Operator Theory

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Chapter II FUNDAMENTAL PROPERTIES OF BOUNDED LINEAR OPERATORS

2 . 1 Bounded Linear Operators on a Hilbert Space

2.1.1 Norm of bounded linear operator

Definition (Linear operator)

A mapping T from a Hilbert space H to H is said to be a linear operator if T satisfies the following (i) and (ii) :
(i) additive : $T(x + y) = Tx + Ty$ for any $x, y \in H$.
(ii) homogeneous: $T(\alpha x) = \alpha Tx$ for any $x \in H$ and any complex number α.

Identity operator

The identity operator I is defined by $Ix = x$ for all $x \in H$.

Zero operator

The zero operator 0 is defined by $0x = 0$ for all $x \in H$.

Definition

A linear operator T on a Hilbert space H is said to be bounded if there exists $c > 0$ such that

$$\|Tx\| \leq c\|x\| \text{ for all } x \in H.$$

$\|T\|$ is defined by

$$(1) \quad \|T\| = \inf\{c > 0 : \|Tx\| \leq c\|x\| \text{for all } x \in H\}.$$

$\|T\|$ is said to be the operator norm of T.

Definition (B(H))

B(H) is defined as the set of all bounded linear operators on a Hilbert space H.

Needless to say, B(H) can be regarded as an extension of the set of all 2×2 matrices.

Theorem

For any bounded linear operator T,

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$

Proof

Put $b = \sup\{\|Tx\| : \|x\| = 1\}$.
If T is bounded, then

$$\|Tx\| \leq \|T\|\|x\| = \|T\| \text{ for } \|x\| = 1,$$

Therefore $b \leq \|T\|$ by the definition (1).
Conversely for any vector $x \in H$,

$$\|Tx\| = \left\|T\left(\|x\|\frac{x}{\|x\|}\right)\right\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\|\|x\| = b\|x\|.$$

Therefore $\|T\| \leq b$.

$$\text{Hence } \|T\| = b = \sup\{\|Tx\| : \|x\| = 1\}.$$

Theorem

For any bounded linear operator T, $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$

Proof

Since $\{x : \|x\| = 1\} \subseteq \{x : \|x\| \leq 1\}$

$$\sup\{\|Tx\| : \|x\| \leq 1\} \geq \sup\{\|Tx\| : \|x\| = 1\} = \|T\|.$$

Conversely

$$\begin{aligned} \sup\{\|Tx\| : \|x\| \leq 1\} &\leq \sup\{\frac{\|Tx\|}{\|x\|} : \|x\| \leq 1\} \\ &= \sup\{\|Ty\| : \|y\| = 1\} \\ &= \|T\| \text{ Since } \|T\| = \sup\{\|Tx\| : \|x\| = 1\} \end{aligned}$$

Hence $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$.

Theorem

For any bounded linear operator T, the following formula holds:

$$\|T\| = \sup\{|(Tx, y)| : \|x\| = \|y\| = 1\}.$$

Proof

By Schwarz's inequality, $|(Tx, y)| \leq \|Tx\|\|y\| = \|Tx\|$ for $\|y\| = 1$.

Therefore, $\sup\{|(Tx, y)| : \|x\| = \|y\| = 1\} \leq \sup\{\|Tx\| : \|x\| = 1\}$

Therefore, using the result $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$, we get

$$\sup\{|(Tx, y)| : \|x\| = \|y\| = 1\} \leq \|T\|.$$

On the otherhand,

$$\sup\{|(Tx, y)| : \|x\| = \|y\| = 1\} \geq \sup\{|(Tx, \frac{Tx}{\|Tx\|})| : \|x\| = 1\}$$

$$(\text{ Since } \|\frac{Tx}{\|Tx\|}\| = \frac{\|Tx\|}{\|Tx\|} = 1)$$

Therefore

$$\sup\{|(Tx, y)| : \|x\| = \|y\| = 1\} = \sup\{\|Tx\| : \|x\| = 1\} = \|T\|$$

$$(\text{ Since } |(Tx, \frac{Tx}{\|Tx\|})| = \frac{1}{\|Tx\|}\|Tx\|^2 = \|Tx\| \text{ and } \|T\| = \sup\{\|Tx\| : \|x\| = 1\})$$

Hence

$$\|T\| = \sup\{|(Tx, y)| : \|x\| = \|y\| = 1\}.$$

Theorem

For any linear operator T on a Hilbert space H, the following statements are mutually equivalent:
(i) T is bounded.
(ii) T is continuous on the whole space H.
(iii) T is continuous on some point x_0 on H.

Proof

To prove (i) $\implies$ (ii)

Assume that T is bounded.
Then $\|Tx\| \leq \|T\|\|x\|$ for all $x \in H$.
Let $x_0 \in H$.
Let $\{x_n\}$ be any sequence in H converging to x_0.
Hence $\|x_n - x_0\| \to 0$.

$$\text{Then } \|Tx_n - Tx_0\| = \|T(x_n - x_0)\| \leq \|T\|\|x_n - x_0\| \to 0.$$

That is, $x_n \to x_0 \implies Tx_n \to Tx_0$. Hence T is continuous at x_0.
Since x_0 in H is arbitrary,
T is continuous on the whole space H.
Hence (i) $\implies$ (ii)

To Prove (ii) $\implies$ (iii)

Since T is continuous on the whole space H,
T is continuous on all points on H.
Hence T is continuous on some point x_0 on H.

To Prove (iii) $\implies$ (i)

Assume that T is continuous on some point x_0 on H.
To prove that T is bounded.
On the contrary, assume that T is not bounded.
Then for each natural number n, there exists a nonzero vector x_n such that

$$\|Tx_n\| > n\|x_n\|.$$

Put $y_n = \frac{x_n}{n\|x_n\|}$.
Then $\|y_n\| = \frac{1}{n}$
Therefore $x_0 + y_n \to x_0$,
but $\|T(x_0 + y_n) - Tx_0\| = \|Ty_n\| = \frac{\|Tx_n\|}{n\|x_n\|} > \frac{n\|x_n\|}{n\|x_n\|} = 1$.
This shows that T is not continuous at x_0 which is contrary to (iii).
Hence T is bounded.

Hence (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i)
Hence (i),(ii) and (iii) are equivalent.

Theorem

Let S and T be bounded linear operators on a Hilbert space H. Then the following properties hold:
(i) $\|\alpha T\| \leq |\alpha|\|T\|$ for any $\alpha \in C$.
(ii) $\|S + T\| \leq \|S\| + \|T\|$.
(iii) $\|ST\| \leq \|S\|\|T\|$.

Proof

(i)Consider

$$\begin{aligned}
\|\alpha T\| &= \sup\{\|(\alpha T)x\|/\|x\| = 1\}. \\
&= \sup\{|\alpha|\|Tx\|/\|x\| = 1\} \\
&= |\alpha| \sup\{\|Tx\|/\|x\| = 1\} \\
&= |\alpha|\|T\|
\end{aligned}$$

Hence $\|\alpha T\| = |\alpha|\|T\|$

Consider

$$\begin{aligned}
\|S+T\| &= \sup\{\|(S+T)x\|/\|x\|=1\}. \\
&= \sup\{\|Sx+Tx\|/\|x\|=1\} \\
&\leq \sup\{\|Sx\|+\|Tx\|/\|x\|=1\} \\
&\leq \sup\{\|Sx\|/\|x\|=1\}+\sup\{\|Tx\|/\|x\|=1\} \\
&= \|S\|+\|T\|
\end{aligned}$$

Hence $\|S+T\| \leq \|S\|+\|T\|$

Consider

$$\begin{aligned}\|ST\| &= \sup\{\|(ST)x\|/\|x\| = 1\}.\\ &= \sup\{\|S(Tx)\|/\|x\| = 1\}\end{aligned}$$

$$\text{Since S is bounded}, \|Sx\| \leq \|S\|\|x\|, \text{ for all x inH.}$$

Hence $\|S(Tx)\| \leq \|S\|\|Tx\|$ for all $x \in H$.

$$\text{Since T is bounded}, \|Tx\| \leq \|T\|\|x\|, \text{ for all x inH.}$$

$$\text{Hence } \|S(Tx)\| \leq \|S\|\|T\|\|x\| \text{ for all } x \in H.$$

$$\text{Hence } \sup\{\|S(Tx)\|/\|x\| = 1\} \leq \|S\|\|T\|$$

$$\text{Hence } \|ST\| \leq \|S\|\|T\|$$

2.1.2 Adjoint operator

In what follows, an operator means a bounded linear operator on a complex Hilbert space H without specified.

