$ in $\mathbb{R}$. Also, if $\phi = 0$ on $\partial\mathbb{R}$, we shall show that $\phi = 0$ in $\overline{\mathbb{R}} = \mathbb{R} U \partial\mathbb{R}$. Recalling Green's first identity, i.e., Eq. (2.20), we get

$$\iiint\limits_{\mathbb{R}} (\nabla\phi)^2 \, dV = \iint\limits_{\partial\mathbb{R}} \phi\frac{\partial\phi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\phi \, dV$$

and using the above facts we have, at once, the relation

$$\iiint_{\mathbb{R}} (\nabla\phi)^2\, dV = 0$$

Since $(\nabla\phi)^2$ is positive, it follows that the integral will be satisfied only if $\nabla\phi = 0$. This implies that ϕ is a constant in $\mathbb{R}$. Since ϕ is continuous in $\overline{\mathbb{R}}$ and ϕ is zero on $\partial\mathbb{R}$, it follows that $\phi = 0$ in $\mathbb{R}$.

Theorem 2.2 If ϕ is a harmonic function in $\mathbb{R}$ and $\partial\phi/\partial n = 0$ on $\partial\mathbb{R}$, then ϕ is a constant in $\overline{\mathbb{R}}$.

Proof Using Green's first identity and the data of the theorem, we arrive at

$$\iiint_{\mathbb{R}} (\nabla\phi)^2\, dV = 0$$

implying $\nabla\phi = 0$, i.e., ϕ is a constant in $\mathbb{R}$. Since the value of ϕ is not known on the boundary $\partial\mathbb{R}$ while $\partial\phi/\partial n = 0$, it is implied that ϕ is a constant on $\partial\mathbb{R}$ and hence on $\overline{\mathbb{R}}$.

Theorem 2.3 If the Dirichlet problem for a bounded region has a solution, then it is unique.

Proof If ϕ_1 and ϕ_2 are two solutions of the interior Dirichlet problem, then

$$\nabla^2\phi_1 = 0 \quad \text{in } \mathbb{R}; \qquad \phi_1 = f \quad \text{on } \partial\mathbb{R}$$

$$\nabla^2\phi_2 = 0 \quad \text{in } \mathbb{R}; \qquad \phi_2 = f \quad \text{on } \partial\mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \qquad \text{in } \mathbb{R};$$

$$\psi = \phi_1 - \phi_2 = f - f = 0 \qquad \text{on } \partial\mathbb{R}$$

Therefore,

$$\nabla^2\psi = 0 \text{ in } \mathbb{R}, \qquad \psi = 0 \quad \text{on } \partial\mathbb{R}$$

Now using Theorem 2.1, we obtain $\psi = 0$ on $\overline{\mathbb{R}}$, which implies that $\phi_1 = \phi_2$. Hence, the solution of the Dirichlet problem is unique.

Theorem 2.4 If the Neumann problem for a bounded region has a solution, then it is either unique or it differs from one another by a constant only.

Proof Let ϕ_1 and ϕ_2 be two distinct solutions of the Neumann problem. Then we have

$$\nabla^2 \phi_1 = 0 \text{ in } \mathbb{R}; \quad \frac{\partial \phi_1}{\partial n} = f \quad \text{on } \partial\mathbb{R}.$$

$$\nabla^2 \phi_2 = 0 \text{ in } \mathbb{R}; \quad \frac{\partial \phi_2}{\partial n} = f \quad \text{on } \partial\mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2 \psi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0 \quad \text{in } \mathbb{R}$$

$$\frac{\partial \psi}{\partial n} = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0 \qquad \text{on } \partial\mathbb{R}$$

Hence from Theorem 2.2, ψ is a constant on $\overline{\mathbb{R}}$, i.e., $\phi_1 - \phi_2 =$ constant. Therefore, the solution of the Neumann problem is not unique. Thus, the solutions of a certain Neumann problem can differ from one another by a constant only.

