CHAPTER 3

# Parabolic Differential Equations

## 3.1 OCCURRENCE OF THE DIFFUSION EQUATION

The diffusion phenomena such as conduction of heat in solids and diffusion of vorticity in the case of viscous fluid flow past a body are governed by a partial differential equation of parabolic type. For example, the flow of heat in a conducting medium is governed by the parabolic equation

$$\rho C \frac{\partial T}{\partial t} = \operatorname{div}(K \nabla T) + H(\mathbf{r}, T, t) \tag{3.1}$$

where $\rho$ is the density, $C$ is the specific heat of the solid, $T$ is the temperature at a point with position vector $\mathbf{r}$, $K$ is the thermal conductivity, $t$ is the time, and $H(\mathbf{r}, T, t)$ is the amount of heat generated per unit time in the element $dV$ situated at a point $(x, y, z)$ whose position vector is $\mathbf{r}$. This equation is known as diffusion equation or heat equation. We shall now derive the heat equation from the basic concepts.

Let $V$ be an arbitrary domain bounded by a closed surface $S$ and let $\bar{V} = V \cup S$. Let $T(x, y, z, t)$ be the temperature at a point $(x, y, z)$ at time $t$. If the temperature is not constant, heat flows from a region of high temperature to a region of low temperature and follows the Fourier law which states that heat flux $\mathbf{q}(\mathbf{r}, t)$ across the surface element $dS$ with normal $\hat{n}$ is proportional to the gradient of the temperature. Therefore,

$$\mathbf{q}(\mathbf{r}, t) = -K \nabla T(\mathbf{r}, t) \tag{3.2}$$

where $K$ is the thermal conductivity of the body. The negative sign indicates that the heat flux vector points in the direction of decreasing temperature. Let $\hat{n}$ be the outward unit normal vector and $\mathbf{q}$ be the heat flux at the surface element $dS$. Then the rate of heat flowing out through the elemental surface $dS$ in unit time as shown in Fig. 3.1 is

$$dQ = (\mathbf{q} \cdot \hat{n})\, dS \tag{3.3}$$

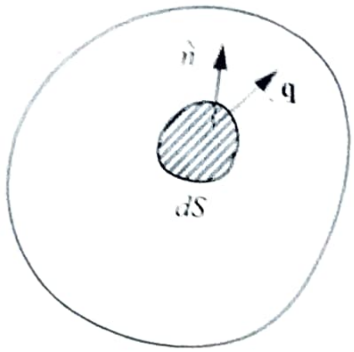


**Fig. 3.1** The heat flow across a surface.

Heat can be generated due to nuclear reactions or movement of mechanical parts as in inertial measurement unit (IMU), or due to chemical sources which may be a function of position, temperature and time and may be denoted by $H(\mathbf{r}, T, t)$. We also define the specific heat of a substance as the amount of heat needed to raise the temperature of a unit mass by a unit temperature. Then the amount of heat $dQ$ needed to raise the temperature of the elemental mass $dm = \rho\, dV$ to the value $T$ is given by $dQ = C\rho T\, dV$. Therefore,

$$Q = \iiint_V C\rho T\, dV$$

$$\frac{dQ}{dt} = \iiint_V C\rho \frac{\partial T}{\partial t} dV$$

The energy balance equation for a small control volume $V$ is: The rate of energy storage in $V$ is equal to the sum of rate of heat entering $V$ through its bounding surfaces and the rate of heat generation in $V$. Thus,

$$\iiint_V C\rho \frac{\partial T(\mathbf{r}, t)}{\partial t} dV = -\iint_S \mathbf{q} \cdot \hat{n}\, dS + \iiint_V H(\mathbf{r}, T, t)\, dV \tag{3.4}$$

Using the divergence theorem, we get

$$\iiint_V \left[ C\rho \frac{\partial T}{\partial t}(\mathbf{r}, t) + \operatorname{div} \mathbf{q}(\mathbf{r}, t) - H(\mathbf{r}, T, t) \right] dV = 0 \tag{3.5}$$

Since the volume is arbitrary, we have

$$\rho C \frac{\partial T(\mathbf{r}, t)}{\partial t} = -\operatorname{div} \mathbf{q}(\mathbf{r}, t) + H(\mathbf{r}, T, t) \tag{3.6}$$

Substituting Eq. (3.2) into Eq. (3.6), we obtain

$$\rho C \frac{\partial T(\mathbf{r}, t)}{\partial t} = \nabla \cdot [K \nabla T(\mathbf{r}, t)] + H(\mathbf{r}, T, t) \tag{3.7}$$

If we define thermal diffusivity of the medium as

$$\alpha = \frac{K}{\rho C}$$

then the differential equation of heat conduction with heat source is

$$\frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t} = \nabla^2 T(\mathbf{r},t) + \frac{H(\mathbf{r},T,t)}{K} \tag{3.8}$$

In the absence of heat sources, Eq. (3.8) reduces to

$$\frac{\partial T(\mathbf{r},t)}{\partial t} = \alpha \nabla^2 T(\mathbf{r},t) \tag{3.9}$$

This is called Fourier heat conduction equation or diffusion equation. The fundamental problem of heat conduction is to obtain the solution of Eq. (3.8) subject to the initial and boundary conditions which are called initial boundary value problems, hereafter referred to as IBVPs.

