Operator Theory

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Chapter II FUNDAMENTAL PROPERTIES OF BOUNDED LINEAR OPERATORS

2.2 Partial Isometry operator and Polar Decomposition of an operator

2.2.1 Partial isometry operator and its characterization

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Definition (1)

An operator U on a Hilbert space H is said to be an isometry operator if

$$\|Ux\| = \|x\| \text{for any } x \in H.$$

This is equivalent to,

$$< Ux, Uy >=< x, y > \text{ for any } x\ , y \in H.[\text{by polarization identity}]$$

Definition

An operator U on a Hilbert space H is said to be a unitary operator if U is an isometry operator from H onto H.

Theorem (1)

(i) An operator U on a Hilbert space H is an isometry operator iff $U^*U = I$.

(ii) An operator U on a Hilbert space H is a unitary operator iff $U^*U = UU^* = I$.

Proof

To prove (i)

Let U be an isometry operator on H

$$\Rightarrow \|Ux\| = \|x\|, \text{ for all } x \in H.$$

Hence by polarization identity,

$$\begin{aligned} < Ux, Uy > &= < x, y >, \forall x,\ y \in H \\ \Rightarrow < U^*Ux, y > &= < x, y >, \forall x,\ y \in H \\ \Rightarrow U^*U &= I \end{aligned}$$

Conversely, assume that $U^*U = I$
Hence

$$\|Ux\|^2 = < Ux, Ux > = < U^*Ux, x > = < x, x > = \|x\|^2$$

$$\Rightarrow \|Ux\| = \|x\|.$$

Hence an operator U on a Hilbert space H is an isometry operator iff $U^*U = I$.

To prove (ii)

Let U be an Unitary operator on H.

$$\Rightarrow \text{U is an isometry operator from H onto H.}$$
$$\Rightarrow \|Ux\| = \|x\|, \forall x \in H.$$

From (i), $U^*U = I$ (1)

Since U is onto, for any $x \in H$, there exists $y \in H$ such that $Uy = x$.

$$\text{Therefore} U^*x = U^*Uy = y \text{ and } \|U^*x\| = \|y\| = \|Uy\| = \|x\|$$

Hence U^* is an isometry on H.

Hence by (i),$(U^*)^*U^* = I$

$$\Rightarrow UU^* = I \tag{2}$$

From (1)and (2),

$$U^*U = UU^* = I$$

Conversely, if $U^*U = UU^* = I$, then U is isometry. (by (i))

For any $x \in H$,

$$x = UU^*x = U(U^*x) \in R(U)$$
$$\Rightarrow R(U) = H$$

Hence U is an isometry from H onto H.

Hence U is an unitary operator.

Hence the theorem.

Definition (2)

An operator U on a Hilbert space H is said to be a partial isometry operator if there exists a closed subspace M such that

$$\|Ux\| = \|x\| \text{for any } x \in M$$

and

$$Ux = 0 \text{ for any } x \in M^{\perp},$$

where M is said to be the initial space of U and N = R(U) is said to be the final space of U.
The Projection onto the initial space is said to be the initial projection and the final space is said to be the final projection of U, respectively.

Remark
(1) U is isometry iff U is partial isometry and M = H.
(2) U is unitary iff U is partial isometry and M = N = H.

Theorem (2)

Let U be a partial isometry operator on a Hilbert space with the initial space M and the final space N. Then the following (i), (ii) and (iii) hold

(i) $UP_M = U$ and $U^*U = P_M$

(ii) N is a closed subspace of H.

(iii) U^* is a partial isometry with the inial space N and the final space M, that is

$$U^*P_N = U^* \text{ and } UU^* = P_N$$

Proof

Let U be a partial isometry operator on a Hilbert space H with the initial space M and the final space N.

$\Rightarrow \|Ux\| = \|x\|, \forall\ x \in M$ (1)

$Ux = 0, \forall\ x \in M^\perp$ (2)

and $N = R(U)$ (3)

(1)$\Rightarrow < Ux, Uy > = < x, y >, \forall\ x,\ y \in M$ (4)

To prove (i)

For any $x \in H$, $x = P_M x \oplus z$, for some $z \in M^{\perp}$

$$\text{Therefore } Ux = UP_M x \oplus Uz = UP_M x [\because \ z \in M^{\perp} \Rightarrow Uz = 0]$$

$$\text{Hence for any} x \in H, UP_M x = Ux \Rightarrow UP_M = U (5)$$

Now for any x, $y \in H$

$$\begin{aligned} < U^*Ux, y > &= < Ux, Uy > \\ &= < UP_M x, UP_M y > \qquad [\text{from } (5)] \\ &= < P_M x, P_M y > \qquad [\because P_M x, P_M y \in M \text{ and } (4)] \\ &= < P_M^2 x, y > \\ &= < P_M x, y > \qquad [\because P_M \text{ is projection}] \end{aligned}$$

$$\Rightarrow U^*U = P_M \tag{6}$$

Hence (i) is proved.

To prove (ii)

To prove that N is a closed subspace of H.
Let x be a limit point of $N = R(U)$.(i.e) $x \in \overline{N}$. Now,

$$\begin{aligned} N = R(U) &= R(UP_M) \qquad [by(5)] \\ &= UR(P_M) \\ &= UM \end{aligned}$$

Hence there exists a sequence $\{Uy_n\} \subseteq N$ such that $Uy_n \to x$, where
$\{y_n\} \subset M$ (7)
Hence $\|y_n - y_m\| = \|U(y_n - y_m)\| = \|Uy_n - U_ym\| \to 0$ as $m, n \to \infty$
$\Rightarrow \{y_n\}$ is a cauchy sequence in $M \subset N$.
Since H is complete, $\{y_n\}$ converges in H.
Let $y_n \to y$, Then $Uy_n \to Uy$ (8)
From (7)and (8),

$$x = Uy \in R(U) = N$$

Hence N contains all its limit points. Hence N is a closed subspace H.

To prove (iii)

Since $N = R(U)$, for any $x \in N$, there exists $y \in M$ such that $Uy = x$,
Since $y \in M$, $\|Uy\| = \|y\|$.
Hence $\|x\| = \|Uy\| = \|y\|$ and

$$\begin{aligned} U^*x &= U^*Uy = P_M y \qquad [\text{by}(5)] \\ &= y \qquad [\because y \in M] \end{aligned}$$

Hence $\|U^*x\| = \|y\| = \|x\|$
Hence $\|U^*x\| = \|x\|, \forall\ x \in N$ (9)
For any $x \in N^{\perp}$ and $y \in H$, Consider $< U^*x, y >=< x, Uy >= 0$
(because $x \in N^{\perp}$ and $Uy \in R(U) = N$)
$\Rightarrow U^*x = 0$, for all $x \in N^{\perp}$ (10)

$$\text{Now } R(U^*) = U^*N = U^*R(U) = U^*UH = P_M H = M$$

Hence $R(U^*) = M$ (11)
From (9), (10), (11), U^* is a partial isometry with the initial space N and the final space M.
Hence by (1), $U^*P_N = U^*$ and $(U^*)^*U^* = P_N$.(i.e) $UU^* = P_N$
Hence (iii) is proved

Theorem (3)

Let U be an operator on a Hilbert space H. Then the following statements are mutually equivalent.