2.4.1 The Spherical Mean

Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$ and let $P(x, y, z)$ be any point in $\mathbb{R}$. Also, let $S(P, r)$ represent a sphere with centre at P and radius r such that it lies entirely within the domain $\mathbb{R}$ as depicted in Fig. 2.2. Let u be a continuous function in $\mathbb{R}$. Then the spherical mean of u denoted by $\overline{u}$ is defined as

$$\overline{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q)\, dS \tag{2.21}$$

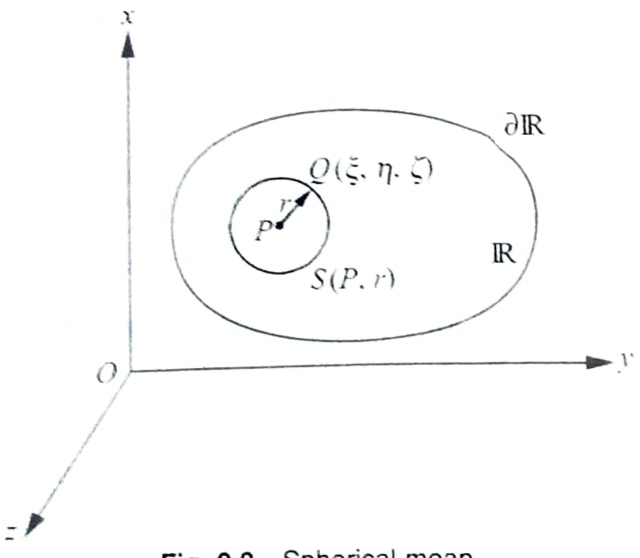

Fig. 2.2 Spherical mean.

where $Q(\xi, \eta, \zeta)$ is any variable point on the surface of the sphere $S(P, r)$ and dS is the surface element of integration. For a fixed radius r, the value $\bar{u}(r)$ is the average of the values of u taken over the sphere $S(P, r)$, and hence it is called the spherical mean. Taking the origin at P, in terms of spherical polar coordinates, we have

$$\xi = x + r \sin\theta \cos\phi$$

$$\eta = y + r \sin\theta \sin\phi$$

$$\zeta = z + r \cos\theta$$

Then, the spherical mean can be written as

$$\bar{u}(r) = \frac{1}{4\pi r^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} u(x + r\sin\theta\cos\phi,\ y + r\sin\theta\sin\phi,\ z + r\cos\theta) r^2 \sin\theta \, d\theta \, d\phi$$

Also, since u is continuous on $S(P, r), \bar{u}$ too is a continuous function of r on some interval $0 < r \leq R$, which can be verified as follows:

$$\bar{u}(r) = \frac{1}{4\pi} \iint u(Q) \sin\theta \, d\theta \, d\phi = \frac{u(Q)}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi = u(Q)$$

Now, taking the limit as $r \to 0$, $Q \to P$, we have

$$\operatorname*{Lt}_{r \to 0} \bar{u}(r) = u(P) \tag{2.22}$$

Hence, $\bar{u}$ is continuous in $0 \leq r \leq R$.

2.4.2 Mean Value Theorem for Harmonic Functions

Theorem 2.5 Let u be harmonic in a region $\mathbb{R}$. Also, let $P(x, y, z)$ be a given point in $\mathbb{R}$ and $S(P, r)$ be a sphere with centre at P such that $S(P, r)$ is completely contained in the domain of harmonicity of u. Then

$$u(P) = \bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q) \, dS$$

Proof Since u is harmonic in $\mathbb{R}$, its spherical mean $\bar{u}(r)$ is continuous in $\mathbb{R}$ and is given by

$$\bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q) \, dS = \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^{\pi} u(\xi, \eta, \zeta) r^2 \sin\theta \, d\theta \, d\phi$$

Therefore,

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} (u_\xi \xi_r + u_\eta \eta_r + u_\zeta \zeta_r) \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} (u_\xi \sin\theta\cos\phi + u_\eta \sin\theta \sin\phi + u_\zeta \cos\theta) \sin\theta \, d\theta \, d\phi \quad (2.23)$$

Noting that $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$ are the direction cosines of the normal $\hat{n}$ on $S(P, r)$,

$$\nabla u = iu_\xi + ju_\eta + ku_\zeta, \qquad \hat{n} = (in_1, jn_2, kn_3),$$

the expression within the parentheses of the integrand of Eq. (2.23) can be written as $\nabla u \cdot \hat{n}$. Thus

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi r^2}\iint_{S(P,r)} \nabla u \cdot \hat{n} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi r^2}\iint_{S(P,r)} \nabla u \cdot \hat{n} \, dS$$

$$= \frac{1}{4\pi r^2}\iiint_{V(P,r)} \nabla\cdot\nabla u \, dV \text{ (by divergence theorem)}$$

$$= \frac{1}{4\pi r^2}\iiint_{V(P,r)} \nabla^2 u \, dV = 0 \text{ (since } u \text{ is harmonic)}$$

Therefore, $\dfrac{d\bar{u}}{dr} = 0$, implying $\bar{u}$ is constant.