## 3.2 BOUNDARY CONDITIONS

The heat conduction equation may have numerous solutions unless a set of initial and boundary conditions are specified. The boundary conditions are mainly of three types, which we now briefly explain.

**Boundary Condition I:** *The temperature is prescribed all over the boundary surface.* That is, the temperature $T(\mathbf{r}, t)$ is a function of both position and time. In other words, $T = G(\mathbf{r}, t)$ which is some prescribed function on the boundary. This type of boundary condition is called the *Dirichlet condition*. Specification of boundary conditions depends on the problem under investigation. Sometimes the temperature on the boundary surface is a function of position only or is a function of time only or a constant. A special case includes $T(\mathbf{r}, t) = 0$ on the surface of the boundary, which is called a *homogeneous boundary condition*.

**Boundary Condition II:** *The flux of heat, i.e., the normal derivative of the temperature $\partial T/\partial n$, is prescribed on the surface of the boundary*. It may be a function of both position and time, i.e.,

$$\frac{\partial T}{\partial n} = f(\mathbf{r}, t)$$

This is called the Neumann condition. Sometimes, the normal derivatives of temperature may be a function of position only or a function of time only. A special case includes

$$\frac{\partial T}{\partial n} = 0 \text{ on the boundary}$$

This homogeneous boundary condition is also called insulated boundary condition which states that the heat flow is zero.

**Boundary Condition III:** *A linear combination of the temperature and its normal derivative is prescribed on the boundary*, i.e.,

$$K\frac{\partial T}{\partial n} + hT = G(\mathbf{r}, t)$$

where $K$ and $h$ are constants. This type of boundary condition is called *Robin's condition*. It means that the boundary surface dissipates heat by convection. Following Newton's law of cooling, which states that the rate at which heat is transferred from the body to the surroundings is proportional to the difference in temperature between the body and the surroundings, we have

$$-K\frac{\partial T}{\partial n} = h(T - T_a)$$

As a special case, we may also have

$$K\frac{\partial T}{\partial n} + hT = 0$$

which is a homogeneous boundary condition. This means that heat is convected by dissipation from the boundary surface into a surrounding maintained at zero temperature.

The other boundary conditions such as the heat transfer due to radiation obeying the fourth power temperature law and those associated with change of phase, like melting, ablation, etc. give rise to non-linear boundary conditions.

## 3.3 ELEMENTARY SOLUTIONS OF THE DIFFUSION EQUATION

Consider the one-dimensional diffusion equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad -\infty < x < \infty,\ t > 0 \tag{3.10}$$

The function

$$T(x, t) = \frac{1}{\sqrt{4\pi\alpha t}}\exp[-(x-\xi)^2/(4\alpha t)] \tag{3.11}$$

where $\xi$ is an arbitrary real constant, is a solution of Eq. (3.10). It can be verified easily as follows:

$$\frac{\partial T}{\partial t} = \frac{1}{\sqrt{4\pi\alpha t}}\left[\frac{(x-\xi)^2}{4\alpha t^2} - \frac{1}{2t}\right]\exp[-(x-\xi)^2/(4\alpha t)]$$

$$\frac{\partial T}{\partial x} = \frac{1}{\sqrt{4\pi\alpha t}}\frac{-2(x-\xi)}{4\alpha t}\exp[-(x-\xi)^2/(4\alpha t)]$$

Therefore,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sqrt{4\pi\alpha t}}\left[-\frac{1}{2\alpha t} + \frac{(x-\xi)^2}{4\alpha^2 t^2}\right]\exp\left[-(x-\xi)^2/(4\alpha t)\right] = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which shows that the function (3.11) is a solution of Eq. (3.10). The function (3.11), known as Kernel, is the elementary solution or the fundamental solution of the heat equation for the infinite interval. For $t > 0$, the Kernel $T(x, t)$ is an analytic function of $x$ and $t$ and it can also be noted that $T(x, t)$ is positive for every $x$. Therefore, the region of influence for the diffusion equation includes the entire $x$-axis. It can be observed that as $|x| \to \infty$, the amount of heat transported decreases exponentially.

In order to have an idea about the nature of the solution to the heat equation, consider a one-dimensional infinite region which is initially at temperature $f(x)$. Thus the problem is described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0 \tag{3.12}$$

$$\text{IC: } T(x, 0) = f(x), \qquad -\infty < x < \infty,\ t = 0 \tag{3.13}$$

Following the method of variables separable, we write

$$T(x, t) = X(x)\beta(t) \tag{3.14}$$

Substituting into Eq. (3.12), we arrive at

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\beta'}{\beta} = \lambda \tag{3.15}$$

where $\lambda$ is a separation constant. The separated solution for $\beta$ gives

$$\beta = Ce^{\alpha\lambda t} \tag{3.16}$$

If $\lambda > 0$, we have $\beta$ and, therefore, $T$ growing exponentially with time. From realistic physical considerations, it is reasonable to assume that $f(x) \to 0$ as $|x| \to \infty$, while $|T(x, t)| < M$ as $|x| \to \infty$. But, for $T(x, t)$ to remain bounded, $\lambda$ should be negative and thus we take $\lambda = -\mu^2$. Now from Eq. (3.15) we have