(α) U is a partial isometry operator.

(α^*) U^* is a partial isometry operator.

(β) $UU^*U = U$.

(β^*) $U^*UU^* = U^*$.

(γ) U^*U is a projection operator.

(γ^*) UU^* is a projection.

Proof

Assume (α) i.e U is a partial isometry operator.
By theorem, " Let U be a partial isometry operator on a Hilbert space H with the initial space M and the final space N. Then

(i) $UP_M = U$ and $U^*U = P_M$

(ii) N is a closed subspace of H.

(iii) U^* is a partial isometry with the intial space N and the final space M, that is

$$U^*P_N = U^* \text{and} UU^* = P_N$$

$UP_M = U$ and $U^*U = P_M$
$\Rightarrow UU^*U = UP_M = U$
Hence $(\alpha) \Rightarrow (\beta)$ (1)

Assume (γ) i.e U^*U is a projection operator.
Put $U^*U = P_M$ then for any $x \in H$,
$\|Ux\|^2 = < Ux, Ux > = < U^*Ux, x > = < P_M x, x > = \|P_M x\|^2$
$\|Ux\|^2 = \|P_M x\|^2 = \|x\|^2, \ \forall x \in M$
and $\|Ux\| = \|P_M x\| = 0, \ \forall x \in M^\perp$
i.e $Ux = 0, \ \forall x \in M^\perp$
Hence $\|Ux\| = \|x\|, \ \forall x \in M$ and $\forall x \in M^\perp$
Hence U is a partial isometry on M.
Hence $(\gamma) \Rightarrow (\alpha)$ (3)
From (1), (2) and (3),
$(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\alpha)$. (a)

Similarly if U^* is a partial isometry operator,
by $(\alpha) \Rightarrow (\beta)$, $U^*(U^*)^*U^* = U^*$
i.e $U^*UU^* = U^*$
Hence $(\alpha^*) \Rightarrow (\beta^*)$ (4)
by $(\beta) \Rightarrow (\gamma)$, $U^*UU^* = U^* \Rightarrow UU^*$ is a projection
Hence $(\beta^*) \Rightarrow (\gamma^*)$ (5)
by $(\gamma) \Rightarrow (\alpha)$, UU^* is a projection operator
$\Rightarrow U^*$ is a partial isometry operator.
Hence $(\gamma^*) \Rightarrow (\alpha^*)$ (6)
From (4), (5) and (6),
$(\alpha^*) \Rightarrow (\beta^*) \Rightarrow (\gamma^*) \Rightarrow (\alpha^*)$. (b)

Assume (β) $\quad \mathrm{UU^*U = U}$
Taking adjoint on both sides
$(\mathrm{UU^*U})^* = \mathrm{U}^*$
$\Rightarrow \mathrm{U^*UU^* = U^*}$
Hence $(\beta) \Rightarrow (\beta^*)$
Similarly $\mathrm{U^*UU^* = U^*}$
$\Rightarrow (\mathrm{U^*UU^*})^* = (\mathrm{U}^*)^*$
$\Rightarrow \mathrm{UU^*U = U}$
Hence $(\beta^*) \Rightarrow (\beta)$
Hence $(\beta) \iff (\beta^*)$ (c)
From (a), (b), (c), it is clear that, (α), (α^*), (β), (β^*), (γ) and (γ^*) are all equivalent.

2.2.2. Polar decomposition of an operator

Theorem (1)

Let M be a dense subspace of a normed space X. Let T be a linear operator from M to a Banach space Y. If T is bounded, then there uniquely exists $\overline{T}$ which is the extention of T from X to Y. i.e $\overline{T}x = Tx$ for all $x \in M$ and $\|\overline{T}\| = \|T\|$

Proof

Let M be a dense subspace of a normed space X.
Then $\Rightarrow \overline{M} = X$
Hence for any $x \in X$, there exists $\{x_n\} \subset M$ such that $x_n \to x$.
$\Rightarrow \|Tx_m - Tx_n\| \leq \|T\|\|x_m - x_n\| \to 0$as $m, n \to \infty$
$\Rightarrow \{Tx_n\}$ is a cauchy sequence in Y.
Since Y is a Banach space, Y is complete.
Hence $\{Tx_n\}$ converges in Y.
Hence there exists $y_0 \in Y$ such that $Tx_n \to y_0$.
This limit point y_0 is determined independently from its choice of $\{x_n\}$ converging to x. i.e y_0 depends only on x.

Put $\overline{T}x = y_0$.
This $\overline{T}$ defines an operator $\overline{T}$ from X to Y.
For any $x \in M \subset X$, we can choose $x_n = x, \ \forall n$.
Then $\overline{T}x = Tx$.
Hence $\overline{T}$ is an extension of T.
Claim:To show that $\overline{T}$ is linear, $\overline{T}$ is bounded and $\|\overline{T}\| = \|T\|$
Let $x_1, x_2 \in X$ and α, β scalars.
By definition of $\overline{T}$,
$\overline{T}(x_1) = y_1$, where $x_{n_1} \to x_1$ and $Tx_{n_1} \to y_1$
$\overline{T}(x_2) = y_2$, where $x_{n_2} \to x_2$ and $Tx_{n_2} \to y_2$
Now $x_{n_1}, x_{n_2} \in M$ and T is linear on M.
$\therefore \ T(\alpha x_{n_1} + \beta x_{n_2}) = \alpha Tx_{n_1} + \beta Tx_{n_2} \to \alpha y_1 + \beta y_2$
Hence

$$\begin{aligned} \overline{T}(\alpha x_1 + \beta x_2) &= \alpha y_1 + \beta y_2 \\ &= \alpha \overline{T}x_1 + \beta \overline{T}x_2 \end{aligned}$$

Hence $\overline{T}$ is linear.