Let T be an operator. For each fixed $y \in H$, consider a function f defined by

$$f(x) = (Tx,\ y) \text{ on } H.$$

According to Riesz's representation theorem, there exists uniquely $u \in H$ such that

$$f(x) = (Tx,\ y) = (x, u) \text{ for all } x \in H.$$

Definition

T^*, the adjoint operator of T, is defined by

$$(Tx,\ y) = (x,\ u) = (x,\ T^*y) \text{ for } x, y \in H.$$

Theorem

Let T be an operator on a Hilbert space H. Then T^* is also an operator on H, and the following properties hold:
(i) $\|T^*\| = \|T\|$.
(ii) $(T_1 + T_2)^* = T_1^* + T_2^*$.
(iii) $(\alpha T)^* = \overline{\alpha} T^*$ for any $\alpha \in C$.
(iv) $(T^*)^* = T$.
(v)$(ST)^* = T^*S^*$.

Proof

T^* is Linear

Let $y_1, y_2 \in H$ and $\alpha, \beta \in C$.
For any $x \in H$,

$$\begin{aligned}(x, T^*(\alpha y_1 + \beta y_2)) &= (Tx, \alpha y_1 + \beta y_2) \\ &= (Tx, \alpha y_1) + (Tx, \beta y_2) \\ &= \overline{\alpha}(Tx, y_1) + \overline{\beta}(Tx, y_2) \\ &= \overline{\alpha}(x, T^*y_1) + \overline{\beta}(x, T^*y_2) \\ &= (x, \alpha T^*y_1) + (x, \beta T^*y_2) \\ &= (x, \alpha T^*y_1 + \beta T^*y_2)\end{aligned}$$

Hence $T^*(\alpha y_1 + \beta y_2) = \alpha y_1 + \beta y_2$.

T^* is bounded

For $y \in H$,

$$\begin{aligned} \|T^*y\|^2 = (T^*y, T^*y) = (TT^*y, y) &\leq \|TT^*y\|\|y\| \leq \|T\|\|T^*y\|\|y\| \\ \Longrightarrow \|T^*y\| &= \|T\|\|y\| \\ \Longrightarrow \ \mathsf{sup}\{\|T^*y\|/\|y\| = 1\} &\leq \|T\| \\ \Longrightarrow \|T^*\| &\leq \|T\| \ldots (1) \end{aligned}$$

Hence T^* is bounded linear operator on H.

To prove (i)

If T is an operator, then T^* is also an operator with $\|T^*\| \leq \|T\|$.
Hence T^{**} is also an operator and $\|(T^*)^*\| \leq \|T^*\|$. ...(2)
For any $x, y \in H$,

$$(y, (T^*)^*x) = (T^*y, x) = \overline{(x, T^*y)} = \overline{(Tx, y)} = (y, Tx)$$

$$\implies (T^*)^* = T \ldots (3)$$

$$\text{From}(2) \text{ and } (3), \|T\| \leq \|T^*\| \ldots (4)$$

$$\text{From}(1) \text{ and } (4), \|(T^*)\| = \|T\|$$

Hence (i) is proved.

To prove (ii)

For $x, y \in H$,

$$\begin{aligned}
(x, (T_1 + T_2)^*) &= ((T_1 + T_2)x, y) \\
&= (T_1 x + T_2 x, y) \\
&= (T_1 x, y) + (T_2 x, y) \\
&= (x, T_1^* y) + (x, T_2^* y) \\
&= (x, T_1^* y + T_2^* y) \\
&= (x, (T_1^* + T_2^*) y)
\end{aligned}$$

$$(T_1 + T_2)^* = (T_1^* + T_2^*)$$

Hence (ii) is proved.

To prove (iii)

For $x, y \in H$, and $\alpha \in C$,

$$(x, (\alpha T)^* y) = ((\alpha T)x, y) = (\alpha Tx, y) = \alpha(Tx, y) = \alpha(x, T^* y) = (x, \overline{\alpha} T^* y)$$

$$\text{Hence } (\alpha T)^* = \overline{\alpha} T^*$$

To prove (iv)

From (3), $(T^*)^* = T$. Hence (iv) is proved.

To prove (v)

For $x, y \in H$,

$$(x, (ST)^*y) = ((ST)x, y) = (S(Tx), y) = (Tx, S^*y) = (x, T^*S^*y)$$

$$\text{Therefore } (ST)^* = T^*S^*.$$

Hence (v) is proved. Hence the theorem.

Corollary

Let T be an operator. Then
(i) $\|T^*T\| = \|TT^*\| = \|T\|^2$.
(ii) $T^*T = 0$ if and only if $T = 0$.

To prove (i)

Since $\|T^*\| = \|T\|$,

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2 \ldots .(1)$$

Conversely,

$$\|Tx\|^2 = (Tx,\, Tx) = (T^*Tx,\, x) \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2$$

Therefore for $x \in H$ with $\|x\| = 1$,

$$\|Tx\|^2 \leq \|T^*T\|$$

$$\implies \sup\{\|Tx\|^2 / \|x\| = 1\} \leq \|T^*T\|$$

$$\text{From (1) and (2), } \|T\|^2 = \|T^*T\| \dots (3)$$

Replacing T by T^* in (3), we get $\|T^*\|^2 = \|(T^*)^*T^*\|$...(4)
Since $\|T^*\| = \|T\|$, we get $\|T\|^2 = \|TT^*\|$...(5)

$$\text{From (3) and (5), } \|T^*T\| = \|TT^*\| = \|T\|^2.$$

To prove (ii)

Assume $T^*T = 0$. Then
$0 = ((T^*T)x, x) = (T^*(Tx), x) = (Tx, Tx) = \|Tx\|^2$.
Hence $Tx = 0$ for all $x \in H$ and hence $T = 0$.

2.1.3 Generalized polarization identity and its application

Definition (1)

A bilinear functional $f(x, y)$ on a complex vector space X is defined as follows:

$$f(x, y) = g_y(x) = h_x(y)$$

is a complex valued function with respect to x and y such that $g_y(x)$ is a linear functional on x and $h_x(y)$ is a conjugate linear functional on y, that is, $h_x(\alpha y) = \overline{\alpha} h_x(y)$ for any $\alpha \in C$.

Theorem (1)

If $f(x, y)$ is a bilinear functional on a complex vector space X, then

$$f(x, y) = \frac{1}{4}\{f(x + y, x + y) - f(x - y, x - y)\}$$

Theorem (2 Generalized polarization identity)

If T is an operator on a Hilbert space H, then

$$\begin{aligned}(Tx, y) &= \{(T(x+y), x+y) - (T(x-y), x-y)\} \\ &\quad + i\{(T(x+iy), x+iy) - (T(x-iy), x-iy)\}\end{aligned}$$

holds for any $x, y \in H$.