$$X'' + \mu^2 X = 0$$

Its solution is found to be

$$X = c_1 \cos \mu x + c_2 \sin \mu x$$

Hence

$$T(x, t, \mu) = (A\cos \mu x + B \sin \mu x)e^{-\alpha\mu^2 t} \tag{3.17}$$

is a solution of Eq. (3.12), where $A$ and $B$ are arbitrary constants. Since $f(x)$ is in general not periodic, it is natural to use Fourier integral instead of Fourier series in the present case. Also, since $A$ and $B$ are arbitrary, we may consider them as functions of $\mu$ and take $A = A(\mu)$, $B = B(\mu)$. In this particular problem, since we do not have any boundary conditions which limit our choice of $\mu$, we should consider all possible values. From the principles of superposition, this summation of all the product solutions will give us the relation

$$T(x,t) = \int_0^\infty T(x,t,\mu)\,d\mu = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]e^{-\alpha\mu^2 t}\,d\mu \tag{3.18}$$

which is the solution of Eq. (3.12). From the initial condition (3.13), we have

$$T(x,0) = f(x) = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]\,d\mu \tag{3.19}$$

In addition, if we recall the Fourier integral theorem, we have

$$f(t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(x)\cos\omega(t-x)\,dx\right]d\omega \tag{3.20}$$

Thus, we may write

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\cos\mu(x-y)\,dy\right]d\mu$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)(\cos\mu x\cos\mu y + \sin\mu x\sin\mu y)\,dy\right]d\mu$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos\mu x\int_{-\infty}^\infty f(y)\cos\mu y\,dy + \sin\mu x\int_{-\infty}^\infty f(y)\sin\mu y\,dy\right]d\mu \tag{3.21}$$

Let

$$A(\mu) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\cos\mu y\,dy$$

$$B(\mu) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\sin\mu y\,dy$$

Then Eq. (3.21) can be written in the form

$$f(x) = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]\,d\mu \tag{3.22}$$

Comparing Eqs. (3.19) and (3.22), we shall write relation (3.19) as

$$T(x,0) = f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\cos\mu(x-y)\,dy\right]d\mu \tag{3.23}$$

Thus, from Eq. (3.18), we obtain

$$T(x,t)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos\mu(x-y)\exp(-\alpha\mu^2 t)\,dy\right]d\mu \tag{3.24}$$

Assuming that the conditions for the formal interchange of orders of integration are satisfied, we get

$$T(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\left[\int_0^{\infty}\exp(-\alpha\mu^2 t)\cos\mu(x-y)\,d\mu\right]dy \tag{3.25}$$

Using the standard known integral

$$\int_0^{\infty}\exp(-s^2)\cos(2bs)\,ds=\frac{\sqrt{\pi}}{2}\exp(-b^2) \tag{3.26}$$

Setting $s=\mu\sqrt{\alpha t}$, and choosing

$$b=\frac{x-y}{2\sqrt{\alpha t}}$$

Equation (3.26) becomes

$$\int_0^{\infty} e^{-\alpha\mu^2 t}\cos\mu(x-y)\,d\mu=\frac{\sqrt{\pi}}{\sqrt{4\alpha t}}\exp[-(x-y)^2/(4\alpha t)] \tag{3.27}$$

Substituting Eq. (3.27) into Eq. (3.25), we obtain

$$T(x,t)=\frac{1}{\sqrt{4\alpha\pi t}}\int_{-\infty}^{\infty} f(y)\exp[-(x-y)^2/(4\alpha t)]\,dy \tag{3.28}$$

Hence, if $f(y)$ is bounded for all real values of $y$, Eq. (3.28) is the solution of the problem described by Eqs. (3.12) and (3.13).

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**EXAMPLE 3.1** In a one-dimensional infinite solid, $-\infty<x<\infty$, the surface $a<x<b$ is initially maintained at temperature $T_0$ and at zero temperature everywhere outside the surface. Show that

$$T(x,t)=\frac{T_0}{2}\left[\operatorname{erf}\left(\frac{b-x}{\sqrt{4\alpha t}}\right)-\operatorname{erf}\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$$

where erf is an error function.

***Solution*** The problem is described as follows:

$$\text{PDE: } T_t=\alpha T_{xx}, \qquad -\infty<x<\infty$$

$$\text{IC: } \quad T=T_0, \qquad a<x<b$$

$$=0 \text{ outside the above region}$$

The general solution of PDE is found to be

$$T(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} f(\xi) \exp[-(x-\xi)^2/(4\alpha t)]\, d\xi$$

Substituting the IC, we obtain

$$T(x,t) = \frac{T_0}{\sqrt{4\pi\alpha t}} \int_{a}^{b} \exp[-(x-\xi)^2/(4\alpha t)]\, d\xi$$

Introducing the new independent variable $\eta$ defined by

$$\eta = -\frac{x-\xi}{\sqrt{4\alpha t}}$$

and hence

$$d\xi = \sqrt{4\alpha t}\, d\eta$$

the above equation becomes

$$T(x,t) = \frac{T_0}{\sqrt{\pi}} \int_{(a-x)/\sqrt{(4\alpha t)}}^{(b-x)/\sqrt{(4\alpha t)}} e^{-\eta^2}\, d\eta = \frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}} \int_{0}^{(b-x)/\sqrt{(4\alpha t)}} e^{-\eta^2}\, d\eta - \frac{2}{\sqrt{\pi}} \int_{0}^{(a-x)/\sqrt{(4\alpha t)}} e^{-\eta^2}\, d\eta\right]$$