By the continuity if norm,

$$\|\overline{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} \|T\|\|x_n\| = \|T\|\|x\|$$

Hence $\overline{T}$ is bounded and $\|\overline{T}\| \leq \|T\|$ (1)

On the other hand,

$$\begin{aligned} \|T\| &= \sup \|Tx\|/x \in M, \|x\| \leq 1 \\ &\leq \sup \|\overline{T}x\|/x \in X, \|x\| \leq 1 \\ &\leq \|\overline{T}\| \end{aligned}$$

(i.e)$\|T\| \leq \|\overline{T}\|$ (2)

From (1)and (2), $\|\overline{T}\| = \|T\|$

Hence if T is bounded, there exists $\overline{T}$ which is the extension of T from X to Y such that $\|\overline{T}\| = \|T\|$

To prove that $\overline{T}$ is unique
Let $\hat{T}$ be a bounded linear operator and an extension of T from X to Y.
For any $x \in X$, take $\{x_n\} \subset M$ such that $x_n \to x$
By the continuity of $\hat{T}$,

$$\hat{T}x = \lim_{n\to\infty} \hat{T}x_n = \lim_{n\to\infty} Tx_n = \overline{T}x, (\text{by definition of } \overline{T})$$

$\Rightarrow \hat{T}x = \overline{T}x, \forall x \in X.$
$\Rightarrow \hat{T} = \overline{T}.$
Hence $\overline{T}$ is unique.
Hence the theorem.

Theorem

2 Let S and T be bounded linear operators on a Hilbert space H. If $T^*T = S^*S$, then there exists a partial isometry operator U such that the initial space $M = \overline{R(T)}$ and the final space $N = \overline{R(S)}$ and S=UT.

Proof

Let $T^*T = S^*S$. Then for any $x \in H$,

$$\begin{aligned} \Rightarrow \|Tx\|^2 = < Tx, Tx > &= < T^*Tx, x > \\ &= < S^*Sx, x > \\ &= < Sx, Sx > \\ &= \|Sx\|^2 \end{aligned}$$

i.e $\|Tx\|^2 = \|Sx\|^2, \ \forall x \in H$ (1)

Hence if $Tx_1 = Tx_2$, for $x_1, \ x_2 \in H$ then

$\|Sx_1 - Sx_2\| = \|Tx_1 - Tx_2\| = 0 \Rightarrow \ Sx_1 = Sx_2$

i.e $Tx_1 = Tx_2 \Rightarrow Sx_1 = Sx_2, \ \forall x_1, x_2 \in H.$ (2)

Define an operator $V : R(T) \to R(S)$ as

$$VTx = Sx$$

Then

$$V(Tx_1 + Tx_2) = V(T(x_1 + x_2)) = S(x_1 + x_2) = Sx_1 + Sx_2 = VTx_1 + VTx_2$$

$$V(\alpha Tx) = V(T(\alpha x)) = S(\alpha x) = \alpha Sx = \alpha VTx$$

Hence V is linear on R(T).
Then $\|VTx\| = \|Sx\| = \|Tx\|$ by (1).
Therefore if $y \in R(T)$, then $\|Vy\| = \|y\|$ (3)
Hence V is a bounded linear operator and $N = \overline{R(S)}$ is a Banach space.
Hence V can be extended to $\overline{V}$ from $M = \overline{R(T)}$ onto N.
i.e for $y \in M, \exists \{y_n\} \subset R(T) \ni \ y_n \to y$ and $Vy_n \to \overline{V}y$

and $\|\overline{V}y\| = \lim_{n\to\infty} \|Vy_n\| = \lim_{n\to\infty} \|y_n\| = \|y\|$ (4)
Now, define U as

$$Ux = \overline{V}P_M x, \forall x \in H.$$

Then for $x \in M = \overline{R(T)}$,

$$\|Ux\| = \|\overline{V}P_M x\| = \|\overline{V}x\| = \|x\|$$

($\because x \in M, P_M x = x$ and $\|\overline{V}y\| = \|y\|$ by (4))
and for $x \in M^\perp$,

$$\|Ux\| = \|\overline{V}P_M x\| = \|P_M x\| = 0$$

($\because \|\overline{V}y\| = \|y\|$ by (4) and $\because x \in M^\perp, P_M x = 0$)
Hence

$$\begin{aligned} \|Ux\| &= \|x\|, \ \forall x \in M \\ \text{and } Ux &= 0, \forall x \in M^\perp. \end{aligned}$$

Hence U is a partial isometry with the initial space M.

For any $x \in H$, consider

$$UTx = \overline{V}P_M Tx = \overline{V}Tx = VTx = Sx$$

Hence S=UT
Moreover $VR(T) = R(S)$, $\overline{V}$ is an isometry and
$R(U) = \overline{V}M = \overline{R(S)} = N$.
Hence N is the final space of U.
Hence the theorem.

Theorem (3)

Let T be any operator on a Hilbert space H. Then there exists a partial isometry operator U such that $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and M and N, the initial and final space of U can be expressed as follows:
$M = \overline{R(|T|)} = \overline{R(T^*)}$ and $N = \overline{R(T)}$
Moreover $N(U) = N(|T|)$ and $U^*U|T| = |T|$

Proof

Since $|T|^2 = T^*T$, replacing T by$|T|$ and S by T in the theorem,
"Let S and T be bounded linear operators on a Hilbert space H. If $T^*T = S^*S$, then there exists a partial isometry operator U such that the initial space $M = \overline{R(T)}$ and the final space $N = \overline{R(S)}$ and $S = UT$."
We get, there exists a partial isometry operator U such that
$M = \overline{R(|T|)}$, $N = \overline{R(|T|)}$ and $T = U|T|$.

Therefore,

$$\begin{aligned} \|Ux\| &= \|x\|, \forall x \in M = \overline{R(|T|)} \\ Ux &= 0, \forall x \in M^{\perp} = \overline{R(|T|)}^{\perp} \end{aligned}$$

Also $N(U)^{\perp} = \overline{R(|T|)} = N(|T|)^{\perp}$
$\Rightarrow N(U) = N(|T|)$
Since $U^*T = U^*U|T| = |T|$,

$$T^*U = (U^*T)^* = |T|^* = |T|$$

Hence $R(|T|) = R(T^*U) \subset R(T^*)$
On the other hand, since $T^* = (U|T|)^* = |T|U^*$,

$$R(T^*) \subset R(|T|)$$

Hence $R(|T|) = R(T^*)$
Hence the theorem.

Definition (1)

Let T be an operator on a Hilbert space H. When $T = U|T|$ with $N(U) = N(|T|), T = U|T|$ is said to be the polar decomposition of T. If the kernel condition $N(U) = N(|T|)$ is not necessarily satisfied, $T = U|T|$ is said to be a decomposition of T.

Theorem (4)

Let $T = U|T|$ be the polar decomposition of an operator T on a Hilbert space H. Then the following (i)and (ii) hold

(i) $N(|T|) = N(T)$

(ii) $|T^*|^q = U|T|^q U^*$ for any positive number q.