Proof:

Define f on $H \times H$ as

$$f(x, y) = (Tx, y) \text{ for all } x, y, \in H.$$

Now for fixed $y \in H$, define g_y on H as

$$g_y(x) = f(x,\ y) \text{ for all } x \in H.$$

Then for $x_1, x_2 \in H$,

$$\begin{aligned} g_y(x_1 + x_2) &= f(x_1 + x_2, y) \\ &= (T(x_1 + x_2), y) \\ &= (Tx_1 + Tx_2, y) \\ &= (Tx_1, y) + (Tx_2, y) \\ &= f(x_1, y) + f(x_2, y) \\ &= g_y(x_1) + g_y(x_2) \end{aligned}$$

Also for $\alpha \in C$ and $x \in H$,

$$\begin{aligned} g_y(\alpha x) &= f(\alpha x, y) \\ &= (T(\alpha x), y) \\ &= (\alpha Tx, y) \\ &= \alpha(Tx, y) \\ &= \alpha f(x, y) \\ &= \alpha g_y(x) \end{aligned}$$

Now for fixed $x \in H$, define h_x on H as

$$h_x(y) = f(x, y) \text{ for all } y \in H.$$

Then for $x_1, x_2 \in H$,

$$\begin{aligned} h_x(y_1 + y_2) &= f(x, y_1 + y_2) \\ &= (Tx, y_1 + y_2) \\ &= (Tx, y_1) + T(x, y_2) \\ &= f(x, y_1) + f(x, y_2) \\ &= h_x(y_1) + h_x(y_2) \end{aligned}$$

Also for $\alpha \in C$ and $y \in H$,

$$\begin{aligned} h_x(\alpha y) &= f(x, \alpha y) \\ &= (Tx, \alpha y) \\ &= \overline{\alpha}(Tx, y) \\ &= \overline{\alpha} f(x, y) \\ &= \overline{\alpha} h_x(y) \end{aligned}$$

Hence $h_x(y)$ is conjugate linear on y on H.
Therefore $f(x, y) = (Tx,\ y)$ is a bilinear functional on a Hilbert space H.
Therefore from the result, "If $f(x, y)$ is a bilinear functinal on a complex vector space X, then

$$\begin{aligned} f(x,y) &= \frac{1}{4}\{f(x+y,\ x+y) - f(x-y,\ x-y)\} \\ &\quad + i\frac{1}{4}\{f(x+iy,\ x+iy) - f(x-iy,\ x-iy)\} \end{aligned}$$

holds for any $x, y \in X$." we get

$$\begin{aligned} (Tx,y) &= \{(T(x+y),\ x+y) - (T(x-y),\ x-y)\} \\ &\quad + i\{(T(x+iy), x+iy) - (T(x-iy), x-iy)\} \end{aligned}$$

holds for any $x, y \in H$.

Theorem (3)

1f T is an operator on a Hilbert space H over the complex scalars C, then the following (i) , (ii) and (iii) are mutually equivalent:

(i) $T = 0$.

(ii) $(Tx, x) = 0$ for all $x \in H$.

(iii) $(Tx, y) = 0$ for all $x, y \in H$.

Proof:

Assume (ii) that $(Tx, x) = 0$ for all xinH.
Hence for all $x, y \in H$,

$$\begin{aligned}(Tx, y) &= \{(T(x+y), x+y) - (T(x-y), x-y)\} \\ &\quad + i\{(T(x+iy), x+iy) - (T(x-iy), x-iy)\} \\ &= 0\end{aligned}$$

Hence (ii) $\Longrightarrow$ (iii)
On the other hand, assume $(Tx, y) = 0$ for all $x, y \in H$.
Taking y=x, we get $(Tx, x) = 0$ for all xinH.
Hence (iii) $\Longrightarrow$ (ii)

Assume (i) that $T = 0$.
Then $Tx = 0$ for every xinH.
Hence $(Tx, y) = 0$ for every x, yinH.
Hence (i) $\implies$ (iii)
Conversely assume that $(Tx, y) = 0$ for every $x, y \in H$.
Then taking $y = Tx$, $(Tx, Tx) = 0$ for every $x \in H$.
$\implies \|Tx\|^2 = 0$ for every $x \in H$.
$\implies Tx = 0$ for every $x \in H$.
Hence $T = 0$ for every $x \in H$.
Hence (iii) $\implies$ (i) Therefore (i) $\implies$ (iii) $\implies$ (ii)
Hence (i) , (ii) and (iii) are mutually equivalent.

Definition

The special types of operators are defined as follows:

self-adjoint operator : $T^* = T$.

normal operator : $T^*T = TT^*$.

quasinormal operator : $T(T^*T) = (T^*T)T$.

projection operator : $T^2 = T$(idempotent) and $T^* = T$.

unitary operator : $T^*T = TT^* = I$.

isometry operator : $T^*T = I$.

positive operator (denoted by $T \geq 0$): $(Tx, x) \geq 0$ for all $x \in H$.

hyponormal operator : $T^*T \geq TT^*$,
where $A \geq B$ means $A - B \geq 0$ for self-adjoint operators A and B.

Theorem (4)

1f T is an operator on a Hilbert space H over the complex scalars C, then the following (i) , (ii) , (iii) and (iv) hold:

(i) T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

(ii) T is self-adjoint if and only if (Tx, x) is real for all $x \in H$.

(iii) T is unitary if and only if $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in H$.

(iv) T is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$ for all $x \in H$.

corollary 5

1f T is an operator on a Hilbert space H over the complex scalars C, then the following (i), (ii) and (iii) are equivalent:

(i) T is isometry.

(ii) $\|Tx\| = \|x\|$ for all $x \in H$.

(iii) $(Tx, Ty) = (x, y)$ for all $x, y \in H$.

Theorem (6 Cartesian form)

If T is an operator, there exist self-adjoint operators A and B such that $T = A + iB$. Necessarily $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$, respectively.

Proof

Define A and B as $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$.
Then $A^* = (\frac{1}{2}(T + T^*))^* = \frac{1}{2}(T^* + T) = A$
and $B^* = (\frac{1}{2i}(T - T^*))^* = \frac{1}{-2i}(T^* - T) = \frac{1}{2i}(T - T^*) = B$.
Hence A and B are both self-adjoint and
$A + iB = \frac{1}{2}(T + T^*) + i\frac{1}{2i}(T - T^*) = T$.
Conversely suppose that $T = C + iD$, where C and D are self-adjoint.
Then $T + T* = C + iD + C - iD = 2C$ and
$T - T* = C + iD - C + iD = 2iD$.
Thus $C = \frac{1}{2}(T + T^*) = A$ and$D = \frac{1}{2i}(T - T^*) = B$.
Hence the result.

2.1.4 Several properties on projection operator

A Hilbert space H can be decomposed into $H = M + M^{\perp}$.
By the theorem, "Let M be a closed subspace of a Hilbert space H. Any vector x in H can be uniquely represented as follows:

$$x = y + z \text{ where } y \in M \text{ and } z \in M^{\perp} \text{ ''}$$

for any $x \in H$, $x = y + z$, where $y \in M$ and $z \in M^{\perp}$.
Define $P : H \to H$ as $Px = y$
This transformation P defines a linear operator from H onto M.
This P is said to be an orthogonal projection of H onto M and it is denoted by P_M.

Definition

$R(T)$, the range of T, is defined by $R(T) = \{Tx : x \in H\}$, and $N(T)$, the kernel of T, is defined by $N(T) = \{x \in H : Tx = 0\}$

Theorem (1)

If P_M is a projection onto a closed subspace M of a Hilbert space H, then P_M is an operator such that $P_M^* = P_M$ and $P_M^2 = P_M$. Conversely if P is an operator such that $P^* = P$ and $P^2 = P$, then $M = R(P)$ is a closed subspace and $P = P_M$, i.e., P is a projection onto M.

Proof

Let P_M be a projection onto a closed subspace M of a Hilbert space H. To prove that P_M is an operator such that $P_M^* = P_M$ and $P_M^2 = P_M$.