Now we introduce the error function defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\eta^2)\, d\eta$$

Therefore, the required solution is

$$T(x,t) = \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{b-x}{\sqrt{4\alpha t}}\right) - \operatorname{erf}\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$$

## 3.4 DIRAC DELTA FUNCTION

According to the notion in mechanics, we come across a very large force (ideally infinite) acting for a short duration (ideally zero time) known as impulsive force. Thus we have a function which is non-zero in a very short interval. The Dirac delta function may be thought of as a generalization of this concept. This Dirac delta function and its derivative play a useful role in the solution of initial boundary value problem (IBVP).

Consider the function having the following property:

$$\delta_\varepsilon(t) = \begin{cases} 1/2\varepsilon, & |t| < \varepsilon \\ 0, & |t| > \varepsilon \end{cases} \tag{3.29}$$

Thus,

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\,dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dt = 1 \tag{3.30}$$

Let $f(t)$ be any function which is integrable in the interval $(-\varepsilon, \varepsilon)$. Then using the Mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t)\,dt = f(\xi), \qquad -\varepsilon < \xi < \varepsilon \tag{3.31}$$

Thus, we may regard $\delta(t)$ as a limiting function approached by $\delta_\varepsilon(t)$ as $\varepsilon \to 0$, i.e.

$$\delta(t) = \operatorname*{Lt}_{\varepsilon\to 0} \delta_\varepsilon(t) \tag{3.32}$$

As $\varepsilon \to 0$, we have, from Eqs. (3.29) and (3.30), the relations

$$\delta(t) = \operatorname*{Lt}_{\varepsilon\to 0} \delta_\varepsilon(t) = \begin{cases} & \text{(in the sense of being very large)} \\ \infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \tag{3.33}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{3.34}$$

This limiting function $\delta(t)$ defined by Eqs. (3.33) and (3.34) is known as Dirac delta function or the unit impulse function. Its profile is depicted in Fig. 3.2. Dirac originally called it an improper function as there is no proper function with these properties. In fact, we can observe that

$$1 = \int_{-\infty}^{\infty} \delta(t)\,dt = \operatorname*{Lt}_{\varepsilon\to 0} \int_{|t|>\varepsilon} \delta_\varepsilon(t)\,dt = \operatorname*{Lt}_{\varepsilon\to 0} 0 = 0$$

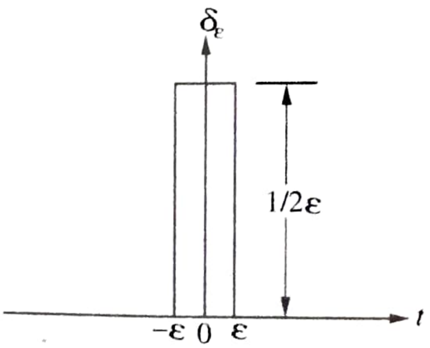


**Fig. 3.2** Profile of Dirac delta function.

Obviously, this contradiction implies that $\delta(t)$ cannot be a function in the ordinary sense. Some important properties of Dirac delta function are presented now:

PROPERTY I:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

PROPERTY II: For any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

***Proof*** Consider the equation

$$\underset{\varepsilon\to 0}{\mathrm{Lt}} \int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = \underset{\xi\to 0}{\mathrm{Lt}}\, f(\xi), \qquad -\varepsilon < \xi < \varepsilon$$

As $\varepsilon \to 0$, we have $\xi \to 0$. Therefore,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

PROPERTY III: Let $f(t)$ be any continuous function. Then

$$\int_{-\infty}^{\infty} \delta(t-a)\,f(t)\,dt = f(a)$$

***Proof*** Consider the function

$$\delta_e(t-a) = \begin{cases} 1/\varepsilon, & a < t < a+\varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

Using the mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t-a)\,f(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(t)\,dt = f(a+\theta\varepsilon), \qquad 0 < \theta < 1$$

Now, taking the limit as $\varepsilon \to 0$, we obtain

$$\int_{-\infty}^{\infty} \delta(t-a)\,f(t)\,dt = f(a)$$

Thus, the operation of multiplying $f(t)$ by $\delta(t-a)$ and integrating over all $t$ is equivalent to substituting $a$ for $t$ in the original function.