To prove (i)

$$
\begin{aligned}
x \in N(|T|) &\iff |T|x = 0 \\
&\iff |T|^2 x = 0 \qquad [\because < |T|^2 x, x > = < |T|x, |T|x > = \||T|x\|] \\
&\iff T^*Tx = 0 \\
&\iff \|Tx\|^2 = 0 \qquad [\because \|Tx\|^2 = < Tx, Tx > = < T^*Tx, x >] \\
&\iff Tx = 0 \\
&\iff x \in N(T)
\end{aligned}
$$

Hence $N(|T|) = N(T)$
Hence (i) is proved.

To prove (ii)

For any positive operator S and for any positive number q,

$$N(S^q) = N(S)(1)$$

Therefore, since $|T|$ is a positive operator,

$$N(|T|^q) = N(|T|)$$

i.e $\overline{R(|T|^q)}^{\perp} = \overline{R(|T|)}^{\perp} \implies \overline{R(|T|^q)} = \overline{R(|T|)}$ (2)
Also U^*U is the initial projection on $M = \overline{R(|T|)}$
i.e $U^*U|T| = |T|$ (3)
Hence

$$U^*U|T|^q = (U^*U|T|)|T|^{q-1} = |T||T|^{q-1} = |T|^q$$

i.e $U^*U|T|^q = |T|^q$ (4)
Using (3),

$$|T^*|^2 = TT^* = (U|T|)(U|T|)^* = U|T||T|U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2$$

Therefore

$$f_n(|T^*|^2) = f_n(U|T|U^*)^2 = Uf_n(|T|^2)U^*$$

for any polynomial $f_n(t)$. (6)

Take $f_n(t) \to t^{1/2}$

Then by (6), $|T^*| = U|T|U^*$

(since the square root $S^{1/2}$ of a positive operator S is approximated uniformly by polynomials of S.)

By induction,

$|T^*|^{\frac{n}{m}} = U|T|^{\frac{n}{m}}U^*$ holds for any natural number m and n

Let $\frac{n}{m} \to q$,

then $|T^*|^q = U|T|^qU^*$ for any positive number q

Hence (ii) is proved.

Theorem

Let $T = U|T|$ be the polar decomposition of an operator T on a Hilbert space H. Then$T^* = U^*|T^*|$ is also the polar decomposition of an operator T^*

Proof

Since $T = U|T|$ is the polar decomposition of T,
$N(U) = N(|T|)$ holds (1)
Now,

$$\begin{aligned} T^* &= (U|T|)^* \\ &= |T|^* U^* \\ &= |T| U^* \qquad [\because |T|^* = |T|] \\ &= U^* U |T| U^* \qquad [\because U^* U |T| = |T|] \\ &= U^* |T^*| \qquad [\because |T^*| = U|T|U^*] \end{aligned}$$

Hence $T^* = U^*|T^*|$

Hence to prove that $T^* = U^*|T^*|$ is tha polar decomposition of T^*, it is sufficient to prove that $N(U^*) = N(|T^*|)$.

$$\begin{aligned}
\text{Now } x \in N(U^*) &\iff U^*x = 0 \\
&\iff \|U^*x\|^2 = 0 \\
&\iff UU^*x = 0 \qquad [\because \|U^*\|^2 =< U^*x, U^*x >=< UU^*x, x \\
&\iff |T|U^*x = 0 \qquad [\because N(U) = N(|T|)] \\
&\iff T^*x = 0 \qquad [\because T^* = |T|U^*] \\
&\iff |T^*|x = 0 \\
&\iff x \in N(|T^*|)
\end{aligned}$$

Hence $N(U^*) = N(|T^*|)$
Hence $T^* = U^*|T^*|$ is the polar decomposition of T^*.
Hence the theorem

2.3 Polar Decomposition of an operator and its Application

2.3.1 Invariant subspace and reducing subspace

An operator T on a Hilbert space H can be decomposed into $T = UP$, where U is a partial isometry and $P = |T| = (T^*T)^{1/2}$ with $N(U) = N(P)$, $N(X)$ denote the kernel of an operator X, the kernel condition $N(U) = N(P)$ uniquely determines U and P of the polar decomposition $T = UP$.

Definition (1)

If an operator T commutes S and S^*, then T is said to doubly commutes with S.
i.e $TS = ST, \ TS^* = S^*T$.

Definition (2)

Let T be an operator on a Hilbert space H.

(i) A closed subspace M of a Hilbert space H is said to be invariant under T if $TM \subset M$.
i.e $Tx \in M$ whenever $x \in M$

(ii) A closed subspace M of a Hilbert space H is said to reduce T if $TM \subset M$ and $TM^{\perp} \subset M^{\perp}$. i.e M and $M^{\perp}$ are both invariant under T.

Theorem (1)

Let T be an operator on Hilbert space H and M be a closed subspace of H. Then the following conditions are mutually equivalent:

(i) $TM \subset M$

(ii) $T^*M^\perp \subset M^\perp$

(iii) $TP = PTP$, where P is the projection onto M.

Proof

Let T be an operator on Hilbert space H, M be a closed subspace of H and P be the projection onto M.

To prove that (i)$\Rightarrow$ (iii)
Assume $TM \subset M$ to prove that $TP = PTP$
If $x \in H$, then $Px \in M$
$\Rightarrow TPx \in M \quad [\because TM \subset M]$
Hence $PTPx = TPx \quad [\because P$ is projection on M$]$
Hence PTP=TP
Hence (i)$\Rightarrow$(iii) (1)
Conversely, assume that $PTP = TP$
Let $y \in M$
$\because$ P is Projection of H onto M, $\exists\, x \in H \ni Px = y$
then $Ty = TPx = PTPx = P(TPx) \in M$.
i.e $y \in M \Rightarrow Ty \in M$
Hence $TM \subset M$
Hence (iii)$\Rightarrow$(i) (2)
From (1)and (2),
(i) $\iff$ (iii) (3)

To prove that (ii) $\iff$ (iii).
by (3), (i) $\iff$ (iii)
i.e $TM \subset M \iff TP = PTP$.
Hence

$$\begin{aligned} T^*M^{\perp} \subset M^{\perp} &\iff T^*(I-P) = (I-P)T^*(I-P) \\ &\iff T^* - T^*P = T^* - PT^* - T^*P + PT^*P \\ &\iff PT^* = PT^*P \\ &\iff (PT^*)^* = (PT^*P)^* \\ &\iff TP = PTP \end{aligned}$$

Hence (ii) $\iff$ (iii) (4)
From (3) and (4), (i) $\iff$ (ii) $\iff$ (iii). Hence the theorem.