To prove: P_M is linear

Let $x_1, x_2 \in H$ and $\alpha, \beta \in C$.
Since $H = M \oplus M^\perp$,
$x_1 = y_1 \oplus z_1$ and $x_2 = y_2 \oplus z_2$, where $y_1, y_2 \in M$ and $z_1, z_2 \in M^\perp$.
Then $P_M x_1 = y_1$, $P_M x_2 = y_2$ Consider

$$\begin{aligned} P_M(\alpha x_1 + \beta x_2) &= P_M(\alpha(y_1 \oplus z_1) + \beta(y_2 \oplus z_2)) \\ &= P_M((\alpha y_1 \oplus \alpha z_1) + (\beta y_2 \oplus \beta z_2)) \\ &= P_M((\alpha y_1 + \beta y_2) \oplus (\alpha z_1 + \beta z_2)) \\ &= (\alpha y_1 + \beta y_2) = \alpha P_M x_1 + \beta P_M x_2 \\ &\quad (\text{ Since } (\alpha y_1 + \beta y_2) \in M \text{ and } (\alpha z_1 + \beta z_2) \in M^\perp) \end{aligned}$$

Hence P_M is an linear operator.

To prove: P_M is bounded

Let $x \in H$. Then $x = y + z$, where $y \in M$ and $z \in M^{\perp}$.
Now by definition, $P_M x = y$.
Therefore, $\|P_M x\|^2 = \|y\|^2 \leq \|y\|^2 + \|z\|^2 = \|x\|^2$
Hence $\|P_M x\| \leq \|x\|$.
Hence P_M is bounded.
Therefore P_M is a bounded linear operator.

To prove $P_M^* = P_M$

Let $x_1, x_2 \in H$. Then $x_1 = y_1 \oplus z_1$ and $x_2 = y_2 \oplus z_2$, where $y_1, y_2 \in M$ and $z_1, z_2 \in M^{\perp}$.
Also $P_M x_1 = y_1$, $P_M x_2 = y_2$

$$\begin{aligned}
\langle P_M x_1, x_2\rangle &= \langle y_1, x_2\rangle \\
&= \langle y_1, y_2 + z_2\rangle \\
&= \langle y_1, y_2\rangle + \langle y_1, z_2\rangle \\
&= \langle y_1, y_2\rangle + 0 \\
&= \langle y_1, y_2\rangle + \langle z_1, y_2\rangle \\
&= \langle y_1 + z_1, y_2\rangle \\
&= \langle x_1, P_M x_2\rangle \\
&= \langle P_M^* x_1, x_2\rangle
\end{aligned}$$

Hence $P_M^* = P_M$.

To prove $P_M^2 = P_M$.

If $x \in H$, then $P_M x \in M \subseteq H$.
Therefore, $P_M(P_M x) = P_M x$
$\implies P_M^2 x = P_M x$.

Hence P_M is an operator such that $P_M^* = P_M$ and $P_M^2 = P_M$.

Conversely, assume that P is an operator such that $P = P^* = P^2$.
Let $M = R(P)$.
To prove: $M = R(P)$ is a closed subspace and $P = P_M$, i.e., P is a projection onto M.
Let x be a limit point of $M = R(P)$.
Hence there exists a sequence $\{Px_n\}$ of points in $M = R(P)$ such that

$$Px_n \to x.$$

$$\implies P^2x_n \to Px.(\text{ Since Pis continuous. })$$

$$\implies Px_n \to Px. \text{ (Since } P^2 = P.)$$

Hence $Px = x$.
Therefore $x \in R(P) = M$.
Hence $M = R(P)$ contains all its limit points.
Hence M is closed.

Consider

$$\begin{aligned}
\langle (I-P)x, Px\rangle &= \langle x - Px, Px\rangle \\
&= \langle x, Px\rangle - \langle Px, Px\rangle \\
&= \langle x, Px\rangle - \langle x, P^*Px\rangle \\
&= \langle x, Px\rangle - \langle x, P^2x\rangle \\
&= \langle x, Px\rangle - \langle x, Px\rangle \\
&= 0
\end{aligned}$$

Therefore $(I-P)(x) \perp Px$.
Hence $x = Px \oplus (I-P)x$, where $Px \in M$ and $(I-P)x \in M^{\perp}$.
Therefore $P_M x = Px$ for all $x \in M$.
Hence $P = P_M$ is a projection onto M. Hence the theorem.

Theorem (2)

If an operator P is a projection, then

(i) $\|x\|^2 = \|Px\|^2 + \|(I - P)x\|^2$.

(ii) $(Px, x) = \|Px\|^2 \leq \|x\|^2$.

(iii) $I \geq P \geq 0$.

Proof

Let P be a projection operator.

$$\implies \mathrm{P} = \mathrm{P}^* = \mathrm{P}^2.$$

To prove (i)

$$\begin{aligned} \|\mathrm{Px}\|^2 + \|(\mathrm{I}-\mathrm{P})\mathrm{x}\|^2 &= (\mathrm{Px}, \mathrm{Px}) + ((\mathrm{I}-\mathrm{P})\mathrm{x}, (\mathrm{I}-\mathrm{P})\mathrm{x}) \\ &= (\mathrm{Px}, \mathrm{Px}) + (\mathrm{x}, \mathrm{x}) - (\mathrm{x}, \mathrm{Px}) - (\mathrm{Px}, \mathrm{x}) + (\mathrm{Px}, \mathrm{Px}) \\ &= (\mathrm{P}^2\mathrm{x}, \mathrm{x}) + \|\mathrm{x}\|^2 - (\mathrm{Px}, \mathrm{x}) - (\mathrm{Px}, \mathrm{x}) + (\mathrm{P}^2\mathrm{x}, \mathrm{x}) \\ &= (\mathrm{Px}, \mathrm{x}) + \|\mathrm{x}\|^2 - 2(\mathrm{Px}, \mathrm{x}) + (\mathrm{Px}, \mathrm{x}) \\ &= \|\mathrm{x}\|^2 \end{aligned}$$

Hence (i) is proved.

To prove (ii)

For any $x \in H$,

$$\begin{aligned}(Px, x) = (P^2x, x) &= (P^*Px, x)\\ &= (Px, Px)\\ &= \|Px\|^2\\ &\leq \|Px\|^2 + \|(I-P)x\|^2\\ &= \|x\|^2\end{aligned}$$

Hence $(Px, x) = \|Px\|^2 \leq \|x\|^2$.

To prove (iii)

For any $x \in H$,

$$((I - P)x, x) = (x, x) - (Px, x) = \|x\|^2 - \|Px\|^2 \geq 0.$$

Hence $I - P \geq 0$. Therefore $I \geq P$.
Also $(Px, x) = \|Px\|^2 \geq 0$ and hence $P \geq 0$.
Therefore $I \geq P \geq 0$.

Theorem (3)

Let M_1 and M_2 be two closed subspaces, and let P_1 and P_2 be two projections onto M_1 and M_2, respectively. Then the following (i) and (ii) hold:

(i) $M_1 \perp M_2 \Leftrightarrow P_1P_2 = 0 \Leftrightarrow P_2P_1 = 0.$

(ii) $M_1 \subseteq M_2 \Leftrightarrow P_1P_2 = P_1 \Leftrightarrow P_2P_1 = P_1 \Leftrightarrow P_1 \leq P_2 \Leftrightarrow \|P_1x\| \leq \|P_2x\|$ for all $x \in H$.