PROPERTY IV: $\delta(-t) = \delta(t)$

PROPERTY V: $\delta(at) = \dfrac{1}{a}\delta(t), \qquad a > 0$

PROPERTY VI: If $\delta(t)$ is a continuously differentiable, Dirac delta function vanishing for large $t$, then

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$$

*Proof* Using the rule of integration by parts, we get

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = [f(t)\,\delta(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt$$

Using Eq. (3.33) and property (III), the above equations becomes

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$$

PROPERTY VII:
$$\int_{-\infty}^{\infty} \delta'(t-a)\,f(t)\,dt = -f'(a)$$

Having discussed the one-dimensional Dirac delta function, we can extend the definition to two dimensions. Thus, for every $f$ which is continuous over the region $S$ containing the point $(\xi, \eta)$, we define $\delta(x-\xi, y-\eta)$ in such a way that

$$\iint_S \delta(x-\xi, y-\eta)\,f(x, y)\,d\sigma = f(\xi, \eta) \tag{3.35}$$

Note that $\delta(x-\xi, y-\eta)$ is a formal limit of a sequence of ordinary functions, i.e.,

$$\delta(x-\xi, y-\eta) = \operatorname*{Lt}_{\varepsilon\to 0}\ \delta_\varepsilon(r) \tag{3.36}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$. Also observe that

$$\iint \delta(x-\xi)\,\delta(y-\eta)\,f(x, y)\,dx\,dy = f(\xi, \eta) \tag{3.37}$$

Now, comparing Eqs. (3.35) and (3.37), we see that

$$\delta(x-\xi, y-\eta) = \delta(x-\xi)\,\delta(y-\eta) \tag{3.38}$$

Thus, a two-dimensional Dirac delta function can be expressed as the product of two one-dimensional delta functions. Similarly, the definition can be extended to higher dimensions.

***EXAMPLE 3.2*** A one-dimensional infinite region $-\infty < x < \infty$ is initially kept at zero temperature. A heat source of strength $g_s$ units, situated at $x = \xi$ releases its heat instantaneously at time $t = \tau$. Determine the temperature in the region for $t > \tau$.

***Solution*** Initially, the region $-\infty < x < \infty$ is at zero temperature. Since the heat source is situated at $x = \xi$ and releases heat instantaneously at $t = \tau$, the released temperature at $x = \xi$ and $t = \tau$ is a $\delta$- function type. Thus, the given problem is a boundary value problem described by

$$\text{PDE:} \quad \frac{\partial^2 T}{\partial x^2} + \frac{g(x,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad -\infty < x < \infty, t > 0$$

$$\text{IC:} \quad T(x,t) = F(x) = 0, \qquad -\infty < x < \infty, t = 0$$

$$g(x,t) = g_s \delta(x-\xi)\, \delta(t-\tau)$$

The general solution to this problem as given in Example 7.25, after using the initial condition $F(x) = 0$, is

$$T(x,t) = \frac{\alpha}{k}\int_{t'=0}^{t} \frac{dt'}{\sqrt{4\pi\alpha(t-t')}} \int_{x'=-\infty}^{\infty} g(x',t') \exp[-(x-x')^2/\{4\alpha(t-t')\}]\, dx' \qquad (3.39)$$

Since the heat source term is of the Dirac delta function type, substituting

$$g(x,t) = g_s \delta(x-\xi)\, \delta(t-\tau)$$

into Eq. (3.39), and integrating we get, with the help of properties of delta function, the relation

$$T(x,t) = \frac{\alpha}{k}\frac{g_s}{\sqrt{4\pi\alpha}} \int_0^t \frac{\exp[-(x-\xi)^2/\{4\alpha(t-t')\}]}{\sqrt{t-t'}}\, \delta(t-\tau)\, dt'$$

Therefore, the required temperature is

$$T(x,t) = \frac{\alpha g_s}{k} \frac{\exp[-(x-\xi)^2/\{4\alpha(t-\tau)\}]}{\sqrt{4\pi\alpha(t-\tau)}} \quad \text{for } t > \tau$$

***EXAMPLE 3.3*** An infinite one-dimensional solid defined by $-\infty < x < \infty$ is maintained at zero temperature initially. There is a heat source of strength $g_s(t)$ units, situated at $x = \xi$, which releases constant heat continuously for $t > 0$. Find an expression for the temperature distribution in the solid for $t > 0$.

***Solution*** This problem is similar to Example 3.2, except that $g(x,t) = g_s(t)\, \delta(x-\xi)$ is a Dirac delta function type. The solution to this IBVP is

$$T(x,t) = \frac{\alpha}{K}\int_{t'=0}^{t} \frac{g_s(t')}{\sqrt{4\pi\alpha(t-t')}} \exp[-(x-\xi)^2/\{4\alpha(t-t')\}]\, dt' \qquad (3.40)$$

It is given as $g_s(t) = \text{constant} = g_s$ (say). Let us introduce a new variable $\eta$ defined by

$$\eta = \frac{x-\xi}{\sqrt{4\alpha(t-t')}} \quad \text{or} \quad t-t' = \frac{1(x-\xi)^2}{\eta^2 4\alpha}$$

Therefore,

$$dt' = \frac{1}{\eta^3}\frac{(x-\xi)^2}{2\alpha}d\eta$$

Thus, Eq. (3.40) becomes

$$T(x,t) = g_s \frac{x-\xi}{2K\sqrt{\pi}}\int_{(x-\xi)/\sqrt{4\alpha t}}^{\infty} \frac{\exp(-\eta^2)}{\eta^2}d\eta$$

However,

$$\frac{d}{d\eta}\left(-\frac{e^{-\eta^2}}{\eta}\right) = \frac{e^{-\eta^2}}{\eta^2} + 2e^{-\eta^2}$$