Theorem (2)

Let T be an operator on Hilbert space H and M be a closed subspace of H. Then the following conditions are mutually equivalent:

(i) M reduces T

(ii) $M^{\perp}$ reduces T

(iii) M reduces T^*

(iv) M is invaraiant under T and T^*

(v) $TP = PT$, where P is the projection onto M.

Proof

Let T be an operator on Hilbert space H , M be a closed subspace of H and P be the projection onto M.

To prove that (i)⇒(ii)

By definition,

M reduces T $\iff$ $TM \subset M$ and $TM^{\perp} \subset M^{\perp}$ $\iff$ $M^{\perp}$ reduces T.

Hence M reduces T iff $M^{\perp}$ reduces T

Hence (i) $\iff$ (ii) (1)

To prove that (i) $\iff$ (iii)

By the result,

$TM \subset M \iff T^*M^{\perp} \subset M^{\perp}$,

we have

$TM \subset M$ and $TM^{\perp} \subset M^{\perp} \iff T^*M^{\perp} \subset M^{\perp}$ and $T^*M \subset M$.

$\Rightarrow$ M reduces T $\iff$ M reduces T^*

Hence (i) $\iff$ (iii)

To prove that (i) $\iff$ (iv)

$$\begin{aligned} \text{M reduces T} &\iff TM \subset M \ \&\ TM^{\perp} \subset M^{\perp} \\ &\iff TM \subset M \ \&\ T^{*}M^{\perp} \subset M^{\perp} \\ &\qquad (\because TM^{\perp} \subset M^{\perp} \iff T^{*}M^{\perp} \subset M^{\perp}) \\ &\iff \text{Mis invariant under T and } T^{*} \end{aligned}$$

Hence (i) $\iff$ (iv)

To prove that (iv) $\iff$ (v)

$$
\begin{aligned}
& \text{M is invariant under T and } T^* \\
\iff \quad & TM \subset M \;\&\; T^*M \subset M \\
\iff \quad & TP = PTP \;\&\; T^*P = PT^*P \text{[by previous theorem]} \\
\iff \quad & TP = PTP \;\&\; (T^*P)^* = (PT^*P)^* \\
\iff \quad & TP = PTP \;\&\; PT = PTP \\
\iff \quad & PT = TP
\end{aligned}
$$

Hence (iv) $\iff$ (v)

Hence the theorem.

2.3.2 A necessary and sufficient condition for $T_1T_2 = T_2T_1$& $T_1T_2^* = T_2^*T_1$.

Theorem (1)

If $T = UP$ is the polar decomposition of an operator T, then U and P commutes with A and A^*, where A denotes any operator which commutes with T and T^*.

Proof

Let $T = UP$ be the polar decomposition of an operator T.

Then $N(U) = N(P)$, where $P = |T| = (T^*T)^{1/2}$ (1)

Let A commutes with T and T^*

i.e $AT = TA$ and $AT^* = T^*A$ (2)

Consider

$$\begin{aligned}(T^*T)A &= T^*(TA) = T^*(AT)\\ &= (T^*A)T\\ &= (AT^*)T\\ &= A(T^*T)\end{aligned}$$

$\Rightarrow P^2A = AP^2$

$\Rightarrow PA = AP \qquad (3)$

$\Rightarrow (PA)^* = (AP)^*$

$\Rightarrow A^*P = PA^* \qquad (4)$

Hence P commutes with A and A^*.

Substituting T=UP in(2), i.e $AT - TA = 0$, we get

$$\begin{aligned} & AUP - UPA = 0 \\ \Rightarrow\ & AUP - UAP = 0 [\because PA = AP] \\ \Rightarrow\ & (AU - UA)P = 0 \\ \Rightarrow\ & AU - UA \text{ annihilates } \overline{R(P)} \end{aligned}$$

If $x \in N(P) = N(U)$, then $Px = 0$ and $Ux = 0$.

$$\begin{aligned} Px = 0 \ \Rightarrow\ & APx = 0 \\ \Rightarrow\ & PAx = 0 \\ \Rightarrow\ & Ax \in N(P) = N(U) \\ \Rightarrow\ & U(Ax) = 0 \\ Ux = 0 \ \Rightarrow\ & A(Ux) = 0 \end{aligned}$$

$\Rightarrow\ (AU - UA)x = 0,\ \forall x \in N(P)$
$\Rightarrow\ AU - UA$ annihilates $N(P)$also.
Hence $AU - UA = 0$ on $H = \overline{R(P)} \oplus N(P)$
Hence $AU = UA$ (5)

Similarly substituting $T^* = PU^*$ in (2)
i.e in $AT^* - T^*A = 0$, we get

$$\begin{aligned} & APU^* - PU^*A = 0 \\ \Rightarrow \quad & PAU^* - PU^*A = 0 \\ \Rightarrow \quad & P(AU^* - U^*A) = 0 \\ \Rightarrow \quad & [P(AU^* - U^*A)]^* = 0 \\ \Rightarrow \quad & (UA^* - A^*U)P = 0 \end{aligned}$$

$\Rightarrow$ $UA^* - A^*U$ annihilates $\overline{R(P)}$

If $x \in N(P) = N(U)$, then $Px = 0$ and $Ux = 0$

$$\begin{aligned} Px = 0 &\Rightarrow A^*Px = 0 \\ &\Rightarrow PA^*x = 0 \qquad [\because\ PA^* = A^*P] \\ &\Rightarrow A^*x \in N(P) = N(U) \\ &\Rightarrow UA^*x = 0 \\ Ux = 0 &\Rightarrow A^*Ux = 0 \end{aligned}$$

$\Rightarrow (UA^* - A^*U)x = 0, \ \forall x \in N(P)$
$\Rightarrow UA^* - A^*U$ annihilates $N(P)$
Hence $UA^* - A^*U = 0$ on $H = \overline{R(P)} \oplus N(P)$
Hence $UA^* = A^*U$
Hence the theorem.

Theorem (2)

Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decomposition of T_1 and T_2 respectively. Then the following conditions are equivalent.