Proof

To prove (i)

Let $M_1 \perp M_2$.
Since P_2 is a projection on M_2, for any $x \in H$, $P_2x \in M_2$.

$$\begin{aligned} &\Rightarrow \quad P_2x \in M_1^{\perp} (\text{ Since } M_1 \perp M_2, M_2 \subseteq M_1^{\perp}.) \\ &\Rightarrow \quad P_1(P_2x) = 0 \text{ (Since } P_1 \text{ is projection onto } M_1) \\ &\Rightarrow \quad P_1P_2x = 0, \text{ for all } x \in H. \\ &\Rightarrow \quad P_1P_2 = 0. \end{aligned}$$

Hence $M_1 \perp M_2 \Rightarrow P_1P_2 = 0 ...(1)$
Now $P_1P_2 = 0 \Leftrightarrow (P_1P_2)^* = 0^* \Leftrightarrow (P_2^*P_1^*) = 0^* \Leftrightarrow P_2P_1 = 0.$
Hence $P_1P_2 = 0 \Leftrightarrow P_2P_1 = 0.$ ...(2)
Now if $P_2P_1 = 0$,
then for any $x_1 \in M_1$, $P_2x_1 = P_2(P_1x_1) = P_2P_1x_1 = 0.$
$\implies \quad x_1 \in M_2^{\perp}.$
Therefore $M_1 \subseteq M_2^{\perp}$ and hence $M_1 \perp M_2$.
Hence $P_2P_1 = 0 \implies M_1 \perp M_2$. ...(3)

To prove (ii)

Assume that $M_1 \subseteq M_2$. Then for any $x \in H$,

$$\begin{aligned} & P_1 x \in M_1 \subseteq M_2. (\text{ Since } P_1 \text{ is projection onto } M_1.) \\ \implies & P_2(P_1 x) = (P_1 x)(\text{ Since } P_2 \text{ is projection onto } M_2.) \\ \implies & P_2 P_1 x = P_1 x, \text{ for all } x \in H. \\ \implies & P_2 P_1 = P_1. \end{aligned}$$

Hence $M_1 \subseteq M_2 \implies P_2 P_1 = P_1 ....(4)$

$$\begin{aligned} \text{Now } P_2 P_1 = P_1 & \Leftrightarrow (P_2 P_1)^* = P_1^* \\ & \Leftrightarrow P_1^* P_2^* = P_1^* \\ & \Leftrightarrow P_1 P_2 = P_1 \\ \text{Hence } P_2 P_1 = P_1 & \Leftrightarrow P_1 P_2 = P_1 \ldots (5) \end{aligned}$$

Let $P_1P_2 = P_1$. Then for any $x \in H$,

$$(P_1x, x) = \|P_1x\|^2 = \|P_1P_2x\|^2 \leq \|P_2x\|^2 = (P_2x\, x).$$

$$\implies (P_1x, x) \leq (P_2x\, x) \text{ and hence } P_1 \leq P_2.$$

Therefore $P_1P_2 = P_1 \implies P_1 \leq P_2$....(6)

Let $P_1 \leq P_2$. Then

$$\begin{aligned} (P_1x, x) &\leq (P_2x\, x), \text{ for all } x \in H. \\ \implies \|P_1x\|^2 &\leq \|P_2x\|^2 \text{ for all } x \in H. \\ \implies \|P_1x\| &\leq \|P_2x\| \text{ for all } x \in H. \\ \text{Hence } P_1 \leq P_2 &\implies \|P_1x\| \leq \|P_2x\| \text{ for all } x \in H.\ldots(7) \end{aligned}$$

Suppose that $\|P_1x\| \leq \|P_2x\|$ for all $x \in H$.
Then for any $x_1 \in M_1$, using the result,
" If P is a projection, then $\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$," we get

$$\|P_2x_1\|^2 + \|(I-P_2)x_1\|^2 = \|x_1\|^2 = \|Px_1\|^2 \leq \|P_2x_1\|^2.$$

$$\begin{aligned} &\Longrightarrow && \|(I-P_2)x_1\|^2 = 0. \\ &\Longrightarrow && (I-P_2)x_1 = 0. \\ &\Longrightarrow && x_1 - P_2x_1 = 0 \\ &\Longrightarrow && x_1 = P_2x_1 \in M_2. \end{aligned}$$

$$\Longrightarrow M_1 \subseteq M_2.$$

$$\text{Hence } \|P_1x\| \leq \|P_2x\| \text{ for all } x \in H. \Longrightarrow M_1 \subseteq M_2. \ldots (8)$$

From (4),(5),(6),(7)and (8), we get
$M_1 \subseteq M_2 \Leftrightarrow P_1P_2 = P_1 \Leftrightarrow P_2P_1 = P_1 \Leftrightarrow P_1 \leq P_2 \Leftrightarrow \|P_1x\| \leq \|P_2x\|$

From (4),(5),(6),(7)and (8), we get
$M_1 \subseteq M_2 \Leftrightarrow P_1P_2 = P_1 \Leftrightarrow P_2P_1 = P_1 \Leftrightarrow P_1 \leq P_2 \Leftrightarrow \|P_1x\| \leq \|P_2x\|$
for all $x \in H$.
Hence the theorem.

Theorem (4)

Let P_1 and P_2 be two projections onto M_1 and M_2 , respectively. Then

(i) $P = P_1P_2$ is a projection iff $P_1P_2 = P_2P_1$.

(ii) If $P_1P_2 = P_2P_1$, then $P = P_1P_2$ is a projection onto $M_1 \cap M_2$.

Proof

To prove (i)

Assume that $P = P_1P_2$ is a Projection. Then

$$\begin{aligned} P^* &= P. \\ \Longrightarrow (P_1P_2)^* &= P_1P_2 \\ \Longrightarrow P_2^*P_1^* &= P_1P_2 \\ \Longrightarrow P_2P_1 &= P_1P_2 \end{aligned}$$

C

onversely,assume that $P_1P_2 = P_2P_1$.
Then $P^* = (P_1P_2)^* = P_2^*P_1^* = P_2P_1 = P_1P_2 = P$.
Also

$$\begin{aligned} P^2 = (P_1P_2)(P_1P_2) &= P_1(P_2P_1)P_2 \\ &= P_1(P_1P_2)P_2 \\ &= (P_1P_1)(P_2P_2) \\ &= P_1^2P_2^2 \\ &= P_1P_2 \\ &= P \end{aligned}$$

Hence $P^* = P$ and $P^2 = P$.
Hence $P = P_1P_2$ is a projection.
Hence $P = P_1P_2$ is a projection iff $P_1P_2 = P_2P_1$. Hence (i) is proved.

To prove (ii)

Assume that $P_1P_2 = P_2P_1$.
Then by (i), $P = P_1P_2$ is a projection.
Let $x \in M_1 \cap M_2$.

$$\begin{aligned}
&\Longrightarrow && x \in M_1 \text{ and } x \in M_2 \\
&\Longrightarrow && x = P_1x \text{ and } x = P_2x \\
&\Longrightarrow && x = P_1x = P_2x \\
&\Longrightarrow && x = P_1(P_2x) = P_1P_2x \\
&\Longrightarrow && x \in R(P_1P_2) \\
&\Longrightarrow && M_1 \cap M_2 \subseteq R(P_1P_2)\ldots.(1)
\end{aligned}$$

Conversely,

$$R(P_1P_2) \subseteq R(P_1) = M_1.$$

$$\text{and } R(P_1P_2) = R(P_2P_1) \subseteq R(P_2) = M_2.$$

Hence $R(P_1P_2) \subseteq M_1 \cap M_2$ ...(2)
From (1) and (2), we get $R(P_1P_2) = M_1 \cap M_2$. Hence $P = P_1P_2$ is a projection onto $M_1 \cap M_2$.
Hence (ii) is proved.

Theorem (5)

Let P_1 and P_2 be two projections onto M_1 and M_2 , respectively such that $P_1P_2 = P_2P_1$.
Then $M_1 + M_2$ is a closed subspace and $P_1 + P_2 - P_1P_2$ is the projection onto $M_1 + M_2$.

Proof

Let P_1 and P_2 be two projections onto M_1 and M_2 such that $P_1P_2 = P_2P_1$.

$$\begin{aligned}
\text{Let } P &= P_1 + P_2 - P_1P_2. \\
\text{Then } P^* &= P_1^* + P_2^* - P_1^*P_2^*. \\
&= P_1 + P_2 - P_2P_1 \\
&= P_1 + P_2 - P_1P_2. \\
&= P
\end{aligned}$$

$$\begin{aligned}\text{Also } P^2 &= PP = (P_1 + P_2 - P_1P_2)(P_1 + P_2 - P_1P_2) \\ &= P_1^2 + P_1P_2 - P_1^2P_2 + P_2P_1 + P_2^2 - P_2P_1P_2 \\ &\quad - P_1P_2P_1 - P_1P_2^2 + (P_1P_2)^2\end{aligned}$$

Since P_1, P_2 and P_1P_2 are projections,

$$\begin{aligned}P_1^2 &= P_1, \\ P_2^2 &= P_2, \\ (P_1P_2)^2 &= P_1P_2, \\ P_2P_1P_2 &= P_1P_2P_2 = P_1P_2, \\ \text{and } P_1P_2P_1 &= P_1P_1P_2 = P_1P_2.\end{aligned}$$

$$\begin{aligned} \text{Hence } P^2 &= P_1 + P_1P_2 - P_1P_2 + P_2P_1 + P_2 - P_1P_2 \\ &\quad - P_1P_2 - P_1P_2 + P_1P_2 \\ &= P_1 + P_2 - P_1P_2 = P \end{aligned}$$

Hence $P^* = P$ and $P^2 = P$.
Hence $P = P_1 + P_2 - P_1P_2$ is a projection.
Since M_1 and M_2 are closed, $M_1 + M_2$ is a closed subspace of H.