Hence,

$$T(x,t) = g_s \frac{x-\xi}{2K\sqrt{\pi}}\left[\left(-\frac{e^{-\eta^2}}{\eta}\right)_{(x-\xi)/\sqrt{4\alpha t}}^{\infty} - 2\int_{(x-\xi)/\sqrt{(4\alpha t)}}^{\infty} e^{-\eta^2}d\eta\right]$$

Recalling the definitions of error function and its complement

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}d\eta, \qquad \text{erf}(\infty) = 1$$

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}}\left(\int_0^\infty \exp(-\eta^2)\,d\eta - \int_0^x \exp(-\eta^2)\,d\eta\right)$$

$$= \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-\eta^2)\,d\eta$$

the temperature distribution can be expressed as

$$T(x,t) = \frac{\alpha g_s}{K}\left[\sqrt{\frac{t}{2\pi}}\exp[-(x-\xi)^2/(4\alpha t)] - \frac{|x-\xi|}{2\alpha}\left(1 - \text{erf}\frac{x-\xi}{\sqrt{4\alpha t}}\right)\right]$$

Alternatively, the required temperature is

$$T(x,t) = \frac{\alpha g_s}{K}\left[\sqrt{\frac{t}{2\pi}}\exp[-(x-\xi)^2/(4\alpha t)] - \frac{|x-\xi|}{2\alpha}\,\text{erfc}\frac{x-\xi}{\sqrt{4\alpha t}}\right]$$

## 3.5 SEPARATION OF VARIABLES METHOD

2

Consider the equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.41}$$

Among the many methods that are available for the solution of the above parabolic partial differential equation, the method of separation of variables is very effective and straightforward. We separate the space and time variables of $T(x, t)$ as follows: Let

$$T(x, t) = X(x)\beta(t) \tag{3.42}$$

be a solution of the differential Eq. (3.41). Substituting Eq. (3.42) into (3.41), we obtain

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\beta'}{\beta} = K, \text{ a separation constant}$$

Then we have

$$\frac{d^2X}{dx^2} - KX = 0 \tag{3.43}$$

$$\frac{d\beta}{dt} - \alpha K \beta = 0 \tag{3.44}$$

In solving Eqs. (3.43) and (3.44), three distinct cases arise:

*Case I* When $K$ is positive, say $\lambda^2$, the solution of Eqs. (3.43) and (3.44) will have the form

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}, \qquad \beta = c_3 e^{\alpha\lambda^2 t} \tag{3.45}$$

*Case II* When $K$ is negative, say $-\lambda^2$, then the solution of Eqs. (3.43) and (3.44) will have the form

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x, \qquad \beta = c_3 e^{-\alpha\lambda^2 t} \tag{3.46}$$

*Case III* When $K$ is zero, the solution of Eqs. (3.43) and (3.44) can have the form

$$X = c_1 x + c_2, \qquad \beta = c_3 \tag{3.47}$$

Thus, various possible solutions of the heat conduction equation (3.41) could be the following:

$$\begin{aligned} T(x, t) &= (c_1' e^{\lambda x} + c_2' e^{-\lambda x}) e^{\alpha\lambda^2 t} \\ T(x, t) &= (c_1' \cos \lambda x + c_2' \sin \lambda x) e^{-\alpha\lambda^2 t} \\ T(x, t) &= c_1' x + c_2' \end{aligned} \tag{3.48}$$

where

$$c_1' = c_1c_3, \qquad c_2' = c_2c_3$$

***EXAMPLE 3.4*** Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi, t \ge 0$, subject to the conditions

(i) $T$ remains finite as $t \to \infty$

(ii) $T = 0$, if $x = 0$ and $\pi$ for all $t$

(iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$

***Solution*** Since $T$ should satisfy the diffusion equation, the three possible solutions are

$$T(x,t) = (c_1e^{\lambda x} + c_2e^{-\lambda x})e^{\alpha\lambda^2 t}$$

$$T(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)e^{-\alpha\lambda^2 t}$$

$$T(x,t) = (c_1x + c_2)$$

The first condition demands that $T$ should remain finite as $t \to \infty$. We therefore reject the first solution. In view of BC (ii), the third solution gives

$$0 = c_1 \cdot 0 + c_2, \qquad 0 = c_1 \cdot \pi + c_2$$

implying thereby that both $c_1$ and $c_2$ are zero and hence $T = 0$ for all $t$. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the third solution also. Thus, the only possible solution satisfying the first condition is

$$T(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)e^{-\alpha\lambda^2 t}$$

Using the BC (ii), we have

$$0 = (c_1 \cos \lambda x + c_2 \sin \lambda x)\big|_{x=0}$$

implying $c_1 = 0$. Therefore, the possible solution is

$$T(x,t) = c_2e^{-\alpha\lambda^2 t} \sin \lambda x$$

Applying the BC: $T = 0$ when $x = \pi$, we get

$$\sin \lambda\pi = 0 \Rightarrow \lambda\pi = n\pi$$

where $n$ is an integer. Therefore,

$$\lambda = n$$

Hence the solution is found to be of the form

$$T(x,t) = ce^{-\alpha n^2 t} \sin nx$$

Noting that the heat conduction equation is linear, its most general solution is obtained by applying the principle of superposition. Thus,