(A) T_1 doubly commutes with T_2

(B) Each of U_1^*, U_1 and P_1 commutes with each of U_2^*, U_2 and P_2

(C) The following five equation are satisfied:

- (C-1) $P_1P_2 = P_2P_1$
- (C-2) $U_1P_2 = P_2U_1$
- (C-3) $P_1U_2 = U_2P_1$
- (C-4) $U_1U_2 = U_2U_1$
- (C-5) $U_1^*U_2 = U_2U_1^*$

Proof

Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decompositions of T_1 and T_2 respectively.
Assume (A) T_1 doubly commutes with T_2.
Taking $A = T_1$ and $T = T_2$ in Theorem 1, we get

U_2 and P_2 commutes T_1 and T_1^*

Now taking $A = U_2$ and $T = T_1$ in the same theorem we get,

U_1 and P_1 commutes with U_2 and U_2^*

Hence $U_1U_2 = U_2U_1, \quad U_1U_2^* = U_2^*U_1, \quad P_1U_2 = U_2P_1, \quad P_1U_2^* = U_2^*P_1$
(1)
Similarly taking $A = P_2$ and $T = T_1$ in the same theorem we get,

U_1 and P_1 commutes with P_2 $[\because P_2^* = P_2]$

Hence $U_1P_2 = P_2U_1, \qquad P_1P_2 = P_2P_1$ (2)

Taking adjoint in (1)

$U_2^*U_1^* = U_1^*U_2^*, \qquad U_2U_1^* = U_1^*U_2$

$U_2^*P_1 = P_1U_2^*, \qquad U_2P_1 = P_1U_2 \qquad (3)$

Taking adjoint in (2)

$P_2U_1^* = U_1^*P_2, \qquad P_1P_2 = P_2P_1 \qquad (4)$

From (1), (2), (3) and (4), it is clear that each of $U_1^*,\ U_1$ and P_1 commutes with each of $U_2^*,\ U_2$ and P_2

Hence (A)⇒(B)

Again from (1), (2), (3) and (4), it is clear that

$$\begin{aligned} P_1P_2 &= P_2P_1 \\ U_1P_2 &= P_2U_1 \\ P_1U_2 &= U_2P_1 \\ U_1U_2 &= U_2U_1 \\ U_1^*U_2 &= U_2U_1^* \end{aligned}$$

Hence (B)⇒(C),

To prove that (C)⇒(A)
Now Assume (C-1)through (C-5)

$$\begin{aligned} \text{Consider } T_1T_2 &= U_1P_1U_2P_2 \\ &= U_1U_2P_1P_2 [\text{from (C-3)}] \\ &= U_2U_1P_2P_1 [\text{from (C-4) and (C-1)}] \\ &= U_2P_2U_1P_1 [\text{from (C-2)}] \\ &= T_2T_1 \end{aligned}$$

$$\begin{aligned} \text{Consider } T_1T_2^* &= U_1P_1P_2U_2^* \\ &= U_1P_2P_1U_2^* [\text{from (C-1)}] \\ &= P_2U_1P_1U_2^* [\text{from (C-2)}] \\ &= P_2U_1(U_2P_1)^* \\ &= P_2U_1(P_1U_2)^* [\text{from (C-3)}] \\ &= P_2U_1U_2^*P_1 \\ &= P_2U_2^*U_1P_1 \\ &= (U_2P_2)^*(U_1P_1) \\ &= T_2^*T_1 \end{aligned}$$

Hence (C)⇒(A)
Hence (A)⇒(B)⇒(C)⇒(A)
Hence the theorem.

Corollary 3

Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decomposition of T_1 and T_2 respectively.
If T_1 doubly commutes with T_2, then $T_1T_2 = (U_1U_2)(P_1P_2)$ is the polar decomposition of T_1T_2, i.e $(P_1P_2) = |T_1T_2|$ and U_1U_2 is the partial isometry of T_1T_2 with $N(U_1U_2) = N(P_1P_2)$.

Proof

Let T_1 doubly commutes with T_2.
Then by the theorem 2,

$$\begin{aligned} T_1T_2 &= (U_1P_1)(U_2P_2) \\ &= U_1(P_1U_2)P_2 \\ &= U_1(U_2P_1)P_2[\text{by (C-3)}] \\ &= (U_1U_2)(P_1P_2) \end{aligned}$$

i.e $T_1T_2 = (U_1U_2)(P_1P_2)$ (1)

Consider

$$\begin{aligned}
(U_1U_2)(U_1U_2)^*(U_1U_2) &= (U_1U_2)(U_2^*U_1^*)(U_1U_2) \\
&= (U_1U_2)(U_1^*U_2^*)(U_1U_2)[\text{by (C-4) } U_1U_2 = U_2U_1 = \\
&= U_1(U_2U_1^*)(U_2^*U_1)U_2 \\
&= U_1(U_1^*U_2)(U_1U_2^*)U_2[\text{by (C-5)}] \\
&= U_1U_1^*(U_2U_1)U_2^*U_2 \\
&= U_1U_1^*(U_1U_2)U_2^*U_2[\text{by (C-4)}] \\
&= (U_1U_1^*U_1)(U_2U_2^*U_2) \\
&= U_1U_2 \\
&\quad (\because U_1 \text{ and } U_2 \text{are partial isometries } U_1U_1^*U_1 = U
\end{aligned}$$

Hence U_1U_2 is a partial isometry.

$$\begin{aligned}
\text{Consider } |T_1T_2|^2 &= (T_1T_2)^*(T_1T_2) \\
&= T_2^*T_1^*T_2T_1 \\
&= T_1^*T_2^*T_1T_2 \\
&= T_1^*T_1T_2^*T_2 \\
&= P_1^2P_2^2 \\
&= (P_1P_2)^2 \\
\Rightarrow |T_1T_2| &= P_1P_2.
\end{aligned}$$

Now to prove that $N(U_1U_2) = N(P_1P_2)$

$$\begin{aligned}
x \in N(U_1U_2) &\iff U_1U_2x = 0 \\
&\iff U_2x \in N(U_1) = N(P_1) \\
&\iff P_1U_2x = 0 \\
&\iff U_2P_1x = 0 \\
&\iff P_1x \in N(U_2) = N(P_2) \\
&\iff P_2P_1x = 0 \\
&\iff P_1P_2x = 0 \\
&\iff x \in N(P_1P_2)
\end{aligned}$$

Hence $N(U_1U_2) = N(P_1P_2)$
Hence the theorem that if T_1 doubly commutes with T_2, then $T_1T_2 = (U_1U_2)(P_1P_2)$ is the polar decomposition of T_1T_2.

Corollary 4(Polar decomposition)

Every operator T can be expressed in the form $U|T|$ where U is a partial isometry with $N(U) = N(|T|)$. This kernal condition uniquely determines U, U and $|T|$ commute with V^*, V and $|A|$ of the polar decomposition $A = V|A|$ of any operator A commuting with T and T^*.

Proof

By the theorem,
"Let T be any operator on a Hilbert space H. Then there exists a partial isometry operator U such that $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and M and N, the initial and final space of U can be expressed as follows:
$M = \overline{R(|T|)} = \overline{R(T^*)}$ and $N = \overline{R(T)}$
Moreover $N(U) = N(|T|)$ and $U^*U|T| = |T|$"

Every operator T can be expressed in the form $U|T|$ where U is a partial isometry with $N(U) = N(|T|)$ and kernal condition uniquely determines U.
Put $T_2 = T$ and $T_1 = A$, in Theorem 1, then we get,
U and $|T|$ commute with V^*, V and $|A|$ of the polar decomposition $A = V|A|$ of any operator A commuting with T and T^*.
Hence the theorem.