Now the continuity of $\bar{u}$ at $r = 0$ gives, from Eq.(2.22), the relation

$$\bar{u}(r) = u(P) = \frac{1}{4\pi r^2}\iint_{S(P,r)} u(Q)\, dS \quad (2.24)$$

2.4.3 Maximum-Minimum Principle and Consequences

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Theorem 2.6 Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$. Also, let u be a function which is continuous in a closed region $\overline{\mathbb{R}}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$. Further, if u is not constant everywhere on $\overline{\mathbb{R}}$, then the maximum and minimum values of u must occur only on the boundary $\partial\mathbb{R}$.

Proof Suppose u is a harmonic function but not constant everywhere on $\overline{\mathbb{R}}$. If possible, let u attain its maximum value M at some interior point P in $\mathbb{R}$. Since M is the maximum of u which is not a constant, there should exist a sphere $S(P, r)$ about P such that some of the values of u on $S(P, r)$ must be less than M. But by the mean value property, the value of u at P is the average of the values of u on $S(P, r)$, and hence it is less than M. This contradicts the assumption that $u = M$ at P. Thus u must be constant over the entire sphere $S(P, r)$.

Let Q be any other point inside $\mathbb{R}$ which can be connected to P by an arc lying entirely within the domain $\mathbb{R}$. By covering this arc with spheres and using the Heine-Borel theorem to choose a finite number of covering spheres and repeating the argument given above, we can arrive at the conclusion that u will have the same constant value at Q as at P. Thus u cannot attain a maximum value at any point inside the region $\mathbb{R}$. Therefore, u can attain its maximum value only on the boundary $\partial\mathbb{R}$. A similar argument will lead to the conclusion that u can attain its minimum value only on the boundary $\partial\mathbb{R}$.

Some important consequences of the maximum-minimum principle are given in the following theorems.

Theorem 2.7 (Stability theorem). The solutions of the Dirichlet problem depend continuously on the boundary data.

Proof Let u_1 and u_2 be two solutions of the Dirichlet problem and let f_1 and f_2 be given continuous functions on the boundary $\partial\mathbb{R}$ such that

$$\nabla^2 u_1 = 0 \text{ in } \mathbb{R}; \quad u_1 = f_1 \quad \text{on } \partial\mathbb{R},$$

$$\nabla^2 u_2 = 0 \text{ in } \mathbb{R}; \quad u_2 = f_2 \quad \text{on } \partial\mathbb{R}$$

Let $u = u_1 - u_2$. Then,

$$\nabla^2 u = \nabla^2 u_1 - \nabla^2 u_2 = 0 \quad \text{in } \mathbb{R}; \qquad u = f_1 - f_2 \quad \text{on } \partial\mathbb{R}$$

Hence, u is a solution of the Dirichlet problem with boundary condition $u = f_1 - f_2$ on $\partial\mathbb{R}$. By the maximum-minimum principle, u attains the maximum and minimum values on $\partial\mathbb{R}$. Thus at any interior point in $\mathbb{R}$, we shall have, for a given $\varepsilon > 0$,

$$-\varepsilon < u_{\min} \le u \le u_{\max} < \varepsilon$$

Therefore,

$$|u| < \varepsilon \quad \text{in } \mathbb{R}, \text{ implying } |u_1 - u_2| < \varepsilon$$

Hence, if

$$|f_1 - f_2| < \varepsilon \quad \text{on } \partial\mathbb{R}, \quad \text{then } |u_1 - u_2| < \varepsilon \quad \text{on } \mathbb{R}$$

Thus, small changes in the initial data bring about an arbitrarily small change in the solution. This completes the proof of the theorem.