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha n^2 t} \sin nx$$

Using the third condition, we get

$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$$

which is a half-range Fourier-sine series and, therefore,

$$c_n = \frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right]$$

Integrating by parts, we obtain

$$c_n = \frac{2}{\pi}\left[\left(-x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left\{-(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

or

$$c_n = \frac{4\sin(n\pi/2)}{n^2\pi}$$

Thus, the required solution is

$$T(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\alpha n^2 t}\sin(n\pi/2)}{n^2}\sin nx$$

***EXAMPLE 3.5*** A uniform rod of length $L$ whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

***Solution*** The initial boundary value problem IBVP of heat conduction is given by

$$\text{PDE: } \frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 \le x \le L,\ t > 0$$

$$\text{BCs: } \theta(0,t) = 0, \qquad t \ge 0$$

$$\frac{\partial \theta}{\partial x}(L,t) = 0, \qquad t > 0$$

$$\text{IC: } \theta(x,0) = \theta_0, \qquad 0 \le x \le L$$

From Section 3.5, it can be noted that the physically meaningful and non-trivial solution is

$$\theta(x,t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Using the first boundary condition, we obtain $A = 0$. Thus the acceptable solution is

$$\theta = Be^{-\alpha\lambda^2 t}\sin\lambda x$$

$$\frac{\partial\theta}{\partial x} = \lambda Be^{-\alpha\lambda^2 t}\cos\lambda x$$

Using the second boundary condition, we have

$$0 = \lambda Be^{-\alpha\lambda^2 t}\cos\lambda L$$

implying $\cos\lambda L = 0$. Therefore,

The eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \frac{(2n+1)\pi}{2L}, \qquad n = 0, 1, 2, \ldots$$

Thus, the acceptable solution is of the form

$$\theta = B\exp[-\alpha\{(2n+1)/2L\}^2\pi^2 t]\sin\left(\frac{2n+1}{2L}\right)\pi x$$

Using the principle of superposition, we obtain

$$\theta(x,t) = \sum_{n=0}^{\infty} B_n\exp[-\alpha\{(2n+1)/2L\}^2\pi^2 t]\sin\left(\frac{2n+1}{2L}\pi x\right)$$

Finally, using the initial condition, we have

$$\theta_0 = \sum_{n=0}^{\infty} B_n\sin\left(\frac{2n+1}{2L}\pi x\right)$$

which is a half-range Fourier-sine series and, thus,

$$B_n = \frac{2}{L}\int_0^L \theta_0\sin\left(\frac{2n+1}{2L}\pi x\right)dx$$

$$= \frac{2}{L}\left[-\theta_0\frac{2L}{(2n+1)\pi}\left\{\cos\left(\frac{2n+1}{2L}\pi x\right)\right\}_0^L\right]$$

$$= -\frac{4\theta_0}{(2n+1)\pi}[\cos\{(2n+1)\pi/2\} - \cos 0] = \frac{4\theta_0}{(2n+1)\pi}$$

Thus, the required temperature distribution is

$$\theta(x,t) = \sum_{n=0}^{\infty} \frac{4\theta_0}{(2n+1)\pi} \exp[-\alpha\{(2n+1)/2L\}^2 \pi^2 t] \sin\left(\frac{2n+1}{2L}\pi x\right)$$

**EXAMPLE 3.6** A conducting bar of uniform cross-section lies along the $x$-axis with ends at $x = 0$ and $x = L$. It is kept initially at temperature 0° and its lateral surface is insulated. There are no heat sources in the bar. The end $x = 0$ is kept at 0°, and heat is suddenly applied at the end $x = L$, so that there is a constant flux $q_0$ at $x = L$. Find the temperature distribution in the bar for $t > 0$.

***Solution*** The given initial boundary value problem can be described as follows:

$$\text{PDE:} \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\text{BCs:} \quad T(0,t) = 0, \qquad t > 0$$

$$\frac{\partial T}{\partial x}(L,t) = q_0, \qquad t > 0$$

$$\text{IC:} \quad T(x,0) = 0, \qquad 0 \le x \le L$$

Prior to applying heat suddenly to the end $x = L$, when $t = 0$, the heat flow in the bar is independent of time (steady state condition). Let

$$T(x,t) = T_{(s)}(x) + T_1(x,t)$$

where $T_{(s)}$ is a steady part and $T_1$ is the transient part of the solution. Therefore,

$$\frac{\partial^2 T_{(s)}}{\partial x^2} = 0$$

whose general solution is

$$T_{(s)} = Ax + B$$

when $x = 0$, $T_{(s)} = 0$, implying $B = 0$. Therefore,

$$T_{(s)} = Ax$$

Using the other BC: $\dfrac{\partial T_{(s)}}{\partial x} = q_0$, we get $A = q_0$. Hence, the steady state solution is

$$T_{(s)} = q_0 x$$

For the transient part, the BCs and IC are redefined as

(i) $T_1(0,t) = T(0,t) - T_{(s)}(0) = 0 - 0 = 0$

(ii) $\partial T_1(L,t)/\partial x = \partial T(L,t)/\partial x - \partial T_s(L,t)/\partial x = q_0 - q_0 = 0$