To show that P is the projection onto $M_1 + M_2$.

Since P1 and P_2 are projections onto M_1 and M_2 respectively,

$$\begin{aligned} R(P_1) &= M_1, \\ R(P_2) &= M_2, \\ P_1x_1 &= x_1 \text{ for } x_1 \in M_1 \text{ and } P_2x_2 = x_2 \text{ for } x_2 \in M_2. \end{aligned}$$

$$\begin{aligned}
\text{Therefore } Px_1 &= (P_1 + P_2 - P_1P_2)x_1 \\
&= P_1x_1 + P_2x_1 - P_1P_2x_1 \\
&= P_1x_1 + P_2x_1 - P_2P_1x_1 \\
&= x_1 + P_2x_1 - P_2x_1 \\
&= x_1. \\
\text{Similarly } Px_2 &= x_2.
\end{aligned}$$

Therefore $x_1 + x_2 = Px_1 + Px_2 = P(x_1 + x_2) \in R(P)$.
$\implies M_1 + M_2 \subseteq R(P)$.
Conversely,
Since $P = P_1 + P_2 - P_1P_2 = P_1 + P_2 - P_2P_1 = P_1 + P_2(I - P_1)$
$R(P) \subseteq R(P_1) + R(P_2(I - P_1)) = R(P_1) + R(P_2) = M_1 + M_2$.
Hence $R(P) = M_1 + M_2$
Hence P is the projection onto $M_1 + M_2$. Hence the theorem.

Theorem (6)

Let P_1 and P_2 be two projections onto M_1 and M_2 , respectively. Then

(i) $P = P_1 + P_2$ is a projection iff $M_1 \perp M_2$.

(ii) If $P = P_1 + P_2$ is a projection, then P is the projection onto $M_1 \oplus M_2$.

To prove (i)

Let P_1 and P_2 be two projections onto M_1 and M_2 , respectively.
If $P = P_1 + P_2$ is a projection, then

$$\begin{aligned} P^2 &= P \\ \implies (P_1+P_2)^2 &= P_1+P_2 \\ \implies P_1^2+P_2P_1+P_1P_2+P_2^2 &= P_1+P_2 \\ P_2P_1+P_1P_2 &= 0 \ldots (1) \end{aligned}$$

$$
\begin{aligned}
\Longrightarrow \; P_2(P_2P_1 + P_1P_2)P_2 &= 0 \\
\Longrightarrow \; P_2^2P_1P_2 + P_2P_1P_2^2 &= 0 \\
\Longrightarrow \; P_2P_1P_2 + P_2P_1P_2 &= 0 \\
\Longrightarrow \; 2P_2P_1P_2 &= 0 \\
\Longrightarrow \; P_2P_1P_2 &= 0 \\
\Longrightarrow \; P_2P_1P_1P_2 &= 0 \\
\Longrightarrow \; (P_1P_2)^*(P_1P_2) &= 0 \\
\Longrightarrow \; (P_1P_2)^2 &= 0 \\
\Longrightarrow \; P_1P_2 &= 0 \\
\Longrightarrow \; M_1 &\perp M_2
\end{aligned}
$$

Conversely, let $M_1 \perp M_2$. Then

$$\begin{aligned} P_1P_2 &= \; = 0 = P_2P_1 \\ \text{Therefore } P^2 &= (P_1+P_2)^2 \\ &= P_1 + P_1P_2 + P_2P_1 + P_2 \\ &= P_1 + P_2 = P \\ \text{Also } P^* = (P_1+P_2)^* & \\ &= P_1^* + P_2^* \\ &= P_1 + P_2 = P \end{aligned}$$

Hence P is a projection. Hence(i) is proved.

Let $P = P_1 + P_2$ is a projection, then by (i) $M_1 \perp M_2$.
Therefore to prove that P is a projection onto $M_1 \oplus M_2$,
it is enough to prove that $M = R(P) = M_1 + M_2$.
Let $y \in M = R(P)$

$$\begin{aligned} \implies y &= Px, \text{ for some } x \in H. \\ \implies y &= (P_1 + P_2)x = P_1x + P_2x \in M_1 + M_2, \\ \implies y &\in M_1 \oplus M_2. \text{ (Since } M_1 \perp M_2.) \\ \text{Hence } M &\subseteq M_1 \oplus M_2 \ldots (1) \end{aligned}$$

Conversely, for any $x = x_1 \oplus x_2 \in M_1 \oplus M_2$,

$$\begin{aligned} Px &= P_1x \oplus P_2x \\ &= P_1(x_1 + x_2) + P_2(x_1 + x_2) \\ &= P_1x_1 + P_1x_2 + P_2x_1 + P_2x_2 \\ &= P_1x_1 + P_2x_2 \\ &= x_1 + x_2 \\ &= x \end{aligned}$$

$\implies x \in R(P) \subseteq M$. ...(2)
Hence $M_1 \oplus M_2 \subseteq M$.
(1) and (2) $\implies M = M_1 + M_2$.
Hence $P = P_1 + P_2$ is the projection onto $M_1 + M_2$.
Hence (ii) is proved. Hence the theorem.

2.1.5 Generalized Schwarz inequality and square root of positive operator

Definition (1)

A sequence $\{Tn\}$ of operators on a Hilbert space H is said to be uniformly operator convergent if there exists an operator T such that

$$\|T_n - T\| \to 0 \text{ as } n \to \infty,$$

and denoted briefly by $T_n \implies T(u)$.

A sequence $\{Tn\}$ of operators on a Hilbert space H is said to be strongly operator convergent if there exists an operator T such that

$$\|T_n x - Tx\| \to 0 \text{ for all } x \in H \text{ as } n \to \infty,$$

and denoted briefly by $T_n \implies T(s)$.

Definition (1)

A sequence {Tn} of operators on a Hilbert space H is said to be weakly operator convergent if there exists an operator T such that

$$(T_n x, y) - (Tx, y) \to 0 \text{ for all } x, y \in H \text{ as } n \to \infty,$$

and denoted briefly by $T_n \implies T(w)$.

Remark

$$T_n \implies T(u) \text{implies} T_n \implies T(s),$$

and

$$T_n \implies T(s) \text{implies} T_n \implies T(w).$$

Definition (2)

Let A be an operator on a Hilbert space H and denote (A) by

$$(A) = \{B : AB = BA, \text{ where} B \text{ is an operator on } H\}.$$

Remark

(i) $(A^n) \supseteq (A)$ for any natural number n.
(ii) $(p(A)) \supseteq (A)$ holds for any polynomial p(t) on t.