Theorem 2.8 Let $\{f_n\}$ be a sequence of functions, each of which is continuous on $\overline{\mathbb{R}}$ and harmonic on $\mathbb{R}$. If the sequence $\{f_n\}$ converges uniformly on $\partial\mathbb{R}$, then it converges uniformly on $\overline{\mathbb{R}}$.

Proof Since the sequence $\{f_n\}$ converges uniformly on $\partial\mathbb{R}$, for a given $\varepsilon > 0$, we can always find an integer N such that

$$|f_n - f_m| < \varepsilon \quad \text{for } n, m > N$$

Hence, from stability theorem, for all $n, m > N$, it follows immediately that

$$|f_n - f_m| < \varepsilon \quad \text{in } \mathbb{R}$$

Therefore, $\{f_n\}$ converges uniformly on $\overline{\mathbb{R}}$.

EXAMPLE 2.1 Show that if the two-dimensional Laplace equation $\nabla^2 u = 0$ is transformed by introducing plane polar coordinates r, θ defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, it takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solution In many practical problems, it is necessary to write the Laplace equation in various coordinate systems. For instance, if the boundary of the region $\partial\mathbb{R}$ is a circle, then it is natural to use polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$. Therefore,

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x)$$

$$r_x = \cos\theta, \qquad r_y = \sin\theta, \qquad \theta_x = -\frac{\sin\theta}{r}, \qquad \theta_y = \frac{\cos\theta}{r}$$

since

$$u = u(r, \theta) \qquad u_x = u_r r_x + u_\theta \theta_x = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)$$

Similarly,

$$u_y = u_r r_y + u_\theta \theta_y = \left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)$$

Now for the second order derivatives,

$$u_{xx} = (u_x)_x = (u_x)_r r_x + (u_x)_\theta \theta_x = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_r \cos\theta + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_\theta \left(-\frac{\sin\theta}{r}\right)$$

Therefore,

$$u_{xx} = \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right)\cos\theta$$

$$+ \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \qquad (2.25)$$

Similarly, we can show that

$$u_{yy} = \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right)\sin\theta$$

$$+ \left(u_{r\theta}\sin\theta + u_r \cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) \qquad (2.26)$$

By adding Eqs. (2.25) and (2.26) and equating to zero, we get

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad (2.27)$$

which is the Laplace equation in polar coordinates. One can observe that the Laplace equation in Cartesian coordinates has constant coefficients only, whereas in polar coordinates, it has variable coefficients.

EXAMPLE 2.2 Show that in cylindrical coordinates r, θ, z defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the Laplace equation $\nabla^2 u = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Solution The Laplace equation in Cartesian coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The relations between Cartesian and cylindrical coordinates give

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x), \qquad z = z$$

Since

$$u = u(r, \theta, z)$$

$$u_x = u_r r_x + u_\theta \theta_x + u_z z_x = u_r \cos\theta - u_\theta \left(\frac{\sin\theta}{r}\right)$$

$$u_y = u_r r_y + u_\theta \theta_y + u_z z_y = u_r \sin\theta + u_\theta \left(\frac{\cos\theta}{r}\right)$$

$$u_z = u_r r_z + u_\theta \theta_z + u_z = u_z$$

for the second order derivatives, we find

$$u_{xx} = (u_x)_x = (u_x)_r r_x + (u_x)_\theta \theta_x + (u_x)_z z_x$$

$$= \left[u_r \cos\theta - u_\theta \left(\frac{\sin\theta}{r}\right)\right]_r \cos\theta + \left[u_r \cos\theta - u_\theta \left(\frac{\sin\theta}{r}\right)\right]_\theta \left(-\frac{\sin\theta}{r}\right)$$

$$= \left(u_{rr} \cos\theta - u_{r\theta} \frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right) \cos\theta$$

$$+ \left(u_{r\theta} \cos\theta - u_r \sin\theta - u_{\theta\theta} \frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \tag{2.28}$$