(iii) $T_1(x,0) = T(x,0) - T_{(s)}(x) = -q_0 x,\ 0 < x < L.$

Thus, for the transient part, we have to solve the given PDE subject to these conditions. The acceptable solution is given by Eq. (3.48), i.e.

$$T_1(x,t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Applying the BC (i), we get $A = 0$. Therefore,

$$T_1(x,t) = Be^{-\alpha\lambda^2 t}\sin\lambda x$$

and using the BC (ii), we obtain

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=L} = B\lambda e^{-\alpha\lambda^2 t}\cos\lambda L = 0$$

implying $\lambda L = (2n-1)\dfrac{\pi}{2},\ n = 1, 2, \ldots$ Using the superposition principle, we have

$$T_1(x,t) = \sum_{n=1}^{\infty} B_n \exp[-\alpha\{(2n-1)/2L\}^2\pi^2 t]\sin\left(\frac{2n-1}{2L}\pi x\right)$$

Now, applying the IC (iii), we obtain

$$T_1(x,0) = -q_0 x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n-1}{2L}\pi x\right)$$

Multiplying both sides by $\sin\left(\dfrac{2m-1}{2L}\pi x\right)$ and integrating between 0 to $L$ and noting that

$$\int_0^L B_n \sin\left(\frac{2n-1}{2L}\right)\pi x \sin\left(\frac{2m-1}{2L}\pi x\right)dx = \begin{cases} 0, & n \neq m \\ \dfrac{B_m L}{2}, & n = m \end{cases}$$

we get at once, after integrating by parts, the equation

$$-q_0 \frac{4L^2}{(2m-1)^2\pi^2}\left[\sin\left(\frac{2m-1}{2}\pi\right)\right] = B_m \frac{L}{2}$$

or

$$-q_0 \frac{4L^2}{(2m-1)^2\pi^2}(-1)^{m-1} = B_m \frac{L}{2}$$

which gives

$$B_m = \frac{(-1)^m 8Lq_0}{(2m-1)^2\pi^2}$$

Hence, the required temperature distribution is

$$T(x,t) = q_0 x + \frac{8Lq_0}{\pi^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^m}{(2m-1)^2}\exp[-\alpha\{(2m-1)/L\}^2\pi^2 t]\sin\left(\frac{2m-1}{2L}\pi x\right)\right]$$

*EXAMPLE 3.7* The ends $A$ and $B$ of a rod, 10 cm in length, are kept at temperatures 0°C and 100°C until the steady state condition prevails. Suddenly the temperature at the end $A$ is increased to 20°C, and the end $B$ is decreased to 60°C. Find the temperature distribution in the rod at time $t$. $t > 0$

***Solution*** The problem is described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < 10$$

$$\text{BCs: } T(0,t) = 0, \qquad T(10,t) = 100$$

Prior to change in temperature at the ends of the rod, the heat flow in the rod is independent of time as steady state condition prevails. For steady state,

$$\frac{d^2T}{dx^2} = 0$$

whose solution is

$$T_{(s)} = Ax + B$$

When $x = 0$, $T = 0$, implying $B = 0$. Therefore,

$$T_{(s)} = Ax$$

When $x = 10$, $T = 100$, implying $A = 10$. Thus, the initial steady temperature distribution in the rod is

$$T_{(s)}(x) = 10x$$

Similarly, when the temperature at the ends $A$ and $B$ are changed to 20 and 60, the final steady temperature in the rod is

$$T_{(s)}(x) = 4x + 20$$

which will be attained after a long time. To get the temperature distribution $T(x,t)$ in the intermediate period, counting time from the moment the end temperatures were changed, we assume that

$$T(x,t) = T_1(x,t) + T_{(s)}(x)$$

where $T_1(x,t)$ is the transient temperature distribution which tends to zero as $t \to \infty$. Now, $T_1(x,t)$ satisfies the given PDE. Hence, its general solution is of the form

$$T(x,t) = (4x+20) + e^{-\alpha\lambda^2 t}(B\cos\lambda x + c\sin\lambda x)$$

Using the BC: $T = 20$ when $x = 0$, we obtain

$$20 = 20 + Be^{-\alpha\lambda^2 t}$$

implying $B = 0$. Using the BC: $T = 60$ when $x = 10$, we get

$$\sin 10\lambda = 0,\ \text{implying}\ \lambda = \frac{n\pi}{10}, \qquad n = 1, 2, \ldots$$

The principle of superposition yields

$$T(x,t) = (4x+20) + \sum_{n=1}^{\infty} c_n \exp[-\alpha(n\pi/10)^2 t]\sin\left(\frac{n\pi}{10}\right)x$$

Now using the IC: $T = 10x$, when $t = 0$, we obtain

$$10x = 4x + 20 + \sum c_n \sin\left(\frac{n\pi}{10}x\right)$$

or

$$6x - 20 = \sum c_n \sin\left(\frac{n\pi}{10}x\right)$$

where

$$c_n = \frac{2}{10}\int_0^{10}(6x-20)\sin\left(\frac{n\pi}{10}x\right)dx = -\frac{1}{5}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]$$

Thus, the required solution is

$$T(x,t) = 4x + 20 - \frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]\exp\left[-\alpha\left(\frac{n\pi}{10}\right)^2 t\right]\sin\left(\frac{n\pi}{10}x\right)$$