Definition (3)

A sequence $\{A_n\}$ of self-adjoint operators is said to be bounded monotone increasing if there exists an operator A such that

$$A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq A.$$

A sequence $\{A_n\}$ of self-adjoint operators is said to be bounded monotone decreasing if there exists an operator A such that

$$A_1 \geq A_2 \geq \cdots \geq A_n \geq \cdots \geq A.$$

Theorem (1(Generalized Schwarz inequality))

If A is a positive operator on a Hilbert space H, then

$$|(Ax, y)|^2 \leq (Ax, x)(Ay, y) \text{ for any } x, y \in H.$$

Proof

Put $[x, y] = (Ax, y)$, for all $x, y \in H$.
Then for all $x, y\, z \in H$,
(1)$[x, x] = (Ax, x) \geq 0$, for all $x \in H$.
(2)$[y, x] = (Ay, x) = \overline{(x, Ay)} = \overline{(A^*x, y)} = \overline{(Ax, y)} = \overline{[x, y]}$
(3)$[x+y, z] = (A(x+y), z) = (Ax+Ay, z) = (Ax, z)+(Ay, z) = [x, z]+[y, z]$
(4)$[\lambda x, y] = (A(\lambda x), y) = (\lambda Ax, y) = \lambda(x, y) = \lambda[x, y]$
Hence [ ] satisfies the conditions of inner product except that
$[x, x] = 0 \implies x = 0$,
since $[x, x] = 0 \implies (Ax, x) = 0 \implies A = 0$ but not $x = 0$.

Let $y \neq 0$ and $\lambda \in C$.

$$\begin{aligned} 0 \leq \|x + \lambda y\|^2 &= [x + \lambda y, x + \lambda y] \\ &= [x, x] + [x, \lambda y] + [\lambda y, x] + [\lambda y, \lambda y] \\ &= \|x\|^2 + \overline{\lambda}[x, y] + \lambda[y, x] + \lambda\overline{\lambda}\|y\|^2 \end{aligned}$$

Taking $\lambda = -\frac{[x, y]}{\|y\|^2}$, we get

$$\begin{aligned}
0 &\leq \|x\|^2 + \overline{\left(-\frac{[x, y]}{\|y\|^2}\right)}[x, y] + \left(-\frac{[x, y]}{\|y\|^2}\right)[y, x] + \left(-\frac{[x, y]}{\|y\|^2}\right)\overline{\left(-\frac{[x, y]}{\|y\|^2}\right)} \\
&= \|x\|^2 - \frac{|[x, y]|^2}{\|y\|^2} - \frac{|[x, y]|^2}{\|y\|^2} + \frac{|[x, y]|^2}{\|y\|^2} \\
&= \|x\|^2 - \frac{|[x, y]|^2}{\|y\|^2} \\
&\implies \frac{|[x, y]|^2}{\|y\|^2} \leq \|x\|^2 \\
&\implies |[x, y]|^2 \leq \|x\|^2\|y\|^2 \\
&\implies |[x, y]|^2 \leq [x, x][y, y] \\
&\implies |(Ax, y)|^2 \leq (Ax, x)(Ay, y)
\end{aligned}$$

Hence the inequality.

Theorem (2)

If a sequence $\{An\}$ of self- adjoint operators is bounded monotone increasing, then there exists a self-adjoint operator A such that $A_n \implies A(s)$, that is, A_n strongly converges to A.

Proof

Assume that $\{A_n\}$ is a sequence of self-adjoint bounded monotone increasing operators.
To prove that $A_n \implies A(s)$.
It is sufficient to prove the result in the case

$$0 \leq A_1 \leq A_2 \leq \cdots \leq I.$$

Since H is complete, every cauchy sequence in H converges in H.
Hence it is sufficient to prove that $\{A_n x\}$ is a cauchy sequence.
i.e $\|A_n x - A_m x\| \to 0$ as $m, n \to \infty$, for all $x \in H$.

Assume $n > m$. Using Generalized Schwarz inequality, we get

$$\begin{aligned}\|A_n x - A_m x\|^4 &= ((A_n - A_m)x, (A_n - A_m)x))^2 \\ &\leq ((A_n - A_m)x, x)\,((A_n - A_m)(A_n - A_m)x, (A_n - A_m)x) \\ &\leq ((A_n - A_m)x, x)\,((A_n - A_m)x, (A_n - A_m)x) \\ &= ((A_n - A_m)x, x)\,\|(A_n - A_m)x\|^2 \ldots (1)\end{aligned}$$

Therefore $\|A_n x - A_m x\|^2 \leq ((A_n - A_m)x, x) = (A_n x, x) - (A_m x, x)$.
Since $A_m \leq A_n \leq I$,
$\{(A_n x, x)\}$ and $\{(A_m x, x)\}$ are monotone increasing sequences and their bound is (x, x).
Hence $(A_n x, x) \to (x, x)$ as $n \to \infty$, for all $x \in H$,
and $(A_m x, x) \to (x, x)$ as $m \to \infty$, for all $x \in H$.
Hence $\|A_n x - A_m x\|^2 \to 0$ as $n, m \to \infty$, for all $x \in H$.
i.e $\|A_n x - A_m x\| \to 0$ as $n, m \to \infty$, for all $x \in H$.
Hence there exists an operator A on H such that $A_n \implies A(s)$.
Hence the theorem.

Theorem (3.Square root of a positive operator)

For any positive operator A, there exists the unique positive operator S such that $S^2 = A$ and $(S) \supseteq (A)$ (denoted by $S = A^{\frac{1}{2}}$) .

Proof

Assume that $0 \leq A \leq I$.
Let S_k be defined as follows:
For $k = 1, 2, \ldots$

$$\begin{aligned} S_0 &= 0 \\ \text{and} S_{k+1} &= S_k + \frac{1}{2}(A - S_k^2) \ldots (1) \end{aligned}$$

Since S_n is a polynomial of A,
S_n is a self-adjoint operator such that $(S_n) \supseteq (A)$.

claim: $I \geq S_{k+1}$, for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
S_1 &= S_0 + \frac{1}{2}(A - S_0^2) \\
&= \frac{1}{2}A \\
\text{Therefore } I - S_1 &= I - \frac{1}{2}A \geq 0 \\
\text{Assume that } S_k &\leq I \\
\text{Consider } I - S_{k+1} &= I - (S_k + \frac{1}{2}(A - S_k^2)) \\
&= I - S_k - \frac{1}{2}A + \frac{1}{2}S_k^2 \\
&= \frac{1}{2}[2I - 2S_k - A + S_k^2] \\
&= \frac{1}{2}[(I - S_k)^2 + (I - A)] \ldots (2) \\
&\geq 0
\end{aligned}$$

To prove $S_{k+1} \geq S_k$ for $k = 0, 1, 2 \dots$

$$\begin{aligned}
\text{Consider } S_1 - S_0 &= \frac{1}{2}(A - S_0^2) = \frac{1}{2}A \geq 0 \\
\text{Therefore } S_1 &\geq S_0 \\
\text{Assume that } S_k &\geq S_{k-1} \text{ for some positive integer k.} \\
\text{Consider } S_{k+1} - S_k &= (I - S_k) - (I - S_{k+1}) \\
&= \frac{1}{2}[(I - S_{k-1})^2 + (I - A)] - \frac{1}{2}[(I - S_k)^2 + (I - A)] \\
&= \frac{1}{2}[(I - S_{k-1})^2 - (I - S_k)^2] \\
&= \frac{1}{2}[(I - S_{k-1}) + (I - S_k)][(I - S_{k-1}) - (I - S_k)] \\
&= \frac{1}{2}[(I - S_{k-1}) + (I - S_k)][(S_k - S_{k-1})] \\
&\geq 0.
\end{aligned}$$

Hence by induction,

$$S_{k+1} \geq S_k \text{ for } k = 0, 1, 2 \dots \quad \dots (4)$$

From (3) and (4),

$$0 = S_0 \leq S_1 \leq S_2 \leq \cdots \leq I \quad \ldots(5)$$

Hence $\{S_k\}$ is a sequence of self-adjoint bounded monotone increasing operators. Hence by the theorem,
"If a sequence $\{An\}$ of self- adjoint operators is bounded monotone increasing, then there exists a self-adjoint operator A such that $A_n \implies A(s)$, that is, A_n strongly converges to A."
$\{S_k\}$ has a limit S.
Therefore as $k \to \infty$, in (1)
i.e in $S_{k+1} = S_k + \frac{1}{2}(A - S_k^2)$, we get

$$\begin{aligned} S &= S + \frac{1}{2}(A - S^2) \\ \implies \quad S^2 &= A \quad \ldots(6) \\ \text{Since each } S_k &\geq 0, \quad S \geq 0. \quad \ldots(7) \\ \text{Since each } (S_k) &\supseteq (A), \quad (S) \supseteq (A). \quad \ldots(8) \end{aligned}$$