Corollary 5

Let $T = UP$ be the polar decomposition of an operator T. Then T is normal iff U commutes with P and U is unitary on $N(T)^{\perp}$.

Proof

In the theorem 2,
Put $T = T_1,\ T_2 = T$, then conditions of (A) is equivalent to the normality of A and condition (B) is equivalent to that U commutes with P and $U^*U = UU^*$.
Therefore U is unitary on the initial space of $U = N(T)^{\perp}$.

Theorem (6)

Let T be a normal operator. Then there exists a unitary operator U such that $T = UP = PU$ and both U and P commutes with V^*, V and $|A|$ of the polar decomposition $A = V|A|$ of any operator A commutes with T and T^*

Proof

Let $T = U_1P = PU_1$ be the polar decomposition of a normal operator T.
Let $A = V|A|$ be the polar decomposition of A.
By the result
"Let $T = UP$ be the polar decomposition of an operator T then T is normal iff U commutes with P and U is unitary on $N(T)^{\perp}$.

$$U_1^*U_1 = U_1U_1^* \quad (1)$$

and the initial space M of U_1 coinsides with the final space N.
i.e $U_1M \subset N = M$
Hence M reduces U_1

Hence $U_1P_M = P_MU_1 = P_MU_1P$ (2)
where $P_M = U_1^*U_1$ denotes the projection of H onto M. Put
$U = U_1P_M + I - P_M$

$$\begin{aligned}
\text{Consider } U^*U &= (P_MU_1^* + I - P_M)(U_1P_M + I - P_M)\\
&= P_MU_1^*U_1P_M + U_1P_M - P_MU_1P_M + P_MU_1^* + I - P_M - P\\
&= P_MP_MP_M + U_1P_M - U_1P_M + P_MU_1^* + I - 2P_M - U_1^*P_M\\
&= 2P_M + I - 2P_M\\
&= I
\end{aligned}$$

Similarly, $U^*U = (U_1P_M + I - P_M)(P_MU_1^* + I - P_M) = I$
Hence U is unitary
Since $P_MP = U^*UP = P$ (because $U^*U|T| = |T|$) and $P = P^* = PP_M$,

$$P_MP = P = P^* = PP_M$$

$$\begin{aligned} UP &= (U_1P_M + I - P_M)P \\ &= U_1P_MP + IP - P_MP \\ &= U_1P + P - P \\ &= U_1P \\ &= T \end{aligned}$$

Therefore $T = UP$.
Similarly $T = PU_1 = PU$
Hence $T = UP = PU$
Also by theorem 2,
P commutes with V^*, V and $|A|$.
By the same theorem,
U_1 commutes with V^*, V and $|A|$
$\therefore$ $P_M = U_1^*U_1$ commutes with V^*, V and $|A|$
i.e $P_M|A| = |A|P_M$, $P_MV = VP_M$ and $P_MV^* = V^*P_M$.

$$\begin{aligned}
VU &= V(U_1P_M + I - P_M) \\
&= VU_1P_M + V - VP_M \\
&= U_1VP_M + V - VP_M \\
&= U_1P_MV + V - P_MV \\
&= (U_1P_M + I - P_M)V \\
&= UV
\end{aligned}$$

Similarly $V^*U = UV^*$ and $|A|U = U|A|$
Hence the theorem.

Theorem (7)

Every normal operator T can be written in the form UP, where P is positive and U may be taken to unitary such that U and P commute with each other and with all operators commuting with T and T^*.

Proof

(By Theorem 6)

Theorem (F-P(Fuglede-Putnam))

Let A and B be normal operator. If $AX = XB$ holds for some operator X, then $A^*X = XB^*$.

Proof

Since (i) e^{iS} is a unitary operator for any self adjoint S and
(ii) $AX = XB \Rightarrow A^nX = XB^n$ for any natural number n,

$$e^{i\bar{\lambda}A}X = Xe^{i\bar{\lambda}B} \text{ for any complex number } \lambda.(1)$$

Define $f(\lambda) = e^{i\lambda A^*} X e^{-i\lambda B^*}$, for $\lambda \in C$.

Then

$$\begin{aligned} f(\lambda) &= e^{i\lambda A^*} e^{i\bar{\lambda} A} X e^{-i\bar{\lambda} B} e^{-i\lambda B^*} \text{ Using (1)} \\ &= e^{i(\lambda A^* + \bar{\lambda} A)} X e^{-i(\lambda B^* + \bar{\lambda} B)} \text{ by the normality of A and B (2)} \end{aligned}$$

Since $(\lambda A^* + \bar{\lambda} A)^* = \bar{\lambda} A + \lambda A^*$ and $(-1(\bar{\lambda} B + \lambda B^*))^* = -1(\lambda B^* + \bar{\lambda} B)$, $\lambda A^* + \bar{\lambda} A$ and $-1(\lambda B^* + \bar{\lambda} B)$ are self-adjoint operators, and by (i), $e^{i(\lambda A^* + \bar{\lambda} A)}$ and $e^{-i(\lambda B^* + \bar{\lambda} B)}$ are both unitary operators.
Hence by (2) $f(\lambda)$ is analytic and bounded for all complex number λ.

Hence by Liouville's theorem, $f(\lambda)$ is constant. i.e

$$f(\lambda) = f(0) = e^0 X e^0 = X$$

$$\text{Hence } e^{i\lambda A^*} X e^{-i\lambda B^*} = X, \text{ for any } \lambda.$$

$$\Rightarrow e^{i\lambda A^*} X = X e^{i\lambda B^*}$$

Differentiating both sides w.r.to λ

$$iA^* e^{i\lambda A^*} X = X i B^* e^{i\lambda B^*}$$

$$\Rightarrow A^* e^{i\lambda A^*} X = X B^* e^{i\lambda B^*}, \text{ for all } \lambda \in C.$$

Put $\lambda = 0 \Rightarrow A^* X = X B^*$
Hence the required result.

Corollary 8

Let $T_1 = U_1P_1$ be the polar decomposition of an operator T_1 and let T_2 be a normal operator and $T_2 = U_2P_2$ be the decomposition of T_2 such that P_2 is positive, U_2 is unitary, U_2 and P_2 commute with V^*, V and $|A|$ of the polar decomposition $A = V|A|$ of any operator A commuting with T_2 and T_2^*. then the following conditions are equivalent.