Similarly

$$u_{yy} = (u_y)_y = (u_y)_r r_y + (u_y)_\theta \theta_y + (u_y)_z z_y$$

$$= \left[u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right]_r \sin\theta + \left[u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right]_\theta \left(\frac{\cos\theta}{r}\right)$$

$$= \left(u_{rr} \sin\theta + u_{\theta r} \frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right) \sin\theta$$

$$+ \left(u_{r\theta} \sin\theta + u_r \cos\theta + u_{\theta\theta} \frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) \tag{2.29}$$

$$u_{zz} = u_{zz} \tag{2.30}$$

Adding Eqs. (2.28)–(2.30), we obtain

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} \tag{2.31}$$

EXAMPLE 2.3 Show that in spherical polar coordinates r, θ, ϕ defined by the relations $x = r\sin\theta \cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the Laplace equations $\nabla^2 u = 0$ takes the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

Solution In Cartesian coordinates, the Laplace equation is

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

In spherical coordinates, $u = u(r, \theta, \phi)$, $r^2 = x^2 + y^2 + z^2$, $\cos\theta = z/r$, $\tan\phi = y/x$.

It can be easily verified that

$$\theta_x = \frac{\cos\theta\cos\phi}{r}, \qquad \theta_y = \frac{\cos\theta\sin\phi}{r}, \qquad \theta_z = -\frac{\sin\theta}{r}$$

$$\phi_x = -\frac{\sin\phi}{r\sin\theta}, \qquad \phi_y = \frac{\cos\phi}{r\sin\theta}, \qquad \phi_z = 0$$

Now,

$$u_x = u_r r_x + u_\theta\theta_x + u_\phi\phi_x = u_r \sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}$$

$$u_y = u_r r_y + u_\theta\theta_y + u_\phi\phi_y = u_r \sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + \frac{u_\phi\cos\phi}{r\sin\theta}$$

$$u_z = u_r r_z + u_\theta\theta_z + u_\phi\phi_z = u_r\cos\theta + u_\theta\left(-\frac{\sin\theta}{r}\right)$$

For the second order derivatives,

$$u_{xx} = (u_x)_r r_x + (u_x)_\theta\theta_x + (u_x)_\phi\phi_x$$

$$= \left(u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}\right)_r \cdot (\sin\theta\cos\phi)$$

$$+\left(u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}\right)_\theta \left(\frac{\cos\theta\cos\phi}{r}\right)$$

$$+\left(u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}\right)_\phi \left(-\frac{\sin\phi}{r\sin\theta}\right)$$

$$= (\sin^2\theta\cos^2\phi)u_{rr} + \frac{\cos^2\theta\cos^2\phi}{r^2}u_{\theta\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}u_{\phi\phi}$$

$$+ u_{r\theta}\left(\frac{2\sin\theta\cos\theta\cos^2\phi}{r}\right) + u_{r\phi}\left(-\frac{2\sin\phi\cos\phi}{r}\right)$$

$$+ u_{\theta\phi}\left(-\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right) + u_r\left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right)$$

$$+ u_\phi\left(\frac{\sin\phi\cos\phi}{r^2} + \frac{\cos^2\theta\cos\phi\sin\phi}{r^2\sin^2\theta} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\right)$$

$$+ u_\theta\left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta} - \frac{2\cos\theta\sin\theta\cos^2\phi}{r^2}\right) \tag{2.32}$$

$$u_{yy} = (u_y)_r r_y + (u_y)_\theta\theta_y + (u_y)_\phi\phi_y = \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_r(\sin\theta\sin\phi)$$

$$+\left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_\theta\frac{\cos\theta\sin\phi}{r}$$

$$+\left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_\phi\frac{\cos\phi}{r\sin\theta}$$

$$= (\sin^2\theta\sin^2\phi)u_{rr} + \frac{\cos^2\theta\sin^2\phi}{r^2}u_{\theta\theta} + \frac{\cos^2\phi}{r^2\sin^2\theta}u_{\phi\phi}$$

$$+ u_{r\theta}\left(\frac{2\sin\theta\cos\theta\sin^2\phi}{r}\right) + u_{r\phi}\left(\frac{2\cos\phi\sin\phi}{r}\right)$$

$$+ u_{\theta\phi}\left(\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right) + u_r\left(\frac{\cos^2\theta\sin^2\phi}{r} + \frac{\cos^2\phi}{r}\right)$$

$$+ u_\theta\left(-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2} + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\right)$$

$$+ u_\phi\left(-\frac{\sin\phi\cos\phi}{r^2} - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta} - \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\right) \tag{2.33}$$