To prove that S is unique

Assume that there exists two positive operators S_1 and S_2 such that $S_1^2 = A$, $S_2^2 = A$ and $(S_1) \supseteq (A)$ and $(S_2) \supseteq (A)$.
Consider $S_2A = S_2S_2^2 = S_2^2S_2 = AS_2$. ...(9)
$\implies \quad S_2 \in (A) \subseteq (S_1)$.
$\implies \quad S_1S_2 = S_2S_1$.
Therefore
$(S_1 + S_2)(S_1 - S_2) = S_1^2 + S_2S_1 - S_1S_2 - S_2^2 = S_1^2 - S_2^2 = A - A = 0$.
...(10)
Since $S_1 \geq 0$ and $S_2 \geq 0$ (by (9)), there exists two positive operators R_1 and R_2 such that

$$R_1^2 = S_1 \text{and} R_2^2 = S_2.$$

Consider

$$\begin{aligned} \|R_1y\|^2 + \|R_2y\|^2 &= (R_1y, R_1y) + (R_2y, R_2y) \\ &= (R_1^2y, y) + (R_2^2y, y) \\ &= (S_1y, y) + (S_2y, y) \end{aligned}$$

Put $y = (S_1 - S_2)x$, for any $x \in H$, then

$$
\begin{aligned}
\|R_1 y\|^2 + \|R_2 y\|^2 &= ((S_1 + S_2)(S_1 - S_2)x,\ (S_1 - S_2)x) \\
&= (0,\ (S_1 - S_2)x) \\
&= 0, \text{for any } x \in H. \\
\Longrightarrow \quad \|R_1 y\| &= 0 \text{ and } \|R_2 y\| = 0 \\
\Longrightarrow \quad R_1 y &= 0 \text{ and } R_2 y = 0 \\
\Longrightarrow \quad R_1 y &= 0 \text{ and } R_2 y = 0 \\
\Longrightarrow \quad S_1 y = R_1^2 y &= 0 \text{ and } S_2 y = R_2^2 y = 0 \\
\text{Therefore } \|(S_1 - S_2)x\|^2 &= ((S_1 - S_2)x\,(S_1 - S_2)x) \\
&= ((S_1 - S_2)(S_1 - S_2)x, x) \\
&= ((S_1 - S_2)y, x) \\
&= (0,\ x) = 0,\ \text{ for all } x \in H. \\
\Longrightarrow \quad (S_1 - S_2) &= 0 \\
\Longrightarrow \quad S_1 &= S_2
\end{aligned}
$$

Corollary

If $A \geq 0$ and $B \geq 0$ such that A commutes with B, then $AB \geq 0$.

Proof

Since $A \geq 0$, there exits a unique operator S such that $S \geq 0$, $S^2 = A$ and $(S) \supseteq (A)$.
Since A commutes with B,

$$B \in (A) \subseteq (S).$$

$\implies$ S commutes with B. ie., $SB = BS$.
Since $B \geq 0$, $(Bx, x) \geq 0$ for all $x \in H$.
Therefore for any $x \in H$

$$(ABx, x) = (S^2Bx, x) = (SBx, Sx) = (BSx, Sx) \geq 0.$$

$\implies AB \geq 0$.
Hence the result.

2.1.6 From diagonalization of self-adjoint matrix to spectral representation of self-adjoint operator

Theorem

For any self-adjoint matrix A, there exists a suitable unitary matrix U such that $A = U\Lambda U^*$, where Λ is a diagonal matrix.

Proof

The proof is by induction on the dimension n of matrix A.
(i) When $n = 1$, the result is obvious.
(ii) Assume that the result holds for $n - 1$. i.e., for a self-adjoint matrix B of dimension $n - 1$, there exists a suitable unitary matrix Q such that

$$B = QMQ^* \ldots (1)$$

where M is a diagonal matrix.

Let A be a self-adjoint matrix of dimension n.
Choose an eigenvalue λ_1 of A.
Let e_1 be the normalized eigenvector

$$e_1 = \begin{pmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{pmatrix}$$

corresponding to λ_1.
Take a system $\{e_1, f_2, \ldots, f_n\}$ of linearly independent vectors, and make a system $\{e_1, e_2, \ldots, e_n\}$ of orthonormal vectors by Schmidt orthonormal procedure.

$$\text{Let } P_1 = (e_1, e_2, \ldots, e_n) = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

Then P_1 is a unitary matrix.

and

$$\begin{aligned} P_1^*AP_1 &= \begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \dots & \overline{p_{n1}} \\ \overline{p_{12}} & \overline{p_{22}} & \dots & \overline{p_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{p_{1n}} & \overline{p_{2n}} & \dots & \overline{p_{nn}} \end{pmatrix} \begin{pmatrix} \lambda_1 p_{11} & * & \dots & * \\ \lambda_1 p_{21} & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & * & \dots & * \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix} \end{aligned}$$

As $P_1^*AP_1$ is self-adjoint, the right hand side turns out to be

$$P_1^*AP_1 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

By the hypothesis of induction, we can write $B = QMQ^*$, where Q is a unitary matrix and M is a diagonal one.

$$\text{Put } P_2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{pmatrix}$$

P_2 is also unitary since Q is unitary, and we have

$$A = P_1 \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix} P_1^* = P_1 \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & QMQ^* & \\ 0 & & & \end{pmatrix} P_1^*$$

$$\Longrightarrow A = P_1 \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & Q^* & \\ 0 & & & \end{pmatrix} P_1^*$$

$$\Longrightarrow A = P_1 P_2 \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M & \\ 0 & & & \end{pmatrix} (P_1 P_2)^*,$$

Since P_1 and P_2 are unitary matrices, P_1P_2 is also a unitary matrix.
Since M is a diagonal matrix,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M & \\ 0 & & & \end{pmatrix}.$$

Hence there exists a suitable matrix U such that $A = U\Lambda U^*$, where Λ is a diagonal matrix.
So the proof is complete for a self-adjoint matrix A with dimension n.
Hence the theorem

Remark 1

By the above theorem, if A is a self-adjoint matrix, then A can be decomposed into,

$$A = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} U^* \quad \ldots(1)$$

where $U = (u_1, u_2, \ldots, u_n)$ is a Unitary Matrix and u_j is the normalized eigenvector which corresponds to the eigenvalue λ_j of A for $j = 1, 2, \ldots, n$. (1) can be represented as follows:

$$A = \lambda_1 U \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} U^* + \lambda_2 U \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} U^*$$

$$+ \cdots + \lambda_n U \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} U^*$$

Put $P_1 = U \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} U^*$, $P_2 = U \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} U^*$

...... and $P_n = U \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} U^*$.

Then $P_1, P_2, \dots, P_n$ are projections and

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = \sum_{j=1}^{n} \lambda_j P_j.$$

If we put $E_1 = U \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} U^*$,

$E_2 = U \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} U^*$,

$\cdots$

$E_n = U \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} U^*$,

then $E_1, E_2, \ldots, E_n$ are projections and

$$A = \lambda_1 E_1 + \lambda_2(E_2 - E_1) + \cdots + \lambda_n(E_n - E_{n-1}) = \sum_{j=1}^{n} \lambda_j \Delta E_j,$$

Hence if A ia self-adjoint operator on a Hilbert space H, then A can be expressed as follows:

$$A = \int \lambda dE_\lambda$$

where $\{E_\lambda / \lambda \in R\}$ is a family of projections such that

$$\begin{aligned} E_\lambda &\leq E_\mu \text{if } \lambda \leq \mu \\ E_{\lambda+0} &= E_\lambda, \\ E_{-\infty} &= 0 \\ E_\infty &= I \end{aligned}$$

Thank You