(A) T_1 commutes with T_2

(B) Each of U_1^*, U_1 and P_1 commutes with each of U_2^*, U_2 and P_2

(C) U_1 and P_1 commutes with U_2 and P_2

Proof

Since T_2 is normal, $T_2 T_2^* = T_2^* T_2$ (1)
Assume (A) i.e T_1 commutes with T_2 , $T_1 T_2 = T_2 T_1$ (2)
(or) $T_2 T_1 = T_1 T_2$ (3)
$\therefore$ Taking $A = B = T_2$ in Fuglede-Putam inequality $T_2^* T_1 = T_1 T_2^*$
i.e $T_1 T_2^* = T_2^* T_1$ (4)
$\Rightarrow \; T_2 T_1^* = T_1^* T_2$ (5)
Hence from (2)and (5), the normal operator T_2 commutes with T_1 and T_1^*
Hence U_2 and P_2 commutes with U_1^*, U_1 and P_1
Hence (B) is shown.
Hence (A)$\Rightarrow$(B)

(C) trivially follows from (B)
Hence (B)⇒(C)
Now assume (C) i.e U_1 and P_1 commutes with U_2 and P_2 then

$$\begin{aligned} T_1T_2 &= U_1P_1U_2P_2 \\ &= U_1U_2P_1P_2 \\ &= U_2U_1P_2P_1 \\ &= U_2P_2U_1P_1 \\ &= T_2T_1 \end{aligned}$$

Hence T_1 commutes with T_2
Hence (C)⇒(A)
Hence the theorem.

2.3.3 Polar decompositionof nonnorml operator

Theorem (1)

Suppose that $N(T) \subset N(T^*)$ and let $T = UP$ be the polar decomposition of T. Then there exists an isometry U_1 such that $T = U_1P$ and both U_1 and P commute with V^*, V and $|A|$ of the polar decomposition $A = V|A|$ of any operator A commuting with T and T^*. In case $N(T) = N(T^*)$, U_1 can be chosen to be unitary.

Proof:

Assume that $N(T) \subset N(T^*)$ and $T = UP$ be the polar decomposition of T.

$N(T) \subset N(T^*)$ implies $N(T)^{\perp} \supset N(T^*)^{\perp} = \overline{R(T)}$

Since $T = UP$ is the polar decomposition of T, U is a partial isometry on the initial space $M = \overline{R(T)}$ and $N(U) = N(T)$

$\therefore UM \subseteq M$ and $Ux = 0, \ \forall\, x \in M^{\perp}$

$\Rightarrow \ UM^{\perp} \subseteq M^{\perp}$

Hence M reduces T.

Corollary 2

Let T be a quasinormal operator. Then there exists an isometry U such that $T = UP = PU$ and U and P commute with U^*, V and $|A|$ of the polar decomposition $A = U|A|$ of any operator A commuting with T and T^*.

Theorem (3)

Let $T = U|T|$ be the polar decomposition of an operator T. Then $T = U|T|$ is quasi normal iff $U|T| = |T|U$.

2.3.4 A necessary and sufficient conditions for $T_1T_2 = T_2T_3$ and $T_1^*T_2 = T_2T_3^*$

Theorem (1)

Let $T_k = U_kP_k$ be the polar decomposition of T_k for $k = 1, 2$ and 3. Then the following conditions are equivalent:

(A) $T_1T_2 = T_2T_3$ and $T_1^*T_2 = T_2T_3^*$

(B) $(B-1) P_3P_2 = P_2P_3$,
$(B-2) U_3P_2 = P_2U_3$,
$(B-3) P_1U_2 = U_2P_3$,
$(B-4) U_1U_2 = U_2U_3$ and
$(B-5) U_1^*U_2 = U_2U_3^*$

Corollary 2

Let $T_1 = U_1P_1$ and $T_3 = U_3P_3$ be the decomposition described in Theorem 6 in 2.3.2 of normal operators T_1 andT_3 and let$T_2 = U_2P_2$ be the polar decomposition of an operator T_2. tiondiThen the following conditions are equivalent:

(A) $T_1T_2 = T_2T_3$

(B) (B-1), (B-2), (B-3),(B-4) and (B-5)in Theorem 1 hold.

(C) (B-1), (B-2), (B-3) and (B-4)in Theorem 1 hold.

Lemma 1

Let T_1 and T_2 be operators on Hilbert spaces H_1 and H_2, respectively. If T_1 is unitarily equivalent to T_2 and T_1 has an algebraic definite (or semi definite) property Σ with $\{p_\alpha\}$, then so has T_2

Corollary 3

Let$T_k = U_kP_k$ be the polar decomposition of T_k for $k = 1, 2$ and 3. and let $T_1T_2 = T_2T_3$ and $T_1^*T_2^* = T_2T_3^*$ Then

(1) $\overline{R(T_2)}$ reduces U_1, U_2 and T_1 $N(T_2)$ reduces U_3, P_3 and T_3

(2) $U_{1|\overline{R(T_2)}}$(res $P_{1|\overline{R(T_2)}}$, $T_{1|\overline{R(T_2)}}$) is unitarily equivalent to $U_{3|N(T_2)^\perp}$ (res $P_{3|\overline{N(T_2)}^\perp}$, $T_{3|\overline{N(T_2)}^\perp}$)

(3) When T_2 has a dense range, and if U_3(res. P_3 and T_3) has an algebraic definite property Σ with polynomials $\{p_\alpha,\}$ then so has U_1(res P_1 and T_1)

(4) When T_2 is injective, and if U_1(res. P_1 and T_1) has an algebraic definite property Σ with polynomials $\{p_\alpha,\}$ then so has U_3(res P_3 and T_3)

2.3.5 Hereditary property on the polar decomposition of an operator

Theorem (1)

Let $T = U|T|$ be the polar decomposition of an operator T. Then $T^2 = 0$ iff $U^2 = 0$

Theorem (2)

Let $T = U|T|$ be the polar decomosition of an operator T. Then

1. If T is binormal, then so is U.
2. If T is quasinormal, then so is U;
   $U = \text{isometry} \oplus 0$ on $N(T)^{\perp} \oplus N(T)$
3. If T is normal, then so is U; $U = \text{unitary} \oplus 0$ on $N(T)^{\perp} \oplus N(T)$
4. If T is self-adjoinit, then so is U;
   $U = \text{symmetry} \oplus 0$ on $N(T)^{\perp} \oplus N(T)$
5. If T is positive, then so is U; U=projection.

Remark 1(Berberian)

Let A and B be normal operators and X be an operator on a Hilbertspace. Then the following (i) and (ii) hold and follows from each other

(i) If $AX = XA$, then $A^*X = XA^*$

(ii) If $AX = XB$, then $A^*X = XB^*$

Thank You