Similarly,

$$u_{zz} = (u_z)_r r_z + (u_z)_\theta \theta_z + (u_z)_\phi \phi_z$$

$$= \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \right)_r (\cos\theta) + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \right)_\theta \left(-\frac{\sin\theta}{r} \right)$$

$$= u_{rr} \cos^2\theta - u_{r\theta} \frac{2\sin\theta\cos\theta}{r} + u_{\theta\theta} \frac{\sin^2\theta}{r^2}$$

$$+ u_r \frac{\sin^2\theta}{r} + u_\theta \frac{\cos\theta\sin\theta}{r^2} \tag{2.34}$$

Adding Eqs. (2.32)–(2.34), we obtain

$$\nabla^2 u = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r^2 \sin^2\theta} u_{\phi\phi} + \frac{2}{r} u_r + \frac{\cos\theta}{r^2 \sin\theta} u_\theta = 0$$

which can be rewritten as

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial\phi^2} = 0 \tag{2.35}$$

2.5 SEPARATION OF VARIABLES

The method of separation of variables is applicable to a large number of classical linear homogeneous equations. The choice of the coordinate system in general depends on the shape of the boundary. For example, consider a two-dimensional Laplace equation in Cartesian coordinates.

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{2.36}$$

We assume the solution in the form

$$u(x, y) = X(x)\, Y(y) \tag{2.37}$$

Substituting in Eq. (2.36), we get

$$X''Y + Y''X = 0$$

i.e.

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

where k is a separation constant. Three cases arise.

Case I Let $k = p^2$, p is real. Then

$$\frac{d^2X}{dx^2} - p^2X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + p^2Y = 0$$

whose solution is given by

$$X = c_1e^{px} + c_2e^{-px}$$

and

$$Y = c_3 \cos py + c_4 \sin py$$

Thus, the solution is

$$u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \tag{2.38}$$

Case II Let $k = 0$. Then

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} = 0$$

Integrating twice, we get

$$X = c_5x + c_6$$

and

$$Y = c_7y + c_8$$

The solution is therefore,

$$u(x, y) = (c_5x + c_6)(c_7y + c_8) \tag{2.39}$$

Case III Let $k = -p^2$. Proceeding as in Case I, we obtain

$$X = c_9 \cos px + c_{10} \sin px$$

$$Y = c_{11}e^{py} + c_{12}e^{-py}$$

Hence, the solution in this case is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11}e^{py} + c_{12}e^{-py}) \tag{2.40}$$

In all these cases, $c_i (i = 1, 2, \ldots, 12)$ refer to integration constants, which are calculated by using the boundary conditions.

2.6 DIRICHLET PROBLEM FOR A RECTANGLE

The Dirichlet problem for a rectangle is defined as follows:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le x \le a, \ 0 \le y \le b$$

$$\text{BCs: } u(x, b) = u(a, y) = 0, \qquad u(0, y) = 0, \qquad u(x, 0) = f(x) \tag{2.41}$$

This is an interior Dirichlet problem. The general solution of the governing PDE, using the method of variables separable, is discussed in Section 2.5. The various possible solutions of the Laplace equation are given by Eqs. (2.38–2.40). Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions as depicted in Fig. 2.3.

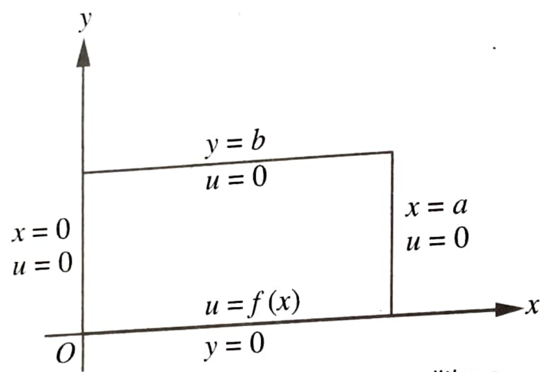

Fig. 2.3 Dirichlet boundary conditions.

Consider the solution given by Eq. (2.38):

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Using the boundary condition: $u(0, y) = 0$, we get

$$(c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0$$

which means that either $c_1 + c_2 = 0$ or $c_3 \cos py + c_4 \sin py = 0$. But $c_3 \cos py + c_4 \sin py \neq 0$; therefore,

$$c_1 + c_2 = 0 \tag{2.42}$$

Again, using the BC; $u(a, y) = 0$, Eq. (2.38) gives

$$(c_1 e^{ap} + c_2 e^{-ap})(c_3 \cos py + c_4 \sin py) = 0$$

implying thereby

$$c_1 e^{ap} + c_2 e^{-ap} = 0 \tag{2.43}$$

To determine the constants c_1, c_2, we have to solve Eqs. (2.42) and (2.43); being homogeneous, the determinant

$$\begin{vmatrix} 1 & 1 \\ e^{ap} & e^{-ap} \end{vmatrix} = 0$$

for the existence of non-trivial solution, which is not the case. Hence, only the trivial solution $u(x, y) = 0$ is possible.

If we consider the solution given by Eq. (2.39) $u(x, y) = (c_5 x + c_6)(c_7 y + c_8)$, the boundary conditions: $u(0, y) = u(a, y) = 0$ again yield a trivial solution. Hence, the possible solutions given by Eqs. (2.38) and (2.39) are ruled out. Therefore, the only possible solution obtained from Eq. (2.40) is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11} e^{py} + c_{12} e^{-py})$$

Using the BC: $u(0, y) = 0$, we get $c_9 = 0$. Also, the other BC: $u(a, y) = 0$ yields

$$c_{10} \sin pa (c_{11} e^{py} + c_{12} e^{-py}) = 0$$

For non-trivial solution, c_{10} cannot be zero, implying $\sin pa = 0$, which is possible if $pa = n\pi$ or $p = n\pi/a$, $n = 1, 2, 3, \ldots$ Therefore, the possible non-trivial solution after using the superposition principle is

$$u(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} [a_n \exp(n\pi y/a) + b_n \exp(-n\pi y/a)] \tag{2.44}$$

Now, using the BC: $u(x, b) = 0$, we get

$$\sin \frac{n\pi x}{a} [a_n \exp(n\pi b/a) + b_n \exp(-n\pi b/a)] = 0$$

implying thereby

$$a_n \exp(n\pi b/a) + b_n \exp(-n\pi b/a) = 0$$

which gives

$$b_n = -a_n \frac{\exp(n\pi b/a)}{\exp(-n\pi b/a)}, \qquad n = 1, 2, \ldots, \infty$$

The solution (2.44) now becomes

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2a_n \sin(n\pi x/a)}{\exp(-n\pi b/a)} \left[\frac{\exp\{n\pi(y-b)/a\} - \exp\{-n\pi(y-b)/a\}}{2} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2a_n}{\exp(-n\pi b/a)} \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\}$$

Let $2a_n/[\exp(-n\pi b/a)] = A_n$. Then the solution can be written in the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\} \quad (2.45)$$

Finally, using the non-homogeneous boundary condition: $u(x, 0) = f(x)$, we get

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh(-n\pi b/a) = f(x)$$

which is a half-range Fourier series. Therefore,

$$A_n \sinh(-n\pi b/a) = \frac{2}{a}\int_0^a f(x) \sin(n\pi x/a)\, dx \quad (2.46)$$

Thus, the required solution for the given Dirichlet problem is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\} \quad (2.47)$$

where

$$A_n = \frac{2}{a}\frac{1}{\sinh(-n\pi b/a)}\int_0^a f(x) \sin(n\pi x/a)\, dx$$

2.7 THE NEUMANN PROBLEM FOR A RECTANGLE

The Neumann problem for a rectangle is defined as follows:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le x \le a,\ 0 \le y \le b$$

$$\text{BCs: } u_x(0, y) = u_x(a, y) = 0, \qquad u_y(x, 0) = 0, \qquad u_y(x, b) = f(x) \quad (2.4\ldots)$$

The general solution of the Laplace equation using the method of variables separable is given in Section 2.5, and is found to be

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

The BC: $u_x(0, y) = 0$ gives

$$0 = c_2 p(c_3 e^{py} + c_4 e^{-py})$$

implying $c_2 = 0$. Therefore,

$$u(x, y) = c_1 \cos px(c_3 e^{py} + c_4 e^{-py}) \quad (2.\ldots